Cheap-Talk with Incomplete Information on Both Sides

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Abstract

We provide a characterization of the set of equilibria of two-person cheap-talk games with incomplete information on both sides. Each equilibrium generates a martingale with certain properties and one can obtain an equilibrium from each such martingale. Moreover, the characterization depends on the number of possible messages. It is shown that for every natural number n, there exist equilibrium payoffs that can be obtained only when the number of possible messages is at least n.

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1 Introduction

Cheap-Talk is a communication that costs nothing and occurs before the players choose their actions. The payoffs to the players depend only on their actions and not on the messages that were sent during the Cheap-Talk phase. Cheap-Talk with Incomplete Information is a three phase game. In the first phase the players receive their private information, in the second phase they talk (communicate) and in the last phase they choose actions and get payoffs. Again, the payoffs depend only on the actions. The players can use the Cheap-Talk to transfer information and to choose an equilibrium from a possible set of equilibria. The players can also ignore the Cheap-Talk and in this case the set of equilibria is the same as in the Incomplete Information game without Cheap-Talk. The Cheap-Talk enlarges the equilibrium set, and a question that rises in this setup is the structure of the new set.

A full characterization in the Two-Player One-Sided Information case was given by Aumann and Hart (1996). In that work they show that the concept of a bi-martingale, first used to characterize the equilibrium set of Repeated Games with Incomplete Information on One Side (Hart 1985, Aumann and Hart 1986), is applicable also in the Cheap-Talk games. They also show that the set of possible messages does not affect the equilibrium set, provided that it includes at least two different messages.

In this work we study the general (finite) two player case. In the Two-Sided Information case we show that bi-martingales are replaced by an appropriate class of "admissible martingales". However, these do not have all the nice properties of bi-martingales. In particular, the size of the set of possible messages does affect the equilibrium set, and in a very strong sense, as an example will show.

In section 2 we discuss some general properties of games with incomplete information and in section 3 we define the notion of admissible martingales and state the main results. In section 4 an example is analyzed. The three sections are independent and can be read in any order.

2 Games with Incomplete Information

2.1 The model

We define two games - G and $\Gamma(M)$. G is a (finite) game of incomplete information, and $\Gamma(M)$ is its cheap-talk extension. In $\Gamma(M)$ the cheap-talk occurs after the players have received their private information.

- $G = (N, (C_n)_{n\in N}, (K_n)_{n\in N}, (p_n)_{n\in N}, (u_n)_{n\in N})$ is defined by the following:
  1. N is a finite set of players. Without loss of generality (and some abuse of notation) we assume that $N = \{1, 2, 3, ..., N\}$.
  2. C_n is a finite set of actions for player $n \in N$. Let $C := \prod_{n\in N} C_n$ and $C_{-n} := \prod_{m\in N\setminus\{n\}} C_m$.
  3. K_n is a finite set of types of player $n \in N$. Each player $n \in N$ knows his own type $k_n \in K_n$. Let $K := \prod_{n\in N} K_n$ and $K_{-n} := \prod_{m\in N\setminus\{n\}} K_m$.

4. $p_n : K_n \to \Delta(K_{-n})$ for [1] all $n \in N$. The belief of type k_n of player n is $p_n(k_n)$. We will denote $p_n(k_n)$ by p_{k_n}, hence $p_{k_n}(k_{-n})$ is the probability assigned by type k_n of player n to the combination of types $k_{-n} \in K_{-n}$.
5. $u_n : C \times K \to \mathbb{R}$ is the payoff function of player n. That is, $u_n(c,k)$ is the payoff of player n for the profile of actions $c \in C$ and profile of types $k \in K$.
6. The game is played as follows: each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n \in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is:
$$E_{k_n}(c) := \sum_{k_{-n} \in K_{-n}} u_n(c,(k_n,k_{-n}))p_{k_n}(k_{-n})$$
7. 1,2,3,4,5 and 6 are common knowledge.

- The game $\Gamma(M) = ((N,(C_n)_{n\in N},(K_n)_{n\in N},(p_n)_{n\in N},(u_n)_{n\in N},M)$ is defined by 1,2,3,4,5 and in addition:

8. A finite set M, the set of possible messages in the talk phase. We assume that $|M| \geq 2$.
9. The game $\Gamma(M)$ has two phases:

   The Talk Phase : This phase is divided into periods t=1,2,3... . For each t and $n \in N$, player n chooses a message $m_t^n \in M$. The choices are made simultaneously.

   The Action Phase : Each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n\in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is (see item 6 above): $E_{k_n}(c) := \sum_{k_{-n}\in K_{-n}} u_n(c,(k_n,k_{-n}))p_{k_n}(k_{-n})$.
10. All players have perfect recall.
11. 1,2,3,4,5,8,9,10 are common knowledge.

Definition 2.1: The beliefs $(p_n)_{n\in N}$ are said to be *(Harsanyi) consistent* if there exists a common prior distribution on K, $P \in \Delta(K)$, such that

$$p_{k_n}(k_{-n}) = \frac{P(k_n,k_{-n})}{P(k_n)} \text{ for all } (k_n,k_{-n}) \in K \text{ and } n \in N$$

(Note that $P(k_n) := \sum_{l_{-n}\in K_{-n}} P(k_n,l_{-n})$). In addition we say that the information is *independent* if $P = \prod_{n\in N} P^n$ for $P^n \in \Delta(K_n)$.
Note: For the results in this paper we do not assume consistency nor independence. These special cases will be useful in our proofs.

The players have perfect recall, hence m_t^n are functions of the history of messages of length $t-1$, $h_{t-1} := ((m_1^1,m_1^2,...,m_1^N),(m_2^1,m_2^2,...,m_2^N),...,(m_{t-1}^1,m_{t-1}^2,...,m_{t-1}^N))$. The actions, chosen by the players in the action phase, are functions of the infinite sequence defined in the talk phase, $h_\infty := ((m_1^1,m_1^2,...,m_1^N),(m_2^1,m_2^2,...,m_2^N),...,(m_t^1,m_t^2,...,m_t^N)...)$. Let $H_t = (M^N)^t$ be the set of histories of length t. Define $H_0 = \{\phi\}$. Let $H_\infty = \prod_{t=1}^{\infty}(M^N)$ be the set of infinite histories. On H_∞, we define, for every t, a finite field $\mathcal{H}_t$ as the field generated by the first t coordinates. That is,

[1] for a finite set X, $\Delta(X)$ is the $|X|-1$ dimensional simplex of probability vectors on X.

h^1_∞ , $h^2_\infty \in H_\infty$ are in the same atom of $\mathcal{H}_t$ if and only if for every $1 \leq u \leq t$ there [2] exists $h^1_\infty(u) = h^2_\infty(u)$. Let $\mathcal{H}_\infty$ be the σ-field generated by $\{\mathcal{H}_t\}_{t=1}^\infty$. Our basic measure space is $(\Omega, \mathcal{A})$ $= (K \times H_\infty \times C \ , 2^K \otimes \mathcal{H}_\infty \otimes 2^C)$. A point in Ω is a triple (k, h_∞, c), where $k \in K$ is a profile of types, $h_\infty \in H_\infty$ is an history of the game (the communication that took place) and $c \in C$ is a profile of actions.

Since $\Gamma(M)$ is a game with perfect recall, we can restrict ourselves to behaviour strategies (see Aumann 1964). To shorten the writing, whenever we write 'strategy' we will mean a behaviour strategy.

Definition 2.2: A *strategy* α^n of player $n \in N$ in G is a function $\alpha^n : K_n \to \Delta(C_n)$.

Let Ψ^n be the set of strategies of player $n \in N$. Denote $\Psi := \prod_{n\in N} \Psi^n$ and $\Psi^{-n} := \prod_{m \in N\setminus\{n\}} \Psi^m$.

Definition 2.3: The subjective payoff of type k_n of player $n \in N$, given a profile of strategies $\alpha = (\alpha^1, \alpha^2, ..., \alpha^N)$ is

$$E^G_{k_n}(\alpha) := \sum_{k_{-n}\in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,...,c_N)\in C} u_n(c, (k_n, k_{-n})) \prod_{m\in N} \alpha^m_{k_m}(c_m)$$

Definition 2.4:
A *strategy* σ^n of player $n \in N$ in $\Gamma(M)$ is $\sigma^n = (\{\sigma^n_t\}_{t\in\mathbb{N}} \ , \sigma^n_\infty)$ such that: [3]

1. $\sigma^n_t : K_n \times H_{t-1} \to \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\sigma^n_\infty : K_n \times H_\infty \to \Delta(C_n)$.
3. σ^n_∞ is $2^K \otimes \mathcal{H}_\infty$-measurable.

Note that in $\Gamma(M)$ the strategy has a talk component, $\{\sigma^n_t\}_{t\in\mathbb{N}}$, and an action component, σ^n_∞. Let Σ^n be the set of strategies of player n for $n \in N$. Let $\Sigma := \prod_{n\in N} \Sigma^n$.

Every pair $(\sigma, k) \in \Sigma \times K$ defines a *probability* measure $\pi_{\sigma,k}$ on $(H_\infty, \mathcal{H}_\infty)$, i.e, let $h_t := ((m^1_1, m^2_1, ..., m^n_1),(m^1_2, m^2_2, ..., m^n_2), ...,(m^1_t, m^2_t, ..., m^n_t))$, then $\pi_{\sigma,k}(h_t)$ is the probability that the first message sent by player 1 is m^1_1, the first message sent by player 2 is m^2_1, the second message sent by player 1 is m^1_2, and so on, given that $k = (k_1, k_2, ..., k_n)$ is the profile of types and player n plays according to σ^n ($\sigma = (\sigma^1, \sigma^2, ..., \sigma^N)$) for all $n \in N$.

Definition 2.5: The subjective expected payoff of type k_n in $\Gamma(M)$, given a profile of strategies $\sigma = (\sigma^1, \sigma^2, ..., \sigma^N)$, is:

$$E^{\Gamma(M)}_{k_n}(\sigma) := \sum_{k_{-n}\in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,...,c_N)\in C} u_n(c, (k_n, k_{-n})) \int_{H_\infty} \prod_{m\in N} \sigma^m_\infty(k_m, h_\infty)(c_m) d\pi_{\sigma,k}(h_\infty)$$

Definition 2.6: $a \in \prod_{n\in N} \mathbb{R}^{K_n} = \mathbb{R}^{\cup_{n\in N} K_n}$ is an *equilibrium payoff vector* in G if there exists α, a profile of strategies, such that

1. $a^{k_n} = E^G_{k_n}(\alpha)$ for all $n \in N$ and $k_n \in K$.

[2] $h_\infty(u)$ is the vector of N messages sent by the players at period u, according to the infinite history h_∞.
[3] $\mathbb{N}$ denotes the set of natural numbers $\{1,2,3,...\}$.

2. $a^{k_n} \geq E^G_{k_n}(\alpha'^n, \alpha^{-n})$ for all $n \in N$, $k_n \in K$ and $\alpha'^n \in \Psi^n$.

Let $\mathcal{E}^G := \{a \in \mathbb{R}^{\bigcup_{n\in N} K_n}$ s.t. a is an equilibrium payoff vector in $G\}$.

Definition 2.7: $a \in \mathbb{R}^{\bigcup_{n\in N} K_n}$ is an *equilibrium payoff vector* in $\Gamma(M)$ if there exists $\sigma \in \Sigma$ (a profile of strategies) such that

1. $a^{k_n} = E^{\Gamma(M)}_{k_n}(\sigma)$ for all $n \in N$ and $k_n \in K$.

2. $a^{k_n} \geq E^{\Gamma(M)}_{k_n}(\sigma'^n, \sigma^{-n})$ for all $n \in N$, $k_n \in K$ and $\sigma'^n \in \Sigma^n$.

Let $\mathcal{E}^{\Gamma(M)} := \{a \in \mathbb{R}^{\bigcup_{n\in N} K_n}$ s.t a is an equilibrium vector payoff in $\Gamma(M)\}$.

2.2 On some properties of games with incomplete information

Definition 2.8: The games $G^1 = (N^1, (C^1_n)_{n\in N^1}, (K^1_n)_{n\in N^1}, (p^1_n)_{n\in N^1}, (u^1_n)_{n\in N^1})$ and $G^2 = (N^2, (C^2_n)_{n\in N^2}, (K^2_n)_{n\in N^2}, (p^2_n)_{n\in N^2}, (u^2_n)_{n\in N^2})$ are *equivalent* if the following holds:

1. $N^1 = N^2 = N$.

2. $C^1_n = C^2_n = C_n$ for all $n \in N$.

3. $K^1_n = K^2_n = K_n$ for all $n \in N$.

