Cheap-Talk with Random Stopping

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Abstract

Cheap-Talk with Random Stopping is a Cheap-Talk game in which after each period of communication, with probability $1-\lambda$, the talk ends and the players play the original game (i.e, choose actions and receive payoffs). In this paper the relations between Cheap-Talk games and Cheap-Talk with Random Stopping are analyzed.

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1 Introduction

Cheap-Talk is a communication that costs nothing and occurs before the players choose their actions. The payoffs to the players depend only on their actions and not on the messages that were sent during the Cheap-Talk phase. Cheap-Talk with Incomplete Information is a three phase game. In the first phase the players receive their private information, in the second phase they talk (communicate) and in the last phase they choose actions and get payoffs. Again, the payoffs depend only on the actions. The players can use the Cheap-Talk to transfer information and to choose an equilibrium from a possible set of equilibria. The players can also ignore the Cheap-Talk and in this case the set of equilibria is the same as in the game without Cheap-Talk.

A weakness of the model is the fact that the actions are chosen after infinitely many periods of communication, which seems incorrect for modeling real-life situations. To overcome this problem, one can discuss variations of the Cheap-Talk model, like Cheap-Talk with a finite number of periods of communication (see Aumann & Hart 1996) or the model of Cheap-Talk with Random Stopping that we discuss here.

Cheap-Talk with Random Stopping is a Cheap-Talk game in which at each period the communication is stopped with probability $1-\lambda$ $(0<\lambda<1)$ and the players choose actions and receive payoffs. We discuss some of the relations between Cheap-Talk with Random Stopping and the original Cheap-Talk model.

The models are defined in section 2 and in section 3 we give the main results. In section 4 two examples are analyzed and in section 5 we discuss Polite-Talk with Random Stopping games.

2 The Model

Cheap-Talk is an extension of a game. Cheap-Talk with complete information is a two stage game. The first stage, the talk-stage, is divided into infinite number of periods. At each period, each player chooses a message, m, from a finite set of messages, M. The players have perfect recall, hence the messages can be a function of all the past history. In the second stage, the action-stage, each player chooses an action. These actions defines the payoffs to the players according to the original game. Polite-Talk is a similar extension in which at each period only one player can send a message. The Cheap-Talk with Random Stopping extension of G, $G(M,\lambda)$, is a Cheap-Talk extension in which at every period, with probability $1-\lambda$, the game is stopped and the players choose actions and receive their payoffs. We define three games - G, $G(M)$ and $G(M,\lambda)$. G is a (finite) game of incomplete information, $G(M)$ is its cheap-talk extension and $G(M,\lambda)$ is its cheap-talk with random stopping extension. In $G(M)$ and $G(M,\lambda)$ the cheap-talk occurs after the players have received their private information.

- $G=(N,(C_n)_{n\in N},(K_n)_{n\in N},(p_n)_{n\in N},(u_n)_{n\in N})$ is defined by the following:
    1. N is a finite set of players. Without loss of generality (and with some abuse of notation) we assume that $N=\{1,2,3,...,N\}$.
    2. C_n is a finite set of actions for player $n\in N$. Let $C:=\prod_{n\in N}C_n$ and $C_{-n}:=\prod_{m\in N\setminus\{n\}}C_m$.
    3. K_n is a finite set of types of player $n\in N$. Each player $n\in N$ knows his own type $k_n\in K_n$. Let $K:=\prod_{n\in N}K_n$ and $K_{-n}:=\prod_{m\in N\setminus\{n\}}K_m$.

4. $p_n : K_n \to \Delta(K_{-n})$ for [1] all $n \in N$. The belief of type k_n of player n is $p_n(k_n)$. We will denote $p_n(k_n)$ by p_{k_n}, hence $p_{k_n}(k_{-n})$ is the probability assigned by type k_n of player n to the combination of types $k_{-n} \in K_{-n}$.
5. $u_n : C \times K \to \mathbb{R}$ is the payoff function of player n. That is, $u_n(c, k)$ is the payoff of player n for the profile of actions $c \in C$ and profile of types $k \in K$.
6. The game is played as follows: each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n \in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is:
$$E_{k_n}(c) := \sum_{k_{-n} \in K_{-n}} u_n(c, (k_n, k_{-n}))p_{k_n}(k_{-n})$$
7. 1,2,3,4,5 and 6 are common knowledge.