4. $E^{G^1}_{k_n}(\alpha) = E^{G^2}_{k_n}(\alpha)$ for all $n \in N$, $k_n \in K_n$ and $\alpha \in \Psi$ (note that 1,2 and 3 implies $\Psi^1 = \Psi^2 = \Psi$).

Remark: Instead of the last condition we could use:

5. For every $n \in N$ and $k_n \in K_n$ there exists $a_n > 0$ and b_n such that $E^{G^1}_{k_n}(\alpha) = a_n E^{G^2}_{k_n}(\alpha) + b_n$ for all $\alpha \in \Psi$.

Definition 2.9: The games $\Gamma^1(M) = (N^1, (C^1_n)_{n\in N^1}, (K^1_n)_{n\in N^1}, (p^1_n)_{n\in N^1}, (u^1_n)_{n\in N^1}, M)$ and $\Gamma^2(M) = (N^2, (C^2_n)_{n\in N^2}, (K^2_n)_{n\in N^2}, (p^2_n)_{n\in N^2}, (u^2_n)_{n\in N^2}, M)$ are *equivalent* if the following holds:

1. $N^1 = N^2 = N$.

2. $C^1_n = C^2_n = C_n$ for all $n \in N$.

3. $K^1_n = K^2_n = K_n$ for all $n \in N$.

4. $E^{\Gamma^1(M)}_{k_n}(\sigma) = E^{\Gamma^2(M)}_{k_n}(\sigma)$ for all $n \in N$, $k_n \in K_n$ and $\sigma \in \Sigma$.

remark: similar definition can be defined for repeated games.

Following Myerson (1991) we can state the following theorem:

Theorem 2.10: *Every finite game with incomplete information is equivalent to a game with consistent and independent incomplete information.*

Proof:
Fix a game $G = (N, (C_n)_{n\in N}, (K_n)_{n\in N}, (p_n)_{n\in N}, (u_n)_{n\in N})$. We define a game, equivalent to G, $\tilde{G}=(N, (C_n)_{n\in N}, (K_n)_{n\in N}, (\tilde{p}_n)_{n\in N}, (\tilde{u}_n)_{n\in N})$ by:

$$\tilde{p}_{k_n}(k_{-n}) := \frac{1}{|K_{-n}|} \quad \text{for all } n \in N,\ k_n \in K_n \text{ and } k_{-n} \in K_{-n}.$$

and

$$\tilde{u}_n(c,k) := |K_{-n}| p_{k_n}(k_{-n}) u_n(c,k) \quad \text{for all } n \in N,\ k_n \in K_n \text{ and } k_{-n} \in K_{-n}.$$

$$E_{k_n}^{\tilde{G}}(\alpha) := \sum_{k_{-n}\in K_{-n}} \tilde{p}_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,\ldots,c_N)\in C} \tilde{u}_n(c,(k_n,k_{-n})) \prod_{m\in N} \alpha_{k_m}^m(c_m)$$

$$= \sum_{k_{-n}\in K_{-n}} \frac{1}{|K_{-n}|} \sum_{c=(c_1,c_2,\ldots,c_N)\in C} |K_{-n}| p_{k_n}(k_{-n}) u_n(c,(k_n,k_{-n})) \prod_{m\in N} \alpha_{k_m}^m(c_m)$$

$$= \sum_{k_{-n}\in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,\ldots,c_N)\in C} u_n(c,(k_n,k_{-n})) \prod_{m\in N} \alpha_{k_m}^m(c_m) = E_{k_n}^G(\alpha)$$

Hence G and $\tilde{G}$ are equivalent and $\tilde{G}$ is a game with independent incomplete information. ∎

In general theorem 2.10 is not correct for specific families of games with incomplete information. For example, zero-sum games with incomplete information are not always equivalent to zero-sum games with independent incomplete information (and indeed the characterization of zero-sum games with independent incomplete information is in general simpler than the characterization of the general (even the consistent) zero-sum games with incomplete information). The transformation in the proof of theorem 2.10 changes zero-sum games with incomplete information into games with independent incomplete information that are not necessarily zero-sum. On the other hand theorem 2.10 is still correct for cheap-talk games and for repeated games as the following theorem shows.

Theorem 2.11: *Every cheap-talk game with incomplete information is equivalent to a cheap-talk game with consistent and independent incomplete information.*

Proof:
Fix a game $\Gamma(M) = (N, (C_n)_{n\in N}, (K_n)_{n\in N}, (p_n)_{n\in N}, (u_n)_{n\in N}, M)$. We define a game, $\tilde{\Gamma}(M) = (N, (C_n)_{n\in N}, (K_n)_{n\in N}, (\tilde{p}_n)_{n\in N}, (\tilde{u}_n)_{n\in N}, M)$, equivalent to $\Gamma(M)$, by $\tilde{p}_{k_n}(k_{-n}) := \frac{1}{|K_{-n}|}$ and $\tilde{u}_n(c,k) := |K_{-n}| p_{k_n}(k_{-n}) u_n(c,k)$ for all $n \in N$, $k_n \in K_n$ and $k_{-n} \in K_{-n}$ (this is the same transformation used in the proof of theorem 2.10).

$$E_{k_n,\sigma}^{\tilde{\Gamma}(M)} = \sum_{k_{-n}\in K_{-n}} \tilde{p}_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,\ldots,c_N)\in C} \tilde{u}_n(c,(k_n,k_{-n})) \int_{H_\infty} \prod_{m\in N} \sigma_\infty^m(k_m,h_\infty)(c_m)\, d\pi_{\sigma,k}(h_\infty)$$

$$= \sum_{k_{-n}\in K_{-n}} \frac{1}{|K_{-n}|} \sum_{c=(c_1,c_2,\ldots,c_N)\in C} |K_{-n}| p_{k_n}(k_{-n}) u_n(c,(k_n,k_{-n})) \int_{H_\infty} \prod_{m\in N} \sigma_\infty^m(k_m,h_\infty)(c_m)\, d\pi_{\sigma,k}(h_\infty)$$

$$= \sum_{k_{-n}\in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1,c_2,\ldots,c_N)\in C} u_n(c,(k_n,k_{-n})) \int_{H_\infty} \prod_{m\in N} \sigma_\infty^m(k_m,h_\infty)(c_m)\, d\pi_{\sigma,k}(h_\infty) = E_{k_n}^{\Gamma(M)}(\sigma)$$

Hence $\Gamma(M)$ and $\tilde{\Gamma}(M)$ are equivalent and $\tilde{\Gamma}(M)$ is a game with independent incomplete information. ▮

Remark: The analogous theorem is correct for repeated games (with and without discounting).

The transformation used in the proof of theorems 2.10 and 2.11 may fail to preserve a known-own-payoffs property (private value assumption, i.e $u_n(c,k) = u_n(c,k_n)$ for all $n \in N$, $c \in C$ and $k = (k_1, k_2, ..., k_N) \in K$). This problem can be solved using the next definition and theorem.

Definition 2.12: G^1 and G^2 are *semi – equivalent* if $\mathcal{E}^{G^1} = \mathcal{E}^{G^2}$. $\Gamma^1(M)$ and $\Gamma^2(M)$ are *semi – equivalent* if $\mathcal{E}^{\Gamma^1(M)} = \mathcal{E}^{\Gamma^2(M)}$.

Theorem 2.13: *every game with incomplete information is semi-equivalent to a game with known own payoffs and with the same information structure (i.e, N, $(K_n)_{n\in N}$ and $(p_n)_{n\in N}$ are not changed).*

Proof: Fix a game $G = (N, (C_n)_{n\in N}, (K_n)_{n\in N}, (p_n)_{n\in N}, (u_n)_{n\in N})$. Let Z be the upper bound of the absolute value of the possible payoffs, that is $Z := \max_{n\in N, c\in C, k\in K}\{|u_n(c,k)|\}$. We define a game, semi-equivalent to G, $\tilde{G}$=$(N, (\tilde{C}_n)_{n\in N}, (K_n)_{n\in N}, (p_n)_{n\in N}, (\tilde{u}_n)_{n\in N})$ by:

1. For all $n \in N$ define $\tilde{C}_n := \cup_{k_n\in K_n} \tilde{C}^{k_n}$, where $\tilde{C}^{k_n}$ is a set isomorphic to C_n with the isomorphism $s_{k_n} : \tilde{C}^{k_n} \to C_n$. Denote $s_n = \cup_{k_n\in K_n} s_{k_n}$ (i.e, $s_n : \tilde{C}_n \to C_n$). Define a function $k^n : \tilde{C}_n \to K_n$ by $k^n(\tilde{c}_n) = k_n$ if $\tilde{c}_n \in \tilde{C}^{k_n}$.

2. For all $n \in N$, $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_N) \in \prod_{m\in N} \tilde{C}_m$ and $k = (k_1, k_2, ..., k_N) \in K$ define:

$$\tilde{u}_n(\tilde{c},k) := \begin{cases} u_n((s_1(\tilde{c}_1), ..., s_N(\tilde{c}_N)), (k^1(\tilde{c}_1), ..., k^{n-1}(\tilde{c}_{n-1}), k_n, k^{n+1}(\tilde{c}_{n+1}), ..., k^N(\tilde{c}_N))) & \tilde{c}_n \in \tilde{C}^{k_n} \\ -(Z+1) & \text{otherwise} \end{cases}$$

$\tilde{G}$ is a game with known own payoff property. We will prove now that G and $\tilde{G}$ are semi-equivalent. Fix $a \in \mathcal{E}^G$. We will show that $a \in \mathcal{E}^{\tilde{G}}$. There exists $\alpha \in \Psi$ satisfying items 1 and 2 of definition 2.6. Define $\tilde{\alpha} \in \tilde{\Psi}$ by:

$$\tilde{\alpha}^n_{k_n}(\tilde{c}_n) := \begin{cases} \alpha^n_{k_n}(s_{k_n}(\tilde{c}_n)) & \tilde{c}_n \in \tilde{C}^{k_n} \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in N$, $k_n \in K_n$ and $\tilde{c}_n \in \tilde{C}_n$. Despite the abuse of notation we will denote by c_n the strategy that assigns probability 1 to the action c_n. Now $E^{\tilde{G}}_{k_n}(\tilde{\alpha}) = a_{k_n}$ for all $n \in N$ and $k_n \in K_n$ and $E^{\tilde{G}}_{k_n}(\tilde{\alpha}^{-n}, \tilde{c}'_n) \le E^G_{k_n}(\alpha^{-n}, s_n(\tilde{c}'_n)) \le a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $\tilde{c}'_n \in \tilde{C}_n$, hence $a \in \mathcal{E}^{\tilde{G}}$. Thus we proved that $\mathcal{E}^G \subset \mathcal{E}^{\tilde{G}}$. to prove that $\mathcal{E}^G \supset \mathcal{E}^{\tilde{G}}$ fix $a \in \mathcal{E}^{\tilde{G}}$. There exists $\tilde{\alpha}$ such that

1. $E^{\tilde{G}}_{k_n}(\tilde{\alpha}) = a_{k_n}$ for all $n \in N$ and $k_n \in K_n$.

2. $E^{\tilde{G}}_{k_n}(\tilde{\alpha}^{-n}, \tilde{c}'_n) \le a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $\tilde{c}'_n \in \tilde{C}_n$.

The second condition implies that $\tilde{\alpha}_{k_n}(\tilde{c}_n) = 0$ for all $\tilde{c}_n \notin \tilde{C}^{k_n}$ (i.e, $\tilde{\alpha}_{k_n}(\tilde{C}^{k_n}) = 1$). Define $\alpha \in \Psi$ by $\alpha_{k_n}(c_{k_n}) := \tilde{\alpha}(s_{k_n}(\tilde{c}_{k_n}))$. $E^G_{k_n}(\alpha) = E^{\tilde{G}}_{k_n}(\tilde{\alpha}) = a$ and $E^G_{k_n}(\alpha^{-n}, c'_n) = E^{\tilde{G}}_{k_n}(\tilde{\alpha}^{-n}, s^{-1}_{k_n}(c'_n)) \le a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $c'_n \in C_n$. Hence $a \in \mathcal{E}^G$. ▮

Remarks:

1. Theorem 2.13 is also correct for cheap-talk games. One can prove it using the same transformation defined in the proof of theorem 2.13.

2. Theorem 2.13 is correct for repeated games with discounting. Here the transformation should be slightly changed, putting $\frac{-(2Z+1)}{\lambda}$ in the definition of $\tilde{u}_n$ instead of $-(Z+1)$, for λ-discounted games.