- The game $G(M) = ((N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N}, M)$ is defined by 1,2,3,4,5 and in addition:

8. A finite set M, the set of possible messages in the Talk phase.
9. The game $G(M)$ has two phases:

   The Talk Phase : This phase is divided into periods t=1,2,3... . For each t and $n \in N$, player n chooses a message $m_t^n \in M$. The choices are made simultaneously. (In Polite-Talk only player number t mod N sends a message).

   The Action Phase : Each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n \in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is (see item 6 above): $E_{k_n}(c) := \sum_{k_{-n} \in K_{-n}} u_n(c, (k_n, k_{-n}))p_{k_n}(k_{-n})$.
10. All players have perfect recall.
11. 1,2,3,4,5,8,9,10 are common knowledge.

In order to define the "Cheap-Talk with Random Stopping" extension, $G(M, \lambda)$, one should replace item 9 with

9*. The game $G(M, \lambda)$ is divided into periods $t = 1, 2, 3, \ldots$. In each period, t, each player $n \in N$ chooses a message $m_t^n \in M$ and an action $c_t^n \in C_n$ (in Polite-Talk with Random Stopping, all the players choose actions but only player number t mod N chooses a message). Let $c_t := (c_t^n)_{n \in N}$. With probability $1 - \lambda$, the game is stopped and the players receive payoffs according to the actions chosen in the last period: The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is (see items 6 and 9 above): $E_{k_n}(c_t) := \sum_{k_{-n} \in K_{-n}} u_n(c_t, (k_n, k_{-n}))p_{k_n}(k_{-n})$. With probability λ the game continues, and the players observe the messages (note that they do not observe the actions if the game continues).

The specific items of the set M does not affect the equilibrium payoffs. The only feature that may matter is the size of the set M, hence we will assume, without loss of generality (but with some abuse of notation) that $M = \{1, 2, 3, ..., |M|\}$. Recall that m_t^n is the message sent by player n at period t and let h_t be the history up to period t, i.e,
$$h_t := ((m_1^1, m_1^2, ..., m_1^N), (m_2^1, m_2^2, ..., m_2^N), ..., (m_t^1, m_t^2, ..., m_t^N))$$

[1]for a finite set X, the set $\Delta(X)$ is the $|X| - 1$ dimensional simplex of probability vectors on X.

($h_0 := \phi$) and let h_∞ be the infinite history,

$$h_\infty := ((m_1^1, m_1^2, ..., m_1^N), (m_2^1, m_2^2, ..., m_2^N), ..., (m_t^1, m_t^2, ..., m_t^N), ...).$$

Denote by H_t the set of histories of length t and denote by H_∞ the set of infinite histories. Denote the set of non-negative integers ($\{0, 1, 2, ...\}$) by $\mathbb{N}_0$. A strategy of player n in a cheap-talk game with complete information is a pair, $\sigma^n := \{\{\sigma_t^n\}_{t \in \mathbb{N}_0}, \sigma_\infty^n\}$ such that

1. $\sigma_t^n : H_t \to \Delta(M)$ (in Polite-Talk σ_t^n is defined only for $t \equiv n \mod N$).
2. $\sigma_\infty^n : H_\infty \to \Delta(C_n)$.

In Cheap-Talk games with incomplete information the players receive their private information before the talk-phase. Recall that K_n is the set of types of player n. A strategy for player n in a cheap-talk game with incomplete information is a pair, $\sigma^n := \{\{\sigma_t^n\}_{t \in \mathbb{N}_0}, \sigma_\infty^n\}$ such that

1. $\sigma_t^n : K_n \times H_t \to \Delta(M)$ (in Polite-Talk σ_t^n is defined only for $t \equiv n \mod N$).
2. $\sigma_\infty^n : K_n \times H_\infty \to \Delta(C_n)$.

In $G(M, \lambda)$, a strategy for player n, σ^n, is a sequence $\sigma^n := \{\sigma_t^n\}_{t=1}^\infty$ such that $\sigma_t^n : K_n \times H_t \to \Delta(M) \times \Delta(C_n)$ (in games with complete information $\sigma_t^n : H_t \to \Delta(M) \times \Delta(C_n)$). Note that $h_t := ((m_1^1, m_1^2, ..., m_1^N), (m_2^1, m_2^2, ..., m_2^N), ..., (m_t^1, m_t^2, ..., m_t^N))$ as in the original Cheap-Talk extension, i.e., the players do not observe the actions, just the messages.

3 Main results

Fix a game G. Denote by $\mathcal{E}$ the set of equilibrium payoffs of the game G. Denote by $\mathcal{E}_M$ and $\mathcal{E}_{M,\lambda}$ the sets of equilibrium payoffs in the Cheap-talk and the Cheap-Talk with Random Stopping extensions of G (respectively).

Theorem 3.1: *$\mathcal{E}_{M,\lambda} \subset \mathcal{E}_M$ for $M \geq 2$ and any $0 < \lambda < 1$.*

Proof: Using jointly controlled lotteries (here we use the assumption that $M \geq 2$) the players can simulate the lottery by which the game (with the random stopping) is being stopped (with probability $(\lambda, 1 - \lambda)$).∎

Remark: Theorem 3.1 is valid for games with complete information as well as for games with incomplete information.

Theorem 3.2: *For games with complete information $\mathcal{E}_{M,\lambda} = \mathcal{E}_M$ for $M \geq 2$ and $\lambda \geq \frac{N}{N+1}$.*

Proof: Denote by conv(A) the convex hull of the set A. $\mathcal{E}_M = \text{conv}(\mathcal{E})$ (Aumann & Hart 1996). Using theorem 3.1 it is enough to show that $\text{conv}(\mathcal{E}) \subset \mathcal{E}_{M,\lambda}$. Choose $e \in \text{conv}(\mathcal{E})$. $\mathcal{E} \subset \mathbb{R}^N$, hence there exist $e_1, e_2, ..., e_{N+1} \in \mathcal{E}$ and $p \in \Delta(\{1, 2, 3, ..., N+1\})$ such that $\sum_{i=1}^{N+1} p(i) e_i = e$ (Carathéodory's theorem). Let $(\alpha_i^1, \alpha_i^2, ..., \alpha_i^N) \in \Delta(C_1) \times \Delta(C_2) \times ... \times \Delta(C_N)$ be equilibrium strategies in G with expected payoffs e_i (for all $1 \leq i \leq N+1$). Define equilibrium strategies $\sigma^1, \sigma^2, ..., \sigma^N$, together with a sequence of probability vectors $p_t \in \Delta(\{1, 2, 3, ..., N+1\})$ by induction on t. Let $p_0 := p$ and $i_0 := \max \arg_{1 \leq i \leq N+1} p(i)$. Choose arbitrary $\beta \in \Delta(M)$. For every $t \in \mathbb{N}_0$ and $h_t \in H_t$ define:

$$\sigma_t^j(h_t) = (\beta, \alpha_{i_t}^j), \quad \forall 1 \leq j \leq N$$

(i.e, the players send an arbitrary message and play the equilibrium e_{i_t} if the game is stopped at period t).

$$p_{t+1}(i) := \begin{cases} \frac{p_t(i)}{\lambda} & i \neq i_t \\ \frac{(p_t(i)-(1-\lambda))}{\lambda} & i = i_t \end{cases}$$

$$i_{t+1} := \max \arg_{1 \leq i \leq N+1} p_{t+1}(i). \text{ Note that } p_{t+1}(i_{t+1}) \geq \frac{1}{N+1} \geq 1-\lambda.$$

$p_t(i)$ is the probability that the equilibrium e_i should be played given that the game was not stopped before period t. Using the strategies $\sigma^1, \sigma^2, ..., \sigma^N$ the players play the equilibrium e_i with probability $p(i)$, hence the expected payoffs are according to e. ▮

Remark: Note that in the proof of theorem 3.2 the players do not use the messages and therefore the theorem that the set of equilibria includes $conv(\mathcal{E})$ is valid also for weaker models as Polite-Talk with Random Stopping or Cheap-Talk with $M = 1$ (which it actually Cheap-Talk without a talk).

Denote by $cl(A)$ the closure of the set A. The next two theorems show that for games with incomplete information theorem 3.2 is not true in general.

Theorem 3.3: *There exists a game with incomplete information on one side, such that*

$$cl\left(\bigcup_{0<\lambda<1} \mathcal{E}_{2,\lambda}\right) \subsetneq \mathcal{E}_{3,\lambda'} \subset \mathcal{E}_2$$

for every $0 < \lambda' < 1$.

I.e, there exists an equilibrium that can be achieved in Cheap-talk with Random Stopping with three messages but can not be achieved in Cheap-talk with Random Stopping with two messages.

Theorem 3.4: *There exists a game with incomplete information on one side, such that*

$$cl\left(\bigcup_{M\in\mathbb{N}, 0<\lambda<1} \mathcal{E}_{M,\lambda}\right) \subsetneq \mathcal{E}_{M'} = \mathcal{E}_2$$

for every $M' \geq 2$.