3. This will not work in the case of repeated games (where the payoff is defined to be the limit of means), as the expected payoff of a player is not affected by the payoff from a finite number of periods. And indeed the theorem is not true for repeated games, as every equilibrium in repeated games with known own payoffs can be achieved by complete revelation of all the information in the first period of the game (Shalev (1994), Koren (1988)). This is not true in general for repeated games.

4. Theorem 2.13 is correct for stochastic games, and for repeated games with absorbing states. Here the $-(Z+1)$ should be replaced by an absorption with probability 1 and payoff $-(Z+1)$. (in discounted games $\frac{-(2Z+1)}{\lambda}$).

5. The finiteness condition is not essential. We only need u_n to be bounded for every $n \in N$.

3 Characterization of the set of equilibria

We characterize the set of equilibria for two player cheap-talk games. Using theorem 2.11 we can assume that the game is of independent incomplete information (using theorem 2.13 we can even assume that the game is with known own payoffs, but this assumption does not make the characterization simpler). In section 3.1 we define the model and in section 3.2 we introduce the concept of admissible-martingales. The geometrical properties of admissible-martingales are discussed in section 3.3 and in section 3.4 we give the main result.

3.1 The model

We repeat the definitions, given in the previous section, because here we deal with two-player games with consistent and independent information, which enable us to simplify the notations. In addition we insert the information structure (p, q) explicitly into the notations of the games. We define two games - $G(p,q)$ and $\Gamma(p,q,M)$. $G(p,q)$ is a game of incomplete information on both sides, and $\Gamma(p,q,M)$ is its cheap-talk extension. In $\Gamma(p,q,M)$ the cheap-talk occurs after the players have received their private information.

- $G(p,q)$ is defined by the following:

  1. Two players: player 1 and player 2.
  2. A finite set of actions I for player 1, and a finite set of actions J for player 2.
  3. Two finite sets, K and L, such that to each pair $(k \in K, l \in L)$ there corresponds a pair of $I \times J$ matrices $(A^{k,l}\ ,\ B^{k,l})$. $A^{k,l} = (A^{k,l}(i,j))_{i \in I, j \in J}$, $B^{k,l} = (B^{k,l}(i,j))_{i \in I, j \in J}$.

4. Two probability vectors: $p \in \Delta(K)$, $p = (p(k))_{k\in K}$ and $q \in \Delta(L)$, $q = (q(l))_{l\in L}$.
5. The game $G(p,q)$ has two phases: [4]

   The Information Phase : Nature chooses $\mathbf{k} \in K$ according to p and $\mathbf{l} \in L$ according to q. The choices are made independently, i.e, $Prob(\mathbf{k} = k \; and \; \mathbf{l} = l) = p(k)q(l)$. $\mathbf{k}$ is told to player 1 and $\mathbf{l}$ is told to player 2.

   The Action Phase : Player 1 chooses $i \in I$ and player 2 chooses $j \in J$. The choices are made simultaneously. The payoff to player 1 is $A^{\mathbf{k},\mathbf{l}}(i,j)$ and the payoff to player 2 is $B^{\mathbf{k},\mathbf{l}}(i,j)$.
6. 1,2,3,4,5 are common knowledge to both players.

- The game $\Gamma(p,q,M)$, a cheap-talk extension of $G(p,q)$, is defined by 1,2,3,4 and:

7. A finite set M, the set of possible messages in the Talk phase. We assume that $|M| \geq 2$.
8. The game $\Gamma(p,q,M)$ has three phases:

   The Information Phase is the same as in $G(p,q)$.

   The Talk Phase : This phase is divided into periods t=1,2,3... . For each t, player 1 chooses a message $m_t^1 \in M$ and player 2 chooses a message $m_t^2 \in M$. The choices are made simultaneously.

   The Action Phase : Player 1 chooses an action $i \in I$ and player 2 chooses an action $j \in J$. The choices are made simultaneously. The payoff to player 1 is $A^{\mathbf{k},\mathbf{l}}(i,j)$ and the payoff to player 2 is $B^{\mathbf{k},\mathbf{l}}(i,j)$.
9. The players have perfect recall.
10. 1,2,3,4,7,8,9 are common knowledge to both players.

The players have perfect recall, so m_t^1 and m_t^2 are functions of the history of length $t-1$, $h_{t-1} := ((m_1^1,m_1^2),(m_2^1,m_2^2),...,(m_{t-1}^1,m_{t-1}^2))$. The actions, chosen by the players in the action phase, are functions of $h_\infty := ((m_1^1,m_1^2),(m_2^1,m_2^2),...,(m_t^1,m_t^2),...)$, the infinite sequence defined in the talk phase. Let $H_t = (M \times M)^t$ be the set of histories of length t. Define $H_0 = \{\phi\}$. Let $H_\infty = \prod_{t=1}^{\infty}(M \times M)$ be the set of infinite histories. On H_∞ , we define for every t, a finite field $\mathcal{H}_t$. h_∞^1 , $h_\infty^2 \in H_\infty$ are in the same atom of $\mathcal{H}_t$ if and only if for every $1 \leq u \leq t$ there [5] exists $h_\infty^1(u) = h_\infty^2(u)$. Let $\mathcal{H}_\infty$ be the σ-field generated by $\{\mathcal{H}_t\}_{t=1}^{\infty}$. Our basic probability space is $(\Omega, \mathcal{A}) = (K \times L \times H_\infty \times I \times J \; , 2^K \otimes 2^L \otimes \mathcal{H}_\infty \otimes 2^I \otimes 2^J)$. A point in Ω is a five-tuple (k,l,h_∞,i,j), where (k,l) is a possible state of nature, $h_\infty \in H_\infty$ is an history of the game (the communication that took place), i is an action of player 1 and j is an action of player 2. Defining sequences of random variables, we will use the following notation: $a_t, b_t, c_t, ...$ will usually be random variables measurable with respect to $(H_t, \mathcal{H}_t)$, and $a_{h_t}, b_{h_t}, c_{h_t}, ...$ will denote $a_t(h_t), b_t(h_t), c_t(h_t), ...$. For $x \in \Delta(I)$ and $y \in \Delta(J)$ we will write $A^{k,l}(x,y)$ instead of $\sum_{i\in I, j\in J} x(i)y(j)A^{k,l}(i,j)$ and $B^{k,l}(x,y)$ instead of $\sum_{i\in I, j\in J} x(i)y(j)B^{k,l}(i,j)$.

Since $\Gamma(p,q,M)$ is a game with perfect recall, we can restrict ourselves to behaviour strategies (see Aumann 1964). To shorten the writing, whenever we write 'strategy' we will mean a behaviour strategy.

[4] This is an equivalent model to the model described in chapter 2. k and l are the types of the players.

[5] $h_\infty(u)$ is the two messages sent by the players at period u, according to the infinite history h_∞.

Definition 3.1:
A *strategy* σ of player 1 in $\Gamma(p,q,M)$ is $\sigma = (\{\sigma_t\}_{t\in\mathbb{N}}, \sigma_\infty)$ such that: [6]

1. $\sigma_t : K \times H_{t-1} \to \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\sigma_\infty : K \times H_\infty \to \Delta(I)$.
3. σ_∞ is $2^K \otimes \mathcal{H}_\infty$-measurable.

A *strategy* τ of player 2 in $\Gamma(p,q,M)$ is $\tau = (\{\tau_t\}_{t\in\mathbb{N}}, \tau_\infty)$ such that:

1. $\tau_t : L \times H_{t-1} \to \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\tau_\infty : L \times H_\infty \to \Delta(J)$.
3. τ_∞ is $2^L \otimes \mathcal{H}_\infty$-measurable.

Let Σ^i be the set of strategies for player i for $i = 1,2$. Denote:

$$Z := \max_{k\in K, l\in L, i\in I, j\in J} \{|A^{k,l}(i,j)|, |B^{k,l}(i,j)|\}.$$

That is, Z is the upper bound of the absolute value of the possible payoffs.

Now we define the equilibrium in the game $G(p,q)$. Later, we will use the equilibrium in $G(p,q)$ in the characterization of the equilibrium in the cheap-talk extension $\Gamma(p,q,M)$.

Definition 3.2:
$a \in [0,Z]^K$ and $b \in [0,Z]^L$ are *equilibrium vector payoffs* in $G(p,q)$ if there exist $\alpha \in (\Delta(I))^K$ and $\beta \in (\Delta(J))^L$ such that:

3.2.1 $a^k = \sum_{l\in L} q(l) A^{k,l}(\alpha^k, \beta^l)$ for all $k \in K$ such that $p(k) > 0$.

3.2.2 $b^l = \sum_{k\in K} p(k) B^{k,l}(\alpha^k, \beta^l)$ for all $l \in L$ such that $q(l) > 0$.

3.2.3 $a^k \geq \sum_{l\in L} q(l) A^{k,l}(\gamma, \beta^l)$ for all $k \in K$ and $\gamma \in \Delta(I)$.

3.2.4 $b^l \geq \sum_{k\in K} p(k) B^{k,l}(\alpha^k, \delta)$ for all $l \in L$ and $\delta \in \Delta(J)$.

Note the difference between definition 3.2 and definition 2.6. In definition 3.2 we allow a^k to be greater then the payoff of type k when $p(k) = 0$.

Going back to the Cheap-Talk extension, we need a few definitions. Every 4-tuple $(\sigma,\tau,k,l) \in \Sigma^1 \times \Sigma^2 \times K \times L$ defines a *probability* measure $\pi_{\sigma,\tau,k,l}$ on $(H_\infty, \mathcal{H}_\infty)$, i.e, for an history $h_t := ((m_1^1, m_1^2),(m_2^1, m_2^2), ..., (m_t^1, m_t^2))$, $\pi_{\sigma,\tau,k,l}(h_t)$ is the probability that the first message sent by player 1 is m_1^1, the first message sent by player 2 is m_1^2, the second message sent by player 1 is m_2^1, and so on, given that $\mathbf{k} = k$, $\mathbf{l} = l$, player 1 plays according to σ and player 2 plays according to τ. We derive from $\pi_{\sigma,\tau,k,l}$ another probability measure on $(H_\infty \times K \times L\ ,\ \mathcal{H}_\infty \otimes 2^K \otimes 2^L)$

$$P_{\sigma,\tau,p,q}(h_t,k,l) := p(k)q(l)\pi_{\sigma,\tau,k,l}(h_t)$$

Note that $P_{\sigma,\tau,p,q}(\mathbf{k} = k, \mathbf{l} = l) = \sum_{h_t\in H_t} P_{\sigma,\tau,p,q}(h_t,k,l) = p(k)q(l)$. Denote by $E_{\sigma,\tau,p,q}$ the expectation with respect to $P_{\sigma,\tau,p,q}$. We will denote $P_{\sigma,\tau,p,q}(\cdot \mid \mathbf{k} = k)$, $E_{\sigma,\tau,p,q}(\cdot \mid \mathbf{k} = k)$, $P_{\sigma,\tau,p,q}(\cdot \mid \mathbf{l} = l)$ and $E_{\sigma,\tau,p,q}(\cdot \mid \mathbf{l} = l)$ by $P^{k\,\cdot}(\cdot)$, $E^{k\,\cdot}(\cdot)$, $P^{\cdot\, l}(\cdot)$ and $E^{\cdot\, l}(\cdot)$ respectively and $P_{\sigma,\tau,p,q}$ by P. Denote by $\mathbf{a}$ and $\mathbf{b}$ the (random) payoff of player 1 and player 2 respectively.

[6] $\mathbb{N}$ denotes the set of natural numbers $\{1,2,3,...\}$.

Definition 3.3:
$a \in [-Z,Z]^K$ and $b \in [-Z,Z]^L$ are *equilibrium payoffs* in $\Gamma(p,q,M)$ if there exist $\sigma \in \Sigma^1$ and $\tau \in \Sigma^2$ such that:

E1 : $a^k = E_{\sigma,\tau,p,q}(\mathbf{a} \mid \mathbf{k} = k)$ for all $k \in K$ such that $p(k) > 0$.

E2 : $b^l = E_{\sigma,\tau,p,q}(\mathbf{b} \mid \mathbf{l} = l)$ for all $l \in L$ such that $q(l) > 0$.

E3 : $a^k \geq E_{\sigma',\tau,p,q}(\mathbf{a} \mid \mathbf{k} = k)$ for all $k \in K$ and $\sigma' \in \Sigma^1$.

E4 : $b^l \geq E_{\sigma,\tau',p,q}(\mathbf{b} \mid \mathbf{l} = l)$ for all $l \in L$ and $\tau' \in \Sigma^2$.