I.e, there exists an equilibrium that can be achieved in Cheap-talk games with two messages but can not be achieved in Cheap-talk with Random Stopping games for any $0 < \lambda < 1$ and a set of messages.

The proofs for theorems 3.3 and 3.4 will be given by the examples in the next section.

4 Examples

In order to prove theorems 3.3 and 3.4 we need a few definitions and lemmas.

Let G be a two-player game with incomplete information on one side and let K be the set of possible types of player 1. Let $\mathbf{k}$ be the (random) type of player 1. For $p \in \Delta(K)$ denote by G_p the game G with $Prob(\mathbf{k} = k) = p(k)$ for $k \in K$. Let σ and τ be equilibrium strategies of player 1 and player 2 respectively. Denote by $P = P_{\sigma,\tau}$ the probability induced by σ and τ. Let $E = E_{\sigma,\tau}$ be the expectation with respect to P. For a finite history h_t, let $p_{h_t} \in \Delta(K)$ be the probability vector defined by $p^k_{h_t} := P(\mathbf{k} = k \mid h_t)$ for all $k \in K$. Let a^k be the random payoff of type k of player 1.

Denote by $a^{k,t}$ the random payoff of type k of player 1 at period t. Let $a_t^k := (1-\lambda)\sum_{r=t}^{\infty}\lambda^r a^{k,r}$, be the total expected payoff of type k of player 1 in $G(M,\lambda)$ from time t on. For $a, c \in \mathbb{R}^K$ we will write $c \geq a$ when $c^k \geq a^k$ for all $k \in K$. Let

$$\mathcal{D} := \{(a,p) \in \mathbb{R}^K \times \Delta(K) \text{ s.t. } a \text{ is an equilibrium vector payoff of player 1 in } G_p\}$$

$$\mathcal{D}^+ := \{(a,p) \in \mathbb{R}^K \times \Delta(K) \text{ s.t. } \exists c \geq a \text{ s.t. } (c,p) \in \mathcal{D} \text{ and } p(k) > 0 \text{ implies } a(k) = c(k)\}$$

Lemma 4.1: *If a is an equilibrium vector payoff of player 1 in $G_p(M,\lambda)$ then there exist equilibrium strategies σ and τ and a bi-martingale $\{(a_t,p_t)\}_{t=0}^{\infty}$ with the following properties:*

1. $(a_0,p_0) = (a,p)$.
2. $\{(a_t,p_t)\}_{t=0}^{\infty}$ *converges a.s. to* $\mathcal{D}^+$.
3. $p_{h_t}^k = P(\mathbf{k} = k \mid h_t)$.
4. $a_{h_t}^k \geq E(a^k \mid h_t, \mathbf{k} = k)$ *a.s. and* $P(h_t \mid \mathbf{k} = k) > 0$ *(i.e.* $P(h_t) > 0$ *and* $p_{h_t}^k > 0$*) implies* $a_{h_t}^k = E(a^k \mid h_t, \mathbf{k} = k)$ *a.s.*

Proof: Theorem 3.1 and Aumann & Hart (1996). ∎

Lemma 4.2: *If $f : \mathbb{R}^K \times \Delta(K) \to \mathbb{R}$ is a bi-convex function satisfying $f(a,p) \leq 0$ for all $(a,p) \in \mathcal{D}^+$ then $f(a,p) \leq 0$ for all $(a,p) \in \mathcal{E}_{M,\lambda}$.*

Proof: Theorem 3.1 and Aumann & Hart (1996). ∎

4.1 Example 1

The following example is a two-player game with incomplete information on one side. There are three types of player 1 and one type of player 2. The payoffs are only a function of the action chosen by player 2 and the type of player 1. In this game, there exists an equilibrium that can be obtained only by full revelation of the information (i.e, the type) of player 1. Let G be the following game:

	A	B	C	D	E	F
$\mathbf{k} = 1$	10 0	-10 0	-10 0	9 0	9 0	0 1
$\mathbf{k} = 2$	-10 0	10 0	-10 0	9 0	0 1	9 0
$\mathbf{k} = 3$	-10 0	-10 0	10 0	0 1	9 0	9 0

$P(\mathbf{k} = 1) = P(\mathbf{k} = 2) = P(\mathbf{k} = 3) = \frac{1}{3}$.

From the moment that player 2 knows the type of player 1 he will play the action A if player 1 is of type 1, B if player 1 is of type 2 and C if player 1 is of type 3. This yields a payoff of 0 to player 1 and 10 to player 2 (from the moment that the information has been revealed). This can be easily obtain in cheap-talk (with or without random stopping) with at least three messages (player 1 sends the first message if he is of type 1, the second message if he is of type 2 and the third message if he is of type 3). In cheap-talk without random stopping, this can also be done with two messages (player 1 reveals his type in two steps), but it can not be done in cheap-talk with random stopping with two messages, as player 1 will always gain from cheating in this setup. The formal proofs are given in the next lemmas and theorems.

Theorem 4.3: *All equilibria in $G(2,\lambda)$ are non revealing for every $0<\lambda<1$.*

Proof: Assume that there is an equilibrium with revelation of information in $G(2,\lambda)$. Let h_s be an history such that $P(h_s)>0$ and such that $p_{h_s}=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and $p_{(h_s,(m_1,m_2))}\neq(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ for $m_1=1,2$. Let $(r^1,r^2,r^3):=p_{(h_s,(1,m_2))}$ and $(q^1,q^2,q^3):=p_{(h_s,(2,m_2))}$.

Lemma 4.4: *W.l.o.g we can assume that $\frac{2}{3}\geq r^1\geq r^2>r^3$.*

Proof: The game is symmetric, hence we can assume that $r^1\geq r^2\geq r^3$. We will show that if $\frac{2}{3}\geq r^1\geq r^2>r^3$ does not hold then $\frac{2}{3}\geq q^3\geq q^2>q^1$ must hold.

1. If $r^1\geq r^2=r^3$ then $r^1>r^2=r^3$ (because $(r^1,r^2,r^3)\neq(\frac{1}{3},\frac{1}{3},\frac{1}{3})$). Therefore $\frac{2}{3}>0.5\geq q^3=q^2>q^1$ (because $(\frac{1}{3},\frac{1}{3},\frac{1}{3})=p_{h_s}=P(m_1=1\mid h_s)\cdot r+P(m_1=2\mid h_s)\cdot q$).
2. If $r^1\geq r^2>r^3$ and $r^1>\frac{2}{3}$ then $r^1>r^2$ and $q^3>q^2>q^1$. If $q^3>\frac{2}{3}$ we have $q^2<\frac{1}{3}$ and $r^2<\frac{1}{3}$, a contradiction to the fact that their average is $\frac{1}{3}$. Hence $\frac{2}{3}\geq q^3\geq q^2>q^1$.

∎

When we say that under some conditions "player 2 plays" a specific action we mean that under these conditions the action is strongly dominant.

Lemma 4.5: *If $\frac{2}{3}\geq p^1\geq p^2>p^3$ then in G_p player 2 plays D and the payoff for type 3 of player 1 is* 1.

Lemma 4.6: *If $p^3>\frac{1}{3}$ then player 2 does not play D and the payoff for type 3 of player 1 is* 0.

W.l.o.g assume that $h_s=h_0$. Recall that $r^k:=P(\mathbf{k}=k\mid h_0,m_0^1=1)$ and $q^k:=P(\mathbf{k}=k\mid h_0,m_0^1=2)$. $r^3<\frac{1}{3}$ (lemma 4.4) and $q^3>\frac{1}{3}$.

Lemma 4.7: $E(a_1^3\mid m_0^1=2,\mathbf{k}=3)=0$.

Proof: The bi-linear function $f(a,p):=a^3\cdot(p^3-\frac{1}{3})$ satisfies $f(a,p)\leq 0$ for every $(a,p)\in\mathcal{D}^+$ (lemma 4.6 and the fact that the payoffs for player 1 are non-negative). Hence $f(a,p)\leq 0$ for every equilibrium (lemma 4.2), hence $q^3>\frac{1}{3}$ implies $E(a_1^3\mid m_0^1=2,\mathbf{k}=3)=0$. ∎

$E(a_1^3\mid m_0^1=1,\mathbf{k}=3)\geq\lambda(1-\lambda)$ (lemma 4.4 , lemma 4.5 and the fact that the minimal payoff of player 1 is 0), hence type 3 of player 1 will get more by sending the message $m_0^1=1$ with probability 1. A contradiction. This ends the proof of theorem 4.3. ∎

Theorem 4.8: *In every equilibrium of $G(2,\lambda)$ $(0<\lambda<1)$ the payoff for player 2 is* 6.

Proof: Theorem 4.3. ∎

Theorem 4.9: *If $M\geq 3$ and $0<\lambda<1$ then there exists an equilibrium in $G(M,\lambda)$ such that the payoff of player 2 is $10-4\lambda$. If $M\geq 2$ there exists an equilibrium in $G(M)$ such that the payoff of player 2 is* 10.