We need some notations. Let $Q := [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L)$. That is, a point in Q is a 4-tuple (a,b,p,q) such that a is a vector payoff of player 1, b is a vector payoff of player 2, $p \in \Delta(K)$ and $q \in \Delta(L)$.
Let

$$EQ := \{(a,b,p,q) \in Q \ s.t. \ (a,b) \text{ are equilibrium vector payoffs in } G(p,q)\}$$

3.2 Basic Definitions

In this section we introduce three definitions, based upon the concept of admissible split. We will try to explain the motivation for these definitions. Assume that $\Gamma(p,q,M)$ is a Cheap-Talk game with independent incomplete information. Let $(a,b) \in \mathbb{R}^K \times \mathbb{R}^L$ be equilibrium payoff vectors in $\Gamma(p,q,M)$, and let σ and τ be strategies (of player 1 and player 2 respectively) that implement the equilibrium. Let a_{m^1,m^2} and b_{m^1,m^2} be the expected payoff vectors after the first period (i.e, after each player sent one message), when player 1 sent m^1 and player 2 sent m^2. For $k \in K$ let $p_{m^1,m^2}(k)$ be the a posteriori probability that player 2 assigns to the event of k being the $\mathbf{k}$ chosen by nature, given σ and m^1. For $l \in L$ let $q_{m^1,m^2}(l)$ be the a posteriori probability that player 1 assigns to the event of l being the $\mathbf{l}$ chosen by nature, given τ and m^2. Clearly $p_{m^1,m^2} = p_{m^1,m'^2}$ for all m'^2 and $q_{m^1,m^2} = q_{m'^1,m^2}$ for all m'^1. Let $\mu(m^1)$ be the probability of player 1 sending the message m^1 in the first period, given σ and let $\lambda(m^2)$ be the probability of player 2 sending the message m^2 in the first period, given τ. $a_{m^1} := \sum_{m^2 \in M} \lambda(m^2) a_{m^1,m^2}$ is the expected payoff vectors of player 1 after sending m^1. Note that $a_{m^1} \leq a$ (i.e, $a^k_{m^1} \leq a^k$ for all $k \in K$) for all $m^1 \in M$, otherwise player 1 will gain more than a by sending m^1 with probability one, in contradiction to the assumption that σ and τ are equilibrium strategies. On the other hand $\sum_{m^1 \in M} \mu(m^1) a_{m^1} = a$, therefore $a_{m^1} = a$ when $\mu(m^1) > 0$. Similarly $b_{m^2} := \sum_{m^1 \in M} \mu(m^1) b_{m^1,m^2} \leq b$ and $b_{m^2} = b$ when $\lambda(m^2) > 0$.

Denote by $[n]$ the set $\{1,2,3,...,n\}$. A split is a convex combination of a scalar or a vector, i.e, if $a \in \mathbb{R}^l$ then $(a_1,a_2,...,a_n;\mu) \in (\mathbb{R}^l)^n \times \Delta([n])$ is a split of a if $\sum_{i=1}^n \mu(i) a_i = a$. For example $3 = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 2 + \frac{2}{4} \cdot 5$, thus we say that the scalar 3 can be split into $0, 2$ and 5 with probabilities $\frac{1}{4}, \frac{1}{4}$ and $\frac{2}{4}$ respectively. Or in shorter writing, $(0,2,5;(\frac{1}{4},\frac{1}{4},\frac{2}{4}))$ is a split of 3. We are going to define two types of splits. We start with an example.

$$(3,2) \text{ is split into } \begin{matrix} & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{4} & 0,0 & 0,3 & 0,4 \\ \frac{1}{4} & 2,0 & 2,3 & 2,4 \\ \frac{2}{4} & 5,0 & 5,3 & 5,4 \end{matrix}$$

This split is called *a split of type* 1 of $(3,2)$ and is read as follows: the vector $(0,0)$ has probability $\frac{1}{4}\cdot\frac{2}{5}$, the vector $(5,3)$ has probability $\frac{2}{4}\cdot\frac{2}{5}$ and so on (see figure 1). In general, $(\{(a_i, b_j)\}_{1\le i,j\le n}; \mu, \lambda)$

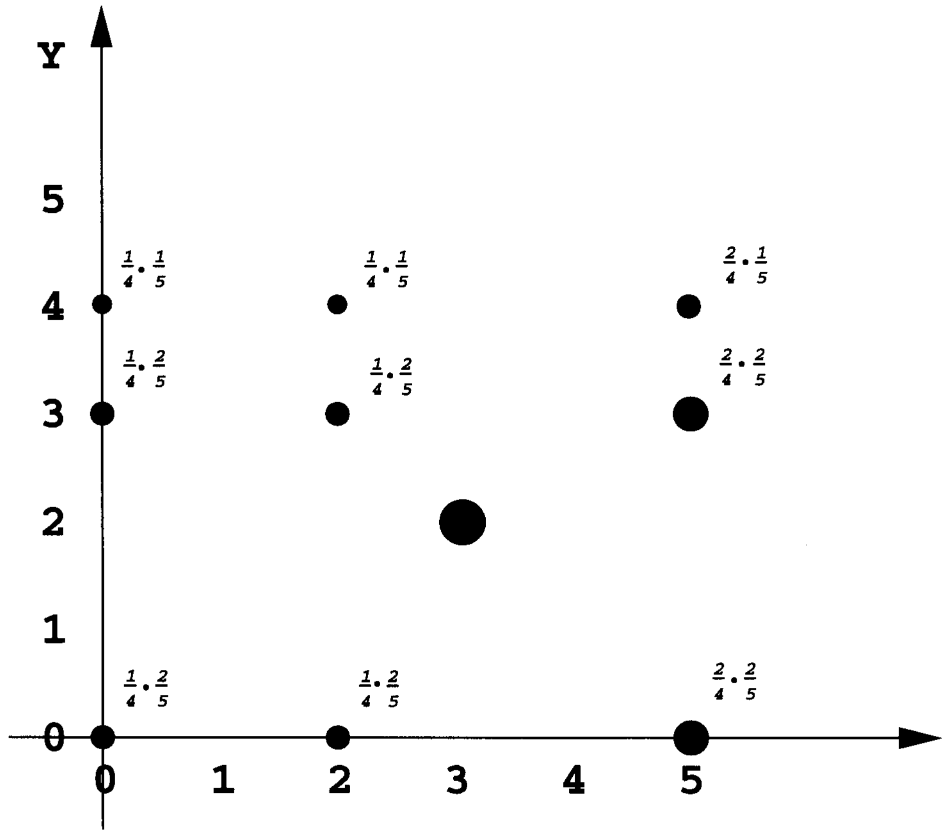

Figure 1: A split type 1 of the point (3,2)

is called *a split of type* 1 of (a,b) if the following holds:

1. $(\{(a_i, b_j)\}_{1\le i,j\le n}; \mu \times \lambda)$ is a split of (a,b).

2. $a_{i,j} = a_{i,j'}$ for all $1 \le i, j, j' \le n$.

3. $b_{i,j} = b_{i',j}$ for all $1 \le i, i', j \le n$.

Note that a split of type 1 can be viewed as a product of two splits (one on the a coordinate with the probability vector $(\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$ and one on the b coordinate with the probability vector $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$). However we need a more complex split, for example:

$$(3,2) \text{ is split into } \begin{array}{c} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{2}{4} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\frac{2}{5} & \quad\frac{2}{5} & \quad\frac{1}{5} \end{array} \\ \begin{pmatrix} 0,0 & 2,3 & 4,4 \\ 2,-2 & 0,7 & 4,0 \\ 5,3 & 5,0 & 2,4 \end{pmatrix} \end{array}$$

See figure 2. This split is not of type 1 but it has the property that the average (according to

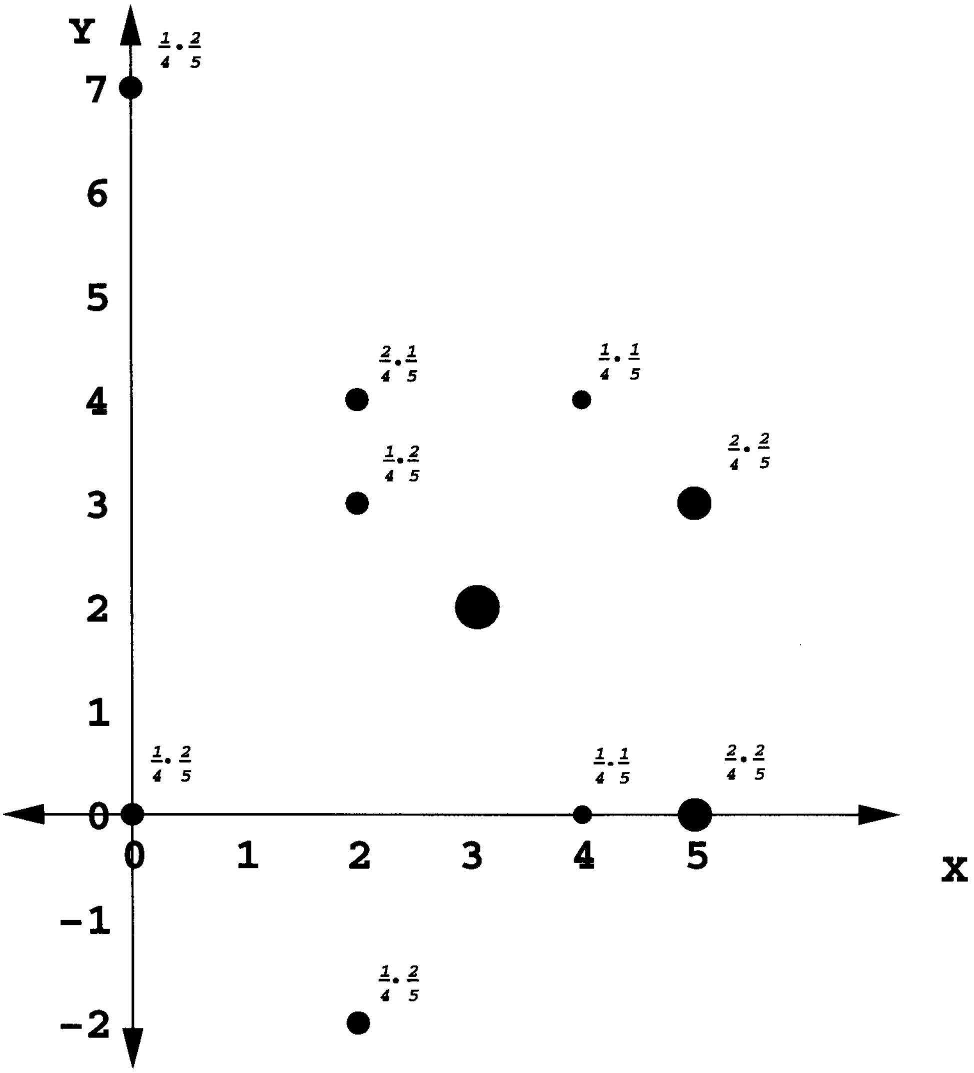

Figure 2: A split of type 2 of the point (3,2)

$\mu = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4}))$ of the first coordinate is constant (and equals to 3) on each column, and the average (according to $\lambda = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5}))$ of the second coordinate is constant (and equals to 2) on each line. This is *a split of type* 2. In general, $(\{(a_i, b_j)\}_{1 \leq i,j \leq n}; \mu, \lambda)$ is called *a split of type* 2 of (a, b) if the following holds:

1. $a = \sum_{i=1}^{n} \mu(i) a_{i,j}$ for all $1 \leq j \leq n$.
2. $b = \sum_{j=1}^{n} \lambda(j) b_{i,j}$ for all $1 \leq i \leq n$.

Note that a split of type 1 is always a split of type 2, and that a split of type 2 is a split. Similarly, splits can be defined for vectors instead of scalars.

In definition 3.4 we introduce a combination of a split of type 1 (of (p, q)) and a split of type 2 (of (b, a)).

Let $x = (a, b, p, q) \in Q$ and let n be a positive integer. Let $S = (\{x_{u,v}\}_{1 \leq u,v \leq n}, \mu, \lambda) \in Q^{n^2} \times \Delta([n]) \times \Delta([n])$, where $x_{u,v} = (a_{u,v}, b_{u,v}, p_{u,v}, q_{u,v})$.

Definition 3.4:
S is called an $n-$ *admissible split* of x if $(\{p_{u,v}, q_{u,v}\}_{1 \leq u,v \leq n}, \mu, \lambda)$ is a split of type 1 of (p, q), and $(\{b_{u,v}, a_{u,v}\}_{1 \leq u,v \leq n}, \mu, \lambda)$ is a split of type 2 of (b, a), i.e,

1. $x = \sum_{u=1}^{n} \sum_{v=1}^{n} \mu(u)\lambda(v)x_{u,v}$
2. (a) $a = \sum_{v=1}^{n} \lambda(v)a_{u,v}$ for all u such that $1 \leq u \leq n$.
   (b) $b = \sum_{u=1}^{n} \mu(u)b_{u,v}$ for all v such that $1 \leq v \leq n$.
   (c) $p_{u,v} = p_{u,v'}$ for all u, v, v' such that $1 \leq u, v, v' \leq n$.
   (d) $q_{u,v} = q_{u',v}$ for all u, u', v such that $1 \leq u, u', v \leq n$.