Proof: We will prove the theorem for $G(3)$ (i.e, $M = \{1,2,3\}$). In the first period player 1 send the message k ($k = 1,2,3$) if $\mathbf{k} = k$. From the second period on, player 2 plays A if he got the message 1, B if he got the message 2 and C if he got the message 3. ∎
We can now complete the proof of theorem 3.3.
Proof of theorem 3.3:
Theorems 4.8 and 4.9 ($\mathcal{E}_{3,\lambda'} \subset \mathcal{E}_2$ follows from Theorem 3.1 and Aumann & Hart 1996). ∎

4.2 Example 2

The following example is also a two-player game with incomplete information on one side. Here there are five types of player 1 and one type of player 2. As in the previous example, the payoff is only a function of the action chosen by player 2 and the type of player 1. G is the following game:

	A	B	C	D	E	F	G	H	I	J
	-9	-9	-9	-9	-4	-9	-9	-9	-9	1
$\mathbf{k} = 1$	1	1	1	1	1	1	1	-1	-1	0
	2	0	0	0	1	0	1	0	1	-4
$\mathbf{k} = 2$	-20	-20	-20	-20	-1	1	1	-1	-1	0
	0	2	0	0	1	0	1	1	0	-4
$\mathbf{k} = 3$	-20	-20	-20	-20	-1	1	1	-1	-1	0
	0	0	2	0	1	1	0	0	1	-4
$\mathbf{k} = 4$	-20	-20	-20	-20	-1	1	1	-1	-1	0
	0	0	0	2	1	1	0	1	0	-4
$\mathbf{k} = 5$	-20	-20	-20	-20	-1	1	1	-1	-1	0

$P(\mathbf{k} = 1) = \frac{1}{2}$, $P(\mathbf{k} = 2) = P(\mathbf{k} = 3) = P(\mathbf{k} = 4) = P(\mathbf{k} = 5) = \frac{1}{8}$.

In this game, in the original cheap talk there is an equilibrium in which player 1 reveals some of the information in two steps. This can not be done in cheap talk with random stopping (with any set of message and $0 < \lambda < 1$) because the information can not be revealed in one step, and player 1 will gain from cheating in the process of revealing the information. This is proved in the next lemmas and theorems.

Theorem 4.10: *If $M \geq 2$ then there exists an equilibrium in $G(M)$ with payoff* 1 *for player 2 and* 0 *for all the types of player 1 (this is also true for Polite-Talk).*

Proof: If $\mathbf{k} = 1$, player 1 sends the message $m_1^1 = 1$ in the first period, and player 2 will play the action J. If $\mathbf{k} > 1$ the communication takes three periods: In the first period player 1 send the message $m_1^1 = 2$ (the message of player 2 is not important). In the second period player 2 sends the message $m_2^2 = 1$ with probability $\frac{1}{2}$ and $m_2^2 = 2$ with probability $\frac{1}{2}$. In the third period, there are four cases (according to m_2^2 and the type of player 1):

1. $m_2^2 = 1$ and $\mathbf{k}$ is 4 or 5 - in this case player 1 sends the message $m_3^1 = 1$ and player 2 plays the action F.

2. $m_2^2 = 1$ and $\mathbf{k}$ is 2 or 3 - in this case player 1 sends the message $m_3^1 = 2$ and player 2 plays the action G.

3. $m_2^2 = 2$ and $\mathbf{k}$ is 3 or 5 - in this case player 1 sends the message $m_3^1 = 1$ and player 2 plays the action H.

4. $m_2^2 = 2$ and $\mathbf{k}$ is 2 or 4 - in this case player 1 sends the message $m_3^1 = 2$ and player 2 plays the action I.

The payoff of player 2 is 1, the expected payoff of player 1 is 0 and no player can gain anything from deviating. ▮

Theorem 4.11: *Fix M, a set of messages, and $0 < \lambda < 1$. For every equilibrium of $G(M, \lambda)$ and any history h_t, if $P(h_t) > 0$ then $P(\mathbf{k} = 1 \mid h_t) = \frac{1}{2}$.*

We will prove this theorem later.

Theorem 4.12: *The expected payoff of player 2 is -1.5 for every equilibrium in $G(M, \lambda)$.*

Proof: Theorem 4.11. ▮
We can now complete the proof of Theorem 3.4.
Proof of Theorem 3.4: Theorems 3.1, 4.10 and 4.12 ($\mathcal{E}_{M'} = \mathcal{E}_2$ follows from Aumann & Hart 1996). ▮

In order to prove theorem 4.11 we need a few lemmas.