S is called an *exact* n-admissible split if it is an n-admissible split and in addition:

3. $\mu(u) > 0$ and $\lambda(v) > 0$ for all u, v such that $1 \leq u, v \leq n$ (The split is into exactly n^2 points).

Remarks :

1. From 1. and 2(c) it follows that $p = \sum_{u=1}^{n} \mu(u)p_{u,v}$ for all $1 \leq v \leq n$.
2. From 1. and 2(d) it follows that $q = \sum_{v=1}^{n} \lambda(v)q_{u,v}$ for all $1 \leq u \leq n$.

Definition 3.5:
Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two finite fields ($\mathcal{F}_2$ is thus a refinement of $\mathcal{F}_1$). Let X^1 and X^2 be Q-valued random variables, measurable with respect to $\mathcal{F}^1$ and $\mathcal{F}^2$ respectively. X^2 is called an *(exact)* $n-$ *admissible split* of X^1 if for every atom f^1 of $\mathcal{F}^1$, such that $P(f^1) > 0$, there exists an (exact) n-admissible split $S = (\{x^2_{u,v}\}_{1 \leq u,v \leq n}, \ \mu, \ \lambda)$ of $x_{f^1} := E(X^1 \mid f^1)$, such that f^1 is partitioned into disjoint $\mathcal{F}^2$-measurable sets $\{f^2_{u,v}\}_{1 \leq u,v \leq n}$ (thus $\cup_{1 \leq u,v \leq n} f^2_{u,v} = f^1$ and $f^2_{u,v} \cap f^2_{u',v'} = \phi$ if $u \neq u'$ or $v \neq v'$) satisfying:

1. $P(f^2_{u,v} \mid f^1) = \mu(u)\lambda(v)$.
2. $X^2 = x^2_{u,v}$ on $f^2_{u,v}$ (i.e, $x^2_{u,v} = E(X^2 \mid f^2_{u,v})$) whenever $P(f^2_{u,v}) > 0$.

Let $\mathbb{N}_0$ be the set of non-negative integers, i.e, $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$.

Definition 3.6:
Let $x = (c, d, w, s) \in Q$. Let $C \subset Q$. An *(exact)* $n-$ *admissible martingale* starting at x and converging to C is a sequence $\{X_t\}_{t \in \mathbb{N}_0} = \{(c_t, d_t, w_t, s_t)\}_{t \in \mathbb{N}_0}$ of Q-valued random variables satisfying:

3.6.1 $X_0 =$ x a.s. (almost surely).

3.6.2 There exists a nondecreasing sequence $\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$ of finite fields ($\mathcal{F}_0 = \{\phi, \Omega\}$) with respect to which $\{X_t\}_{t \in \mathbb{N}_0}$ is a *martingale*, i.e: X_t is measurable with respect to $\mathcal{F}_t$ and $X_t = E(X_{t+1} \mid \mathcal{F}_t)$ a.s.

3.6.3 X_{t+1} is an *(exact)* $n-$ *admissible split* of X_t for every $t \in \mathbb{N}_0$.

3.6.4 Every a.s. limit X_∞ of $\{X_t\}_{t \in \mathbb{N}_0}$ satisfies $X_\infty \in C$ a.s.

3.3 The Geometrical properties of Admissible Martingales

Following Aumann and Hart (Aumann and Hart 1986) we will analyze the relations between admissible martingales and admissible convex functions (definition 3.9). We will actually analyze a more general case. Let $\mathcal{X}$ be a compact convex subset of an Euclidean space. Let A be a set of probability measures on $\mathcal{X}$ with finite support. We will apply this for A equals the set of n-admissible splits.

Definition 3.7: x is an $A-$ *convex combination* of $x_1, x_2, ..., x_n$ if there exists $\mu \in A$ such that $\sum_{i=1}^{n} \mu(x_i) = 1$ and $x = \sum_{i=1}^{n} \mu(x_i)x_i$ (note that an $A-$ *convex combination* is always a convex combination).

Definition 3.8: $C \subset \mathcal{X}$ is an $A-$ *convex set* if it contains all the A-convex combinations of its elements.

Definition 3.9: Let $C \subset \mathcal{X}$ be an A-convex set. $f : C \rightarrow \mathbb{R}$ is an A-convex function if for every $n \in \mathbb{N}$, $\mu \in A$ and $x_1, x_2, ..., x_n \in C$, $\sum_{i=1}^{n} \mu(x_i) = 1$ implies:

$$f(\sum_{i=1}^{n} \mu(x_i)x_i) \leq \sum_{i=1}^{n} \mu(x_i)f(x_i)$$

Definition 3.10: A sequence $\{X_t\}_{t\in\mathbb{N}_0}$ of $\mathcal{X}$-random variables is an $A-$ *martingale* if:

1. There exists a nondecreasing sequence $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ of finite fields with respect to which $\{X_t\}_{t\in\mathbb{N}_0}$ is a martingale, i.e: X_t is measurable with respect to $\mathcal{F}_t$ and $X_t = E(X_{t+1} \mid \mathcal{F}_t)$ a.s.
2. For all $f_t \in \mathcal{F}_t$, with $P(f_t) > 0$, the conditional distribution of X_{t+1} conditioned on f_t belongs to A (that is, for all $f_t \in \mathcal{F}_t$ with $P(f_t) > 0$ the probability distribution μ_{f_t} defined by $\mu_{f_t}(E(X_{t+1} \mid f_{t+1})) := P(f_{t+1} \mid f_t)$ satisfies $\mu_{f_t} \in A$).
3. X_1 is constant a.s.

For $C \subset \mathcal{X}$ let

$$C^* := \{x \in \mathcal{X} \text{ s.t. there exists an A-martingale } \{X_t\}_{t\in\mathbb{N}_0} \text{ converging to } X_\infty$$
$$\text{s.t. } X_\infty \in C \text{ a.s. and } X_1 = x \text{ a.s.}\}$$

Definition 3.11: Let $C \subset B \subset \mathcal{X}$ such that B is an A-convex set. $nsc_C(B)$ is the set of all the points in B that can not be separated from C by any bounded A-convex function which is continuous on C. That is, $x \in nsc_C(B)$ if and only if $f(x) \leq \sup_{c\in C} f(c)$ for all the bounded A-convex functions, f, which are continuous on C.

Theorem 3.12: *Assume that C is a closed set. Then the largest set $D \subset \mathcal{X}$ satisfying $D = nsc_C(D)$ is precisely C^* (i.e $C^* = nsc_C(C^*)$ and $B = nsc_C(B)$ implies $B \subset C^*$).*

Proof: The proof of theorem 4.7 in Aumann & Hart (1986) also applies here.

By choosing A to be the set of n-admissible splits we get that an A-martingales is exactly an admissible martingale, hence the last theorem can be applied to admissible martingales. Similarly bi-convex martingales are also a special case of A-martingales.

3.3.1 Infinite Splits

We can define *Infinite Admissible Splits* by replacing n by ∞ in definition 3.4. The following example shows that there exists a point, which is an infinite admissible combination of a certain set of points, but not an admissible combination of any finite subset (Note that this can not happen for bi-convex combinations).

Example:
Let $Q = [0,1]^2 \times [0,1]^2 \times \Delta(2) \times \Delta(2)$. Let $\mu(i) = \frac{1}{2^i}$ for all $i \in \mathbb{N}$ and $\lambda(j) = \frac{1}{2^j}$ for all $j \in \mathbb{N}$. Let $x := (a,b,p,q) = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{3},\frac{2}{3}),(\frac{1}{3},\frac{2}{3}))$. For all $i,j \in \mathbb{N}$ define $x_{i,j} := (a_{i,j}, b_{i,j}, p_{i,j}, q_{i,j}) \in Q$ by:

$$p_{i,j} = p_i = (\frac{1}{2^i}, 1 - \frac{1}{2^i})$$

$$q_{i,j} = q_j = (\frac{1}{2^j}, 1 - \frac{1}{2^j})$$

$$a^1_{i,j} = \begin{cases} \frac{1}{2^{i-j}} & i \geq j \\ 0 & i < j \end{cases}$$

$$a^2_{i,j} = \begin{cases} \frac{1}{2^{i+5-j}} & i+5 \geq j \\ 0 & i+5 < j \end{cases}$$

$$b^1_{i,j} = \begin{cases} \frac{1}{2^{j-i}} & j \geq i \\ 0 & j < i \end{cases}$$

$$b^2_{i,j} = \begin{cases} \frac{1}{2^{j+5-i}} & j+5 \geq i \\ 0 & j+5 < i \end{cases}$$

Lemma 3.13: $(\{x_{i,j}\}_{i,j \in \mathbb{N}}, \mu, \lambda)$ *is an infinite admissible split of* x.

Lemma 3.14: *No finite split of x can be obtained from any finite subset of* $\{x_{i,j}\}_{i,j \in \mathbb{N}}$.

Proof: Assume that $(\{x_{i_k,j_l}\}_{1 \leq k,l \leq n}, \tilde{\mu}, \tilde{\lambda})$ is an admissible split of x. Let $i_{max} := max_{1 \leq k \leq n} i_k$ and let $j_{max} := max_{1 \leq l \leq n} j_l$. W.l.o.g assume that $i_{max} \geq j_{max}$. From the definition of admissible splits we have that $\frac{1}{2} = a^2 = \sum_{l=1}^n \tilde{\lambda}(j_l) a_{i_{max}, j_l}$ but $a^2_{i_{max},j} \leq \frac{1}{4}$ for all $j \neq i_{max}+5, i_{max}+6$, a contradiction to the fact that $j_l \leq i_{max}$ for all l. ∎

3.4 Main Result

We can now state and prove the main result.

Theorem 3.15:
Let $p > 0$ and $q > 0$. $(a,b) \in \mathbb{R}^K \times \mathbb{R}^L$ are equilibrium payoffs in $\Gamma(p,q,M)$, a cheap-talk extension of $G(p,q)$, if and only if there exists an $|M|$ – admissible martingale starting at (a,b,p,q) and converging to EQ.

We need some definitions and lemmas. Let $p \in \Delta(K)$ and $q \in \Delta(L)$. Denote:
$W^1_q := \{\ a \in [-Z,Z]^K \ s.t.\ \exists \beta \in (\Delta(J))^L \ s.t.\ \sum_{l \in L} q^k(l) A^{k,l}(\gamma, \beta^l) \leq a^k \ \ \forall k \in K \ and\ \gamma \in \Delta(I)\}$.
$W^2_p := \{\ b \in [-Z,Z]^L \ s.t.\ \exists \alpha \in (\Delta(I))^K \ s.t.\ \sum_{k \in K} p^l(k) B^{k,l}(\alpha^k, \delta) \leq b^l \ \ \forall l \in L \ and\ \delta \in \Delta(J)\ \}$.

That is, $a \in W_q^1$ if and only if player 2 can guarantee that player 1 will not get more than a^k for all $k \in K$ simultaneously, and $b \in W_p^2$ if and only if player 1 can guarantee that player 2 will not get more than b^l for all $l \in L$ simultaneously. W_q^1 is upper semi continuous with respect to q, and W_p^2 is upper semi continuous with respect to p.
Denote $W^1 := \{(a,q) \in [-Z,Z]^K \times \Delta(L) \ s.t. \ a \in W_q^1)\}$ and $W^2 := \{(b,p) \in [-Z,Z]^L \times \Delta(K) \ s.t. \ b \in W_p^2)\}$. Denote $IR := \{(a,b,p,q) \in Q$ s.t. $(a,q) \in W^1$ and $(b,p) \in W^2$ $\}$ (i.e, IR is the Cartesian product of W^1 and W^2). That is $(a,b,p,q) \in IR$ if and only if (a,b) are *individually rational* payoffs in $G(p,q)$ (and therefore in $\Gamma(p,q,M)$). Note that $IR \subset EQ$.

Lemma 3.16:
W^1 and W^2 are convex sets.