Lemma 4.13: *If $p^1 > \frac{1}{2}$ then player 2 plays J and the payoff of player 1 is 0.*

Lemma 4.14: *If $p^1 < \frac{1}{2}$ then player 2 does not play J.*

Lemma 4.15:
Player 2 plays F only if $p^4 = p^5 = \frac{1}{2}$.
Player 2 plays G only if $p^2 = p^3 = \frac{1}{2}$.
Player 2 plays H only if $p^3 = p^5 = \frac{1}{2}$.
Player 2 plays I only if $p^2 = p^4 = \frac{1}{2}$.

Lemma 4.16: *$P(h_t) > 0$ and $p^1_{h_t} > \frac{1}{2}$ implies* [2] *$a^1_{h_t} = 0$ and $a^k_{h_t} \geq 0$ for $K > 1$.*

Proof: The function $f(a, p)$ defined by

$$f(a,p) := \begin{cases} |a^1| \cdot (p^1 - \frac{1}{2}) & \text{if } p^1 > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

is a bi-convex function, and $f(a, p) = 0$ for every $(a, p) \in \mathcal{D}^+$ (lemma 4.13). Hence $P(h_t) > 0$ implies $f(a_{h_t}, p_{h_t}) \leq 0$ (lemma 4.2). Therefore $P(h_t) > 0$ and $p^1_{h_t} > \frac{1}{2}$ implies $a^1_{h_t} = 0$. This proofs the first part of the lemma.

The function $f(a, p)$ defined by

$$f(a,p) := \begin{cases} -a^k \cdot (p^1 - \frac{1}{2}) & \text{if } p^1 > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

is a bi-convex function, and $f(a, p) \leq 0$ for every $(a, p) \in \mathcal{D}^+$ (lemma 4.13). Hence $P(h_t) > 0$ implies $f(a_{h_t}, p_{h_t}) \leq 0$ (lemma 4.2). Therefore $P(h_t) > 0$ and $p^1_{h_t} > \frac{1}{2}$ implies $a^k_{h_t} \geq 0$. ▮

[2] $a^k_{h_t}$ was defined in lemma 4.1.

Lemma 4.17: *$k > 1$ and $P(h_t) > 0$ implies $p^k_{h_t}(a^k_{h_t} - a^1_{h_t}) \le 0$.*

Proof: For every action of player 2 the payoff of type 1 of player 1 is greater then or equal to the payoff of the other types of player 1 (see the definition of the game G). Hence $(a, p) \in \mathcal{D}^+$ and $p^k > 0$ implies $a^k \le a^1$. Therefore for every $(a, p) \in \mathcal{D}^+$ there exists $p^k(a^k - a^1) \le 0$. On the other hand $p^k(a^k - a^1)$ is a bi-convex function, hence $P(h_t) > 0$ implies $p^k_{h_t}(a^k_{h_t} - a^1_{h_t}) \le 0$ (lemma 4.2). ∎

Lemma 4.18: *$P(h_t) > 0$, $p^1_{h_t} > \frac{1}{2}$ and $p^k_{h_t} > 0$ implies $a^k_{h_t} = 0$.*

Proof: Lemmas 4.16 and 4.17.∎

Lemma 4.19: *$k > 1$, $P(h_t) > 0$, $p^k_{h_t} > 0$ and $a^1_{h_t} = a^k_{h_t}$ implies that if player 1 plays σ^k (the equilibrium strategy of type k), then player 2 never plays A, B, C, D or E after h_t (i.e. he never plays $A - E$ after any strategy h_s s.t. $P(h_s \mid h_t) > 0$).*

Proof: Playing A, B, C, D or E with positive probability yields (see the definition of the game G and lemma 4.1)

$$a^1_{h_t} \ge E(a^1 \mid \text{ player 1 plays } \sigma^1, h_t) \ge E(a^1 \mid \text{ player 1 plays } \sigma^k, h_t)$$
$$> E(a^k \mid \text{ player 1 plays } \sigma^k, h_t) = a^k_{h_t}$$

a contradiction.∎

Proof of theorem 4.11: Assume that $P(h_s) > 0$, $p^1_{h_s} = \frac{1}{2}$, $P(m^1_s = 1 \mid h_s) > 0$ and $P(\mathbf{k} = 1 \mid h_s, m^1_s = 1) > \frac{1}{2}$. W.l.o.g. assume that $P(\mathbf{k} = 1 \mid h_s, m^1_s = 2) < \frac{1}{2}$ and $P(m^1_s = 2 \mid h_s) > 0$. $P(m^2_s \mid h_s) > 0$ implies $a^1_{(h_s,(1,m^2_s))} = 0$ (lemma 4.16), therefore, $a^1_{(h_s,(2,m^2_s))} = 0$ (the bi-martingale property, lemma 4.1). $P(m^2_s \mid h_s) > 0$ implies $a^k_{(h_s,(1,m^2_s))} \ge 0$ for $k > 1$ (lemma 4.16), hence $a^k_{(h_s,(2,m^2_s))} \ge 0$ for $k > 1$. Hence $p^k_{(h_s,(2,m^2_s))} > 0$ implies $a^k_{(h_s,(2,m^2_s))} = 0$ (lemma 4.17 and the fact that $a^1_{(h_s,(2,m^2_s))} = 0$) and therefore $p^k_{(h_s,(2,m^2_s))} > 0$ implies that if player 1 plays σ^k then player 2 never play A, B, C, D or E after $(h_s, (2, m^2_s))$ (lemma 4.19).

Assume that player 2 plays F after the history $(h_s, (2, m^2_s))$, then $p^4_{(h_s,(2,m^2_s))} = p^5_{(h_s,(2,m^2_s))} = \frac{1}{2}$ (lemma 4.15) and player 2 will play F forever (because he will not play $A - E$ and the other actions are not played when $p^1 = p^2 = p^3 = 0$ (lemmas 4.14 and 4.15) yielding $a^1_{(h_s,(2,m^2_s))} \ge 1$, a contradiction. Assume that player 2 plays H after the history $(h_s, (2, m^2_s))$, then $p^3_{(h_s,(2,m^2_s))} = p^5_{(h_s,(2,m^2_s))} = \frac{1}{2}$ (lemma 4.15) and player 2 will play H forever yielding $a^k_{(h_s,(2,m^2_s))} = -1$, for any k such that $p^k_{(h_s,(2,m^2_s))} > 0$, a contradiction. Similar contradictions are achieved assuming that player 2 plays G or I. Player 2 does not play J after the history $(h_s, (2, m^2_s))$ (lemma 4.14), does not play A, B, C, D or E and all the other actions yield a contradiction.∎

5 Polite-Talk

Recall that polite-talk is a cheap-talk in which at each period only one player can send a message. Let $\mathcal{E}^p_M = \mathcal{E}^p_M(G)$ and $\mathcal{E}^p_{M,\lambda} = \mathcal{E}^p_{M,\lambda}(G)$ be the sets of equilibria in Polite-Talk and in Polite-Talk with Random Stopping games, respectively.

Theorem 5.1: *In Polite-Talk games with complete information $\mathcal{E}^p_M \subset \mathcal{E}^p_{M,\lambda}$ for $\lambda \ge \frac{N}{N+1}$. The inclusion may be strict.*

Proof: $\mathcal{E}^p_{M,\lambda} = conv(\mathcal{E})$ (see the proof of theorem 3.2) and $conv(\mathcal{E}) = \mathcal{E}_M \supset \mathcal{E}^p_M$ (see Aumann & Hart 1996). For an example showing that the inclusion may be strict see Aumann & Hart (1996). ∎

Theorem 5.2: *The two sets $\mathcal{E}^p_M$ and $\mathcal{E}^p_{M,\lambda}$ may be incomparable (i.e, neither $\mathcal{E}^p_M \subset \mathcal{E}^p_{M,\lambda}$ nor $\mathcal{E}^p_M \supset \mathcal{E}^p_{M,\lambda}$) in Polite-Talk games with incomplete information.*

Proof: There exists a game G_1 such that $\mathcal{E}^p_M(G_1) \not\supset \mathcal{E}^p_{M,\lambda}(G_1)$ (Aumann & Hart 1996) and there exists a game G_2 such that $\mathcal{E}^p_M(G_2) \not\subset \mathcal{E}^p_{M,\lambda}(G_2)$ (Example 2). Let G be the game in which with probability $\frac{1}{2}$ the game G_1 is played and with probability $\frac{1}{2}$ the game G_2 is played (i.e, there is a lottery, previous to the game, and its outcome is told to the players). G is equivalent to a game with incomplete information and neither $\mathcal{E}^p_M(G) \subset \mathcal{E}^p_{M,\lambda}(G)$ nor $\mathcal{E}^p_M(G) \supset \mathcal{E}^p_{M,\lambda}(G)$ holds. ∎

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