Proof: We will prove that W^1 is convex. The proof for W^2 is similar. Let $(a_1,q_1) \in W^1$, $(a_2,q_2) \in W^1$, and $0 < \delta < 1$. Let $(a,q) := \delta(a_1,q_1) + (1-\delta)(a_2,q_2)$. We have to show that $(a,q) \in W^1$. $(a_i,q_i) \in W^1$ (for i=1,2), therefore there exists $\beta_i : L \to \Delta(J)$ such that for all $k \in K$ and $\alpha \in \Delta(I)$ we have:

$$\sum_{l \in L} q_i(l) A^{k,l}(\alpha, \beta_i^l) \le a_i^k$$

Define $\beta^l := \frac{\delta q_1(l)}{q(l)} \beta_1^l + \frac{(1-\delta) q_2(l)}{q(l)} \beta_2^l$ (from the definition of q we have $\frac{\delta q_1(l)}{q(l)} + \frac{(1-\delta) q_2(l)}{q(l)} = 1$, therefore $\beta^l \in \Delta(J)$. If $q(l) = 0$ we define β^l arbitrarily). For all $k \in K$ and $\alpha \in \Delta(I)$ we have:

$$\sum_{l \in L} q(l) A^{k,l}(\alpha, \beta^l) = \sum_{l \in L \ s.t. \ q(l)>0} q(l) A^{k,l}(\alpha, \beta^l) = \sum_{l \in L \ s.t. \ q(l)>0} q(l) A^{k,l}(\alpha, \delta \frac{q_1(l)}{q(l)} \beta_1^l + (1-\delta) \frac{q_2(l)}{q(l)} \beta_2^l)$$

$$= \delta \sum_{l \in L \ s.t. \ q(l)>0} q_1(l) A^{k,l}(\alpha, \beta_1^l) + (1-\delta) \sum_{l \in L \ s.t. \ q(l)>0} q_2(l) A^{k,l}(\alpha, \beta_2^l)$$

note that $q(l) = 0$ implies $q_1(l) = q_2(l) = 0$ hence

$$= \delta \sum_{l \in L} q_1(l) A^{k,l}(\alpha, \beta_1^l) + (1-\delta) \sum_{l \in L \ s.t. \ q(l)>0} q_2(l) A^{k,l}(\alpha, \beta_2^l) \le \delta a_1^k + (1-\delta) a_2^k = a^k$$

∎

Corollary 3.17:
IR is a convex set.

Definition 3.18:
Let $h_t \in H_t$, $h_s \in H_s$. Denote by $(h_t, h_s) \in H_{t+s}$ the history h_s following h_t, i.e:

$$(h_t, h_s)(i) := \begin{cases} h_t(i) & \text{for } i \le t \\ h_s(i-t) & \text{for } t < i \le t+s \end{cases}$$

Similarly, $(h_t, h_\infty) \in H_\infty$ denotes the history $h_\infty \in H_\infty$ following h_t. For all $h_\infty \in H_\infty$ and for all $t \in \mathbb{N}_0$ denote the initial t-history, $(h_\infty(1), h_\infty(2), ..., h_\infty(t))$, by $(h_\infty)^t$. $(h_\infty)^t \in H_t$.

In the next lemma we will show that we can assume that every finite history has positive probability with respect to $P_{\sigma,\tau,p,q}$.

Lemma 3.19:
(a,b) is an equilibrium in $\Gamma(p,q,M)$ if and only if there exist σ and τ satisfying conditions E1,E2,E3 and E4 of definition 3.3 and in addition:
3.19.1 $P_{\sigma,\tau,p,q}(h_t) > 0$ *for all* $t \in \mathbb{N}_0$ *and* $h_t \in H_t$.

Proof: It is enough to show that if σ and τ satisfy E1,E2,E3 and E4 then there exist σ' and τ' satisfying E1,E2,E3,E4 and 3.19.1 . We will build σ' and τ', based on σ and τ, such that every message that has zero probability to be sent (after some history and according to σ or τ), will have positive probability to be sent according to σ' or τ'. The idea is to "identify" unsent messages with those which have positive probability of being sent. From the moment that such a message was sent, the two players will continue playing as if another message was sent, one which had positive probability (according to σ or τ).

Formally: We will build a sequence of functions, $F_t : H_t \to H_t$, by induction, together with the definition of σ' and τ'. $F_0(\phi) := \phi$, and we define F_{t+1} based on F_t. Let $h_t \in H_t$. Denote: [7]
$C_{h_t} := \{m^1 \in M \text{ such that } P_{\sigma,\tau,p,q}(m_t^1 = m^1 \mid h_t) > 0\}$ and $\tilde{m}^1_{h_t} := \min\{m \in C_{h_t}\}$.
$D_{h_t} := \{m^2 \in M \text{ such that } P_{\sigma,\tau,p,q}(m_t^2 = m^2 \mid h_t) > 0\}$ and $\tilde{m}^2_{h_t} := \min\{m \in D_{h_t}\}$.
For all $m \in M$ define:

$$R_{h_t}(m) := \begin{cases} m & \text{for } m \in C_{h_t} \\ \tilde{m}^1_{h_t} & \text{otherwise} \end{cases} \quad S_{h_t}(m) := \begin{cases} m & \text{for } m \in D_{h_t} \\ \tilde{m}^2_{h_t} & \text{otherwise} \end{cases}$$

Now define for every $m^1, m^2 \in M, k \in K, l \in L$:

$$F_{t+1}(h_t,(m^1,m^2)) := (\ F_t(h_t)\ ,\ (R_{h_t}(m^1), S_{h_t}(m^2))\ )$$

$$\sigma'_{t+1}(k,h_t)(m^1) := \frac{\sigma_{t+1}(k,F_t(h_t))(R_{h_t}(m^1))}{|R^{-1}_{h_t}(R_{h_t}(m^1))|}$$

$$\tau'_{t+1}(l,h_t)(m^2) := \frac{\tau_{t+1}(l,F_t(h_t))(S_{h_t}(m^2))}{|S^{-1}_{h_t}(S_{h_t}(m^2))|}$$

From the definitions we have 3.19.1 and:

$$\forall h_t\ \forall k \in K\ \forall m^1 \in M \qquad \sum_{m \in R^{-1}_{h_t}(m^1)} \sigma'_{t+1}(k,h_t)(m) = \sigma_{t+1}(k,F_t(h_t))(m^1)$$

$$\forall h_t\ \forall l \in L\ \forall m^2 \in M \qquad \sum_{m \in S^{-1}_{h_t}(m^2)} \tau'_{t+1}(l,h_t)(m) = \tau_{t+1}(l,F_t(h_t))(m^2)$$

Therefore if $P_{\sigma,\tau,p,q}(h_t) = 0$ then $F_t^{-1}(h_t) = \phi$ and otherwise:

[7] M being finite, is assumed to be the set $\{1,2,3,...,|M|\}$ in order to simplify the writing.

$$P_{\sigma',\tau',p,q}(F_t^{-1}(h_t)) = P_{\sigma,\tau,p,q}(h_t) \text{ for all } k \in K \text{ and } l \in L \qquad (1)$$

Define $F_\infty : H_\infty \to H_\infty$ by : for all $h_\infty \in H_\infty$ and $t \in \mathbb{N}_0$ $\quad (F_\infty(h_\infty))^t = F_t((h_\infty)^t)$. F_∞ is well defined. Now define: $\sigma'_\infty(\cdot, h_\infty) = \sigma_\infty(\cdot, F_\infty(h_\infty))$, $\tau'_\infty(\cdot, h_\infty) = \tau_\infty(\cdot, F_\infty(h_\infty))$. From equation (1) we get that $P_{\sigma',\tau',p,q}(\ \sigma'_\infty(\cdot, h_\infty) = \gamma\ ,\ \tau'_\infty(\cdot, h_\infty) = \delta\ ) = P_{\sigma,\tau,p,q}(\ \sigma_\infty(\cdot, h_\infty) = \gamma\ ,\ \tau_\infty(\cdot, h_\infty) = \delta\ )$ for all $k \in K$, $l \in L$, $\gamma \in \Delta(I)$ and $\delta \in \Delta(J)$. For $\sigma'' \in \Sigma^1$ let σ''_F be the strategy σ'' in which for every h_t, every message $m_t^1 \notin C_{h_t}$ is replaced by $\tilde{m}^1_{h_t}$ and every message $m_t^2 \notin D_{h_t}$ is replaced by $\tilde{m}^2_{h_t}$. $P^{\cdot l}_{\sigma'',\tau'}(\tau'_\infty(\cdot, h_\infty) = \delta) = P^{\cdot l}_{\sigma''_F,\tau}(\tau_\infty(\cdot, h_\infty) = \delta)$ for all $\sigma'' \in \Sigma^1$, and $l \in L$. And $P^{k\cdot}_{\sigma',\tau''}(\sigma'_\infty(\cdot, h_\infty) = \gamma) = P^{k\cdot}_{\sigma,\tau''_F}(\sigma_\infty(\cdot, h_\infty) = \gamma)$ for all $\tau'' \in \Sigma^2$, and $k \in K$. Therefore conditions E1,E2,E3,E4 and 3.19.1 are satisfied by σ' and τ'. ∎

Definition 3.20:
Recall that **a** and **b** are the random payoffs to player 1 and player 2, respectively. Let σ and τ be equilibrium strategies in $\Gamma(p, q, M)$. Let $h_t \in H_t, h_\infty \in H_\infty, k \in K, l \in L$. Define: $a_t^k := E(\mathbf{a} \mid h_t, \mathbf{k} = k)$, $b_t^l := E(\mathbf{b} \mid h_t, \mathbf{l} = l)$, $p_t(k) := P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t)$, and $q_t(l) := P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h_t)$. $p_{h_t} = \sum_{h_{t+1}} P_{\sigma,\tau,p,q}(h_{t+1}|h_t)p_{h_{t+1}}$ and $q_{h_t} = \sum_{h_{t+1}} P_{\sigma,\tau,p,q}(h_{t+1}|h_t)q_{h_{t+1}}$, therefore $\{p_t\}_{t\in\mathbb{N}_0}$ and $\{q_t\}_{t\in\mathbb{N}_0}$ are bounded $P_{\sigma,\tau,p,q}$-martingales and p_∞ and q_∞ exist $P_{\sigma,\tau,p,q}$-a.s. (see also corollary 3.24). Recall that m_t^1 and m_t^2 are the messages sent at period t, by player 1 and player 2 respectively. Let a_∞ and b_∞ be the $\mathbb{R}^K$-valued random variable and the $\mathbb{R}^L$-valued random variable defined by

$$a_\infty^k := \sum_{l\in L} q_\infty(l)A^{k,l}(\sigma_\infty(k, h_\infty), \tau_\infty(l, h_\infty))$$

and

$$b_\infty^l := \sum_{k\in K} p_\infty(k)B^{k,l}(\sigma_\infty(k, h_\infty), \tau_\infty(l, h_\infty))$$

Note that a_∞^k is defined only when q_{h_∞} exists and b_∞^l is defined only when p_{h_∞} exists. $a^k_{h_\infty}$ can be think of as the expected payoff of player 1 given h_∞ and $\mathbf{k} = k$. $a_t^k = E(a_\infty^k \mid h_t, \mathbf{k} = k)$ and $b_t^l = E(b_\infty^l \mid h_t, \mathbf{l} = l)$.

Lemma 3.21:
If (a, b) is an equilibrium in $\Gamma(p, q, M)$ and σ and τ satisfy conditions E1,E2,E3,E4 and 3.19.*1 then there exist σ' and τ' satisfying conditions E1,E2,E3,E4,* 3.19.*1 and:*
3.21.1 *(a_∞, b_∞) are equilibrium payoffs in $G(p_\infty, q_\infty)$ for $P_{\sigma,\tau,p,q}$-almost every h_∞.*

Proof: [8] q_∞ exists $P^{k\cdot}$-a.s., for all $k \in K$ (because it exists $P_{\sigma,\tau,p,q}$-a.s. and $p(k) > 0$). Therefore for all $k \in K$ and for $P^{k\cdot}$-almost every h_∞ we can define:

$$\gamma_\infty^k := \min_i\{i \in I \ \ s.t. \ \sum_{l\in L} q_\infty(l)A^{k,l}(i, \tau_\infty(l, h_\infty)) = \max_{\gamma\in\Delta(I)} \sum_{l\in L} q_\infty(l)A^{k,l}(\gamma, \tau_\infty(l, h_\infty)) \}$$

(The minimum in the above definition has no special significance. It is only a tool for choosing one member from the set, when it has more then one member). p_{h_∞} exists for $P^{\cdot l}$-almost every h_∞ for all $l \in L$. Therefore for all $l \in L$ and $P^{\cdot l}$-almost every h_∞ we can define:

$$\delta_\infty^l := \min_j\{j \in J \ \ s.t. \ \sum_{k\in K} p_\infty(k)B^{k,l}(\sigma_\infty(k, h_\infty), j) = \max_{\delta\in\Delta(J)} \sum_{k\in K} p_\infty(k)B^{k,l}(\sigma_\infty(k, h_\infty), \delta) \}$$

[8] To simplify the proof we will assume w.l.o.g. that $I = \{1, 2, 3, ..., |I|\}$ and $J = \{1, 2, 3, ..., |J|\}$.

γ^k_∞ and δ^l_∞ are measurable functions. For all h_∞ and k such that q_{h_∞} exists we have:

$$a^k_{h_\infty} = \sum_{l\in L} q_{h_\infty}(l)A^{k,l}(\sigma_\infty(k,h_\infty),\tau_\infty(l,h_\infty)) \leq \max_{\gamma\in\Delta(I)} \sum_{l\in L} q_{h_\infty}(l)A^{k,l}(\gamma,\tau_\infty(l,h_\infty))$$

hence (note that q_∞ exists a.s.)

$$E(\mathbf{a} \mid \mathbf{k}=k) = E(a^k_\infty \mid \mathbf{k}=k) \leq E(\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty)) \mid \mathbf{k}=k) \tag{2}$$

If player 1 changes his strategy σ, in the action phase of the game, by playing γ^k_∞ instead of $\sigma(k,h_\infty)$, it will change his expected payoff (given $\mathbf{k}=k$ and assuming that player 2 plays τ) from $E(\mathbf{a} \mid \mathbf{k}=k)$ to $E(\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_{h_\infty},\tau_\infty(l,h_\infty)) \mid \mathbf{k}=k)$. σ and τ are equilibrium strategies and player 1 can achieve $E(\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty)) \mid \mathbf{k}=k)$ when player 2 plays τ and given $\mathbf{k}=k$, thus we have

$$E(\mathbf{a} \mid \mathbf{k}=k) \geq E(\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty)) \mid \mathbf{k}=k) \tag{3}$$

and from (2) and (3)

$$E(a^k_\infty \mid \mathbf{k}=k) = E(\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty)) \mid \mathbf{k}=k))$$

hence

$$\frac{1}{p(k)}\int_{H_\infty} p_\infty(k)a^k_\infty dP(h_\infty) = \frac{1}{p(k)}\int_{H_\infty} p_\infty(k)\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty))dP(h_\infty)$$

Therefore

$$p_\infty(k)a^k_\infty = p_\infty(k)\sum_{l\in L} q_\infty(l)A^{k,l}(\gamma^k_\infty,\tau_\infty(l,h_\infty)) \text{ for } P\text{-almost every } h_\infty.$$

Therefore, for all $k\in K$ and for P-almost every h_∞ such that $p_{h_\infty}(k)>0$ we have:

$$\sum_{l\in L} q_\infty(l)A^{k,l}(\sigma_\infty(k,h_\infty),\tau_\infty(l,h_\infty)) = \max_{\gamma\in\Delta(I)} \sum_{l\in L} q_\infty(l)A^{k,l}(\gamma,\tau_\infty(l,h_\infty)) \tag{4}$$

Similarly, for all $l\in L$ and for P-almost every h_∞ such that $q_\infty(l)>0$ we have:

$$\sum_{k\in K} p_\infty(k)B^{k,l}(\sigma_\infty(k,h_\infty),\tau_\infty(l,h_\infty)) = \max_{\delta\in\Delta(J)} \sum_{k\in K} p_\infty(k)B^{k,l}(\sigma_\infty(k,h_\infty),\delta) \tag{5}$$

Now we can define σ' and τ':
For all t let $\sigma'_t := \sigma_t$ and $\tau'_t := \tau_t$, and

$$\sigma'_\infty(k,h_\infty) := \begin{cases} \gamma^k_{h_\infty} & if\ p_{h_\infty}(k)=0\ and\ q_{h_\infty} exists \\ \sigma_\infty(k,h_\infty) & otherwise \end{cases}$$

Here $\gamma^k_{h_\infty}$ stands for the vector in $\Delta(I)$ whose ${\gamma^k_{h_\infty}}^{th}$ coordinate is 1 and the others are 0.

$$\delta'_\infty(h_\infty,l) := \begin{cases} \delta^l_{h_\infty} & if\ q_{h_\infty}(l)=0\ and\ p_{h_\infty} exists \\ \delta_\infty(h_\infty,l) & otherwise \end{cases}$$

We changed the mixed actions chosen by the players only on a subset of $K \times L \times H_\infty$ which has measure zero (with respect to $P = P_{\sigma,\tau,p,q} = P_{\sigma',\tau',p,q}$). Therefore σ' and τ' are also equilibrium strategies with vector payoffs a and b. For σ' and τ' equations (4) and (5) are satisfied for almost every h_∞ for all k and l. These equations are stronger than the conditions for equilibrium in $G(p_\infty, q_\infty)$ (definition 3.2). ∎

Fix (a, b, p, q), σ and τ satisfying conditions E1,E2,E3,E4, 3.19.1 and 3.21.1 .

Definition 3.22:
Denote by $\mu_{h_t}(m^1)$ the probability of player 1 sending the message m^1 after the history h_t and denote by $\lambda_{h_t}(m^2)$ the probability of player 2 sending the message m^2 after the history h_t.

$$\mu_{h_t}(m^1) = \sum_{k \in K} p_{h_t}(k)\sigma_t(k, h_t)(m^1) \quad \textit{and} \quad \lambda_{h_t}(m^2) = \sum_{l \in L} q_{h_t}(l)\tau_t(l, h_t)(m^2)$$

3.19.1 implies that $\mu_{h_t}(m^1) > 0$ and $\lambda_{h_t}(m^2) > 0$ for all $m^1, m^2 \in M$. Define, for $h_t \in H_t, k \in K$ and $l \in L$

$$X^k_{h_t} := \sup_{\sigma'} E_{\sigma',\tau,p,q}(\mathbf{a} \mid h_t)$$

$$Y^l_{h_t} := \sup_{\tau'} E_{\sigma,\tau',p,q}(\mathbf{b} \mid h_t)$$

$X^k_{h_t}$ is the supremum of what player 1 can achieve, given that player 2 plays the strategy τ and given h_t. $Y^l_{h_t}$ is the supremum of what player 2 can achieve, given that player 1 plays the strategy σ and given h_t.

Recall that Z is the upper bound of the possible payoffs. Clearly $X^k_{h_t} \leq Z$ and $Y^l_{h_t} \leq Z$. The next lemma is the main part of the first part of the proof, and in it the admissible martingale is being built.

Lemma 3.23:

1. *For all $h_t \in H_t$ and $k \in K$ there exists $c^k_{h_t} \in \mathbb{R}$ such that:*
   (a) $X^k_{h_t} \leq c^k_{h_t} \leq Z$ *and if* $P^{k\cdot}(h_t) > 0$ *then* $c^k_{h_t} = X^k_{h_t} = E^{k\cdot}(\mathbf{a} \mid h_t)$.
   (b) $c^k_{h_0} = a^k$.
   (c) $c^k_{h_t} = \sum_{m^2 \in M} \lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)}$ *for all* $m^1 \in M$.
2. *For all $h_t \in H_t$ and $l \in L$ there exists $d^l_{h_t} \in \mathbb{R}$ such that:*
   (a) $Y^l_{h_t} \leq d^l_{h_t} \leq Z$ *and if* $P^{\cdot l}(h_t) > 0$ *then* $d^l_{h_t} = Y^l_{h_t} = E^{\cdot l}(\mathbf{b} \mid h_t)$.
   (b) $d^l_{h_0} = b^l$.
   (c) $d^l_{h_t} = \sum_{m^1 \in M} \mu_{h_t}(m^1) d^l_{h_t,(m^1,m^2)}$ *for all* $m^2 \in M$.

Proof: We will prove the first part of the lemma (the other part is similar), using induction on t. For $t = 0$ define $c_{h_0}^k := a^k$. Condition (a) is satisfied for h_0 because σ and τ are equilibrium strategies. Fix $k \in K$, $h_t \in H_t$ and m^1. Assume that $c_{h_t}^k$ is defined correctly and define, simultaneously, $c_{h_t,(m^1,m^2)}^k$ for every $m^2 \in M$. There are two cases to consider.

case 1: $P^{k\cdot}(h_t)\sigma_t(k,h_t)(m^1) = 0$.

In this case $P^{k\cdot}(h_t,(m^1,m^2)) = 0$ for all $m^2 \in M$, hence we have to prove only condition (c) and the first part of (a). Define:

$$U_{m^1} := \sum_{m^2 \in M} \lambda_{h_t}(m^2) X_{h_t,(m^1,m^2)}^k \leq X_{h_t}^k$$

Using the induction hypothesis we have

$$U_{m^1} \leq \max_{m \in M} U_m = X_{h_t}^k \leq c_{h_t}^k \leq Z = \sum_{m^2 \in M} \lambda_{h_t}(m^2) Z \tag{6}$$

and $X_{h_t,(m^1,m^2)}^k \leq Z$ for all $m^2 \in M$. Therefore we can choose (simultaneously for m^2) for all $m^2 \in M$, $c_{h_t,(m^1,m^2)}^k$ such that $X_{h_t,(m^1,m^2)}^k \leq c_{h_t,(m^1,m^2)}^k \leq Z$, and such that (c) is satisfied :

$$c_{h_t,(m^1,m^2)}^k := \begin{cases} X_{h_t,(m^1,m^2)}^k + (Z - X_{h_t,(m^1,m^2)}^k)\frac{c_{h_t}^k - U_{m^1}}{Z - U_{m^1}} & Z > U_{m^1} \\ Z & Z = U_{m^1} \end{cases}$$

The inequality $X_{h_t,(m^1,m^2)}^k \leq c_{h_t,(m^1,m^2)}^k$ follows from the two inequalities: $Z \geq X_{h_t,(m^1,m^2)}^k$ and $c_{h_t}^k \geq U_{m^1}$. The inequality $c_{h_t,(m^1,m^2)}^k \leq Z$ follows from the inequality $Z \geq c_{h_t}^k$ because the later yields $\frac{c_{h_t}^k - U_{m^1}}{Z - U_{m^1}} \leq 1$. Thus, we proved that $X_{h_t,(m^1,m^2)}^k \leq c_{h_t,(m^1,m^2)}^k \leq Z$. To complete this part of the proof we have to show that (c) is satisfied. We have two cases to consider: If $U_{m^1} = Z$ then $c_{h_t}^k = Z$ (see 6) and for all $m^2 \in M$ we have $c_{h_t,(m^1,m^2)}^k = Z$, hence (c) is satisfied. If $U_{m^1} < Z$ then:

$$\sum_{m^2 \in M} \lambda_{h_t}(m^2) c_{h_t,(m^1,m^2)}^k =$$

$$\sum_{m^2 \in M} \lambda_{h_t}(m^2) X_{h_t,(m^1,m^2)}^k + \frac{c_{h_t}^k - U_{m^1}}{Z - U_{m^1}} \left(\sum_{m^2 \in M} \lambda_{h_t}(m^2) Z - \sum_{m^2 \in M} \lambda_{h_t}(m^2) X_{h_t,(m^1,m^2)}^k \right)$$

$$= U_{m^1} + \frac{c_{h_t}^k - U_{m^1}}{Z - U_{m^1}} (Z - U_{m^1}) = c_{h_t}^k$$

and (c) is satisfied.

case 2: $P^{k\cdot}(h_t)\sigma_t(k,h_t)(m^1) > 0$.

In this case $P^{k\cdot}(h_t,(m^1,m^2)) > 0$ for all $m^2 \in M$ (recall that $\lambda_{h_t}(m^2) > 0$). Define:

$$c_{h_t,(m^1,m^2)}^k := E^{k\cdot}(\mathbf{a} \mid (h_t,(m^1,m^2))) \leq X_{h_t,(m^1,m^2)}^k .$$

Denote $(h_t,(m^1,m^2))$ by h_s and we will show that $c_{h_s}^k = X_{h_s}^k$. The idea is that if $E^{k\cdot}(\mathbf{a} \mid h_s) < X_{h_s}^k$ then type k of player 1 can achieve more than $E^{k\cdot}(\mathbf{a})$ by playing σ and switching to a strategy guaranteeing almost $X_{h_s}^k$ after h_s, a contradiction. We will choose an arbitrary strategy σ' and show that player 1 can gain no more than $E^{k\cdot}(\mathbf{a} \mid h_s)$, playing σ' after h_s and given $\mathbf{k}=k$ (otherwise he can get more than $E^{k\cdot}(\mathbf{a})$ by playing σ and switching to σ' after h_s), hence $c^k_{h_s} = X^k_{h_s}$. Formally, let σ' be a strategy of player 1. Define σ'' as follows:

$$\sigma''_r(k', h_r) := \begin{cases} \sigma'_x(k', h_x) & \text{for } h_r = (h_s, h_x) \\ \sigma_r(k', h_r) & \text{otherwise} \end{cases}$$

$$\sigma''_\infty(k', h_\infty) := \begin{cases} \sigma'_\infty(k', h'_\infty) & \text{for } h_\infty = (h_s, h'_\infty) \\ \sigma_\infty(k', h_\infty) & \text{otherwise} \end{cases}$$

Recall that $P^{k\cdot}(h_s) > 0$. σ'' is the strategy of playing σ and switching to σ' if h_s has occurred. Denote $E^{k\cdot}_{\sigma'',\tau}$ by $\tilde{E}^{k\cdot}$ and $P^{k\cdot}_{\sigma'',\tau}$ by $\tilde{P}^{k\cdot}$. Denote the set of strategies different from h_s (i.e, $H_s \backslash \{h_s\}$) by "*not* h_s". σ and τ are equilibrium strategies, therefore

$$E^{k\cdot}(\mathbf{a}) = P^{k\cdot}(h_s)E^{k\cdot}(\mathbf{a}|h_s) + (1 - P^{k\cdot}(h_s))E^{k\cdot}(\mathbf{a} \mid \textit{not } h_s)$$

$$\geq \tilde{P}^{k\cdot}(h_s)\tilde{E}^{k\cdot}(\mathbf{a}|h_s) + (1 - \tilde{P}^{k\cdot}(h_s))\tilde{E}^{k\cdot}(\mathbf{a} \mid \textit{not } h_s)$$

$P^{k\cdot}(h_s) = \tilde{P}^{k\cdot}(h_s) > 0$ and $E^{k\cdot}(\mathbf{a} \mid \textit{not } h_s) = \tilde{E}^{k\cdot}(\mathbf{a} \mid \textit{not } h_s)$ therefore

$$E^{k\cdot}(\mathbf{a} \mid h_s) \geq \tilde{E}^{k\cdot}(\mathbf{a} \mid h_s)$$

This is true for all σ', thus $E^{k\cdot}(\mathbf{a} \mid h_s) \geq X^k_{h_s}$ and therefore $E^{k\cdot}(\mathbf{a}|h_s) = X^k_{h_s}$ (because $E^{k\cdot}(\mathbf{a} \mid h_s) \leq X^k_{h_s}$). Thus $c^k_{h_s} = X^k_{h_s}$. From the induction hypothesis we have $c^k_{h_t} = E^{k\cdot}(\mathbf{a} \mid h_t)$, therefore

$$c^k_{h_t} = \sum_{m^1 \text{ s.t. } \sigma_t(k,h_t)(m^1)>0} \sigma_t(k, h_t)(m^1) \sum_{m^2 \in M} \lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)} \tag{7}$$

On the other hand:

$$c^k_{h_t} = X^k_{h_t} = \max_{m^1} \sum_{m^2 \in M} \lambda_{h_t}(m^2) X^k_{h_t,(m^1,m^2)}$$

hence

$$c^k_{h_t} \geq \max_{m^1 \text{ s.t. } \sigma_t(k,h_t)(m^1)>0} \sum_{m^2 \in M} \lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)} \tag{8}$$

From (7) and (8) follows that for all m^1 such that $\sigma_t(k, h_t)(m^1) > 0$ (in case 2 always $\sigma_t(k, h_t)(m^1) > 0$) we have

$$c^k_{h_t} = \sum_{m^2 \in M} \lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)}$$

∎

Corollary 3.24:
$\{(c_t, d_t, p_t, q_t)\}_{t=0}^{\infty}$ is a martingale with respect to the fields $\{\mathcal{H}_t\}_{t=0}^{\infty}$ and the probability $P{:=}P_{\sigma,\tau,p,q}$. c_t is also a martingale with respect to $P^{k\cdot}$ and d_t is a martingale with respect to $P^{\cdot l}$.

Proof: We will prove this only for c_t and P.

$$E(c_{t+1} \mid h_t) = \sum_{m^1,m^2 \in M} P((h_t,(m^1,m^2)) \mid h_t) c_{h_t,(m^1,m^2)} = \sum_{m^1 \in M} \sum_{m^2 \in M} \mu_{h_t}(m^1)\lambda_{h_t}(m^2) c_{h_t,(m^1,m^2)}$$

$$= \sum_{m^1 \in M} \mu_{h_t}(m^1) \sum_{m^2 \in M} \lambda_{h_t}(m^2) c_{h_t,(m^1,m^2)} = \sum_{m^1 \in M} \mu_{h_t}(m^1) c_{h_t} = c_{h_t} = E(c_t \mid h_t)$$

∎

Define $c_\infty := \lim_{t\to\infty} c_t$ and $d_\infty := \lim_{t\to\infty} d_t$. From the bounded martingales convergence theorem we have that these limits exist a.s.

Lemma 3.25: $\liminf_{t\to\infty} X_t^k \geq a_\infty^k$ *P-a.s. for all* $k \in K$.

Let $A := \{h_\infty \text{ s.t. } \liminf_{t\to\infty} X_t^k < a_\infty^k\}$. We have to show that $P(A) = 0$. $P(A) > 0$ implies that there exists $l \in L$ and $k' \in K$ such that $\pi_{\sigma,\tau,k',l}(A) > 0$ and therefore $\sum_{l' \in L} q(l')\pi_{\sigma,\tau,k',l'}(A) > 0$ hence $P^{k\cdot}_{\sigma^{k'},\tau}(A) > 0$. ($\sigma^{k'}$ is the strategy in which all the types of player 1 play according to σ in the action phase but according to the strategy of type k' in the talk phase of the game). $a'^k_{h_t} := E^{k\cdot}_{\sigma^{k'},\tau}(a^k_\infty \mid h_t) = E^{k\cdot}_{\sigma^{k'},\tau}(\mathbf{a} \mid h_t) \leq X^k_{h_t}$ for all h_t. a'^k_t is a martingale with respect to $P^{k\cdot}_{\sigma^{k'},\tau}$, hence $\liminf_{t\to\infty} X_t^k \geq \liminf_{t\to\infty} a'^k_t = a^k_\infty$ is satisfied $P^{k\cdot}_{\sigma^{k'},\tau}$-a.s. , a contradiction to $P^{k\cdot}_{\sigma^{k'},\tau}(A) > 0$. ∎

Corollary 3.26:
(c_∞, d_∞) *is an equilibrium in* $G(p_\infty, q_\infty)$ *P-a.s.*

Proof: From corollary 3.24 we have that c_∞, d_∞, p_∞ and q_∞ exist a.s. Fix $k \in K$. For all t we have $c_t^k \geq X_t^k$ (lemma 3.23), hence $c_\infty^k \geq a_\infty^k$ P-a.s. (lemma 3.25). If $p_{h_\infty}(k) > 0$ then $p_{(h_\infty)^t}(k) > 0$ for all t, hence $P^{k\cdot}((h_\infty)^t) = \frac{p_{(h_\infty)^t}(k)P((h_\infty)^t)}{p(k)} > 0$ and therefore $c^k_{(h_\infty)^t} = E^{k\cdot}(a^k_\infty|(h_\infty)^t)$ (lemma 3.23). Hence $c^k_\infty = a^k_\infty$ for $P^{k\cdot}$-almost every h_∞ such that $p_{h_\infty}(k) > 0$, which is P-almost every h_∞ such that $p_{h_\infty}(k) > 0$ (because $E^{k\cdot}(H) = \frac{1}{p(k)} \int_H p_{h_\infty}(k) dP$). Now conditions 3.2.1 and 3.2.3 of definition 3.2 follow from 3.21.1 . The proof of conditions 3.2.2 and 3.2.4 is similar. ∎

Thus we get that $\{(c_t, d_t, p_t, q_t)\}_{t=0}^\infty$ is an $|M|$-admissible martingale starting at (a, b, p, q) and converging to EQ: Condition 3.6.1 follows from lemma 3.23 and condition 3.6.2 follows from corollary 3.24. Condition 3.6.3 follows from lemma 3.23 and the fact that the messages of player 2 have no influence on p_t, and similarly player 1 does not affect q_t. Condition 3.6.4 follows from corollary 3.26. This ends the proof of the first part of the theorem.

Next, we assume that $\{(c_t, d_t, w_t, s_t)\}_{t=0}^\infty$ is an n-admissible martingale starting at (a, b, p, q) and converging to EQ, and we build equilibrium strategies σ and τ for a cheap-talk extension, $\Gamma(p, q, M)$ ($M := \{1, 2, 3, ..., n\}$), such that a will be the expected payoff vector for player 1, and b for player 2.

Lemma 3.27:
If there exists an n-admissible martingale starting at (a, b, p, q) *and converging to* EQ*, then there exists an exact n-admissible martingale starting at* (a, b, p, q) *and converging to* EQ.

Proof: This lemma is analogous to lemma 3.19 , and so is its proof. One can transform $S = (\{x_{u,v}\}_{1 \leq u,v \leq n}; \mu, \lambda)$, an n-admissible split of $x \in Q$, into an exact split using the following steps:

1. Choose $1 \leq \tilde{u} \leq n$ and $1 \leq \tilde{v} \leq n$ such that $\mu(\tilde{u}) > 0$ and $\lambda(\tilde{v}) > 0$.

2. Replace $x_{u,v}$ by:

   (a) $x_{\tilde{u},v}$ if $\mu(u) = 0$ and $\lambda(v) > 0$

   (b) $x_{u,\tilde{v}}$ if $\mu(u) > 0$ and $\lambda(v) = 0$

   (c) $x_{\tilde{u},\tilde{v}}$ if $\mu(u) = 0$ and $\lambda(v) = 0$

   (d) $x_{u,v}$ if $\mu(u) > 0$ and $\lambda(v) > 0$

3. If $\mu(u) = 0$ or $u = \tilde{u}$ replace $\mu(u)$ by $\frac{\mu(\tilde{u})}{|\{1 \leq u' \leq n \ s.t. \ \mu(u')=0\}|+1}$

4. If $\lambda(v) = 0$ or $v = \tilde{v}$ replace $\lambda(v)$ by $\frac{\lambda(\tilde{v})}{|\{1 \leq v' \leq n \ s.t. \ \lambda(v')=0\}|+1}$

Given an admissible martingale, one can make it into an exact one by making all the splits into exact ones. ▮

Using lemma 3.27 , we can assume that we have an exact martingale $\{(c_t, d_t, w_t, s_t)\}_{t=1}^{\infty}$ starting at (a, b, p, q) and converging to EQ. Let $x_t := (c_t, d_t, w_t, s_t)$. Let $f^t \in \mathcal{F}_t$. There is an exact split of $E(X_t \mid f^t)$, $S = (\{E(x_{t+1} \mid f^t_{u,v})\}_{1\leq u,v\leq n}; \mu_{f^t}, \lambda_{f^t})$. $\sum_{1\leq u,v\leq n} E(f^t_{u,v} \mid f^t) = 1$, therefore if $E(f^{t+1} \mid f^t) > 0$ then $f^{t+1} \in \{ f^t_{u,v} \mid 1 \leq u, v \leq n \}$. $\mathcal{F}_{t+1} \supset \mathcal{F}_t$, hence for all $f^{t+1} \in \mathcal{F}_{t+1}$ such that $P(f^{t+1}) > 0$ there exists a unique $f^t \in \mathcal{F}_t$ such that $E(f^{t+1} \mid f^t) > 0$, and therefore $f^{t+1} = f^t_{u,v}$ for some $(u, v) \in [n] \times [n]$. From the last two facts we can conclude that to every $f^t \in \mathcal{F}_t$, such that $P(f^t) > 0$, there corresponds a unique sequence from $([n] \times [n])^t$ .This map is one-to-one, since the martingale is exact. $M = [n]$, so to every $f^t \in \mathcal{F}_t$, such that $p(f^t) > 0$ there corresponds an $h_t \in H_t$. Denote by f_{h_t} the $f^t \in \mathcal{F}_t$ corresponding to h_t. We will write h_t instead of f_{h_t}. We will write $\mu_{h_t}, \lambda_{h_t}, c_{h_t}, d_{h_t}, w_{h_t}$ and s_{h_t} instead of $\mu_{f_{h_t}}, \lambda_{f_{h_t}}, c_{f_{h_t}}, d_{f_{h_t}}, w_{f_{h_t}}$ and $s_{f_{h_t}}$ respectively. Now we can define the equilibrium strategies. Define for $h_t \in H_t, m^1 \in M$ and $k \in K$ such that $w_{h_t}(k) > 0$

$$\sigma_t(k, h_t)(m^1) := \mu_{h_t}(m^1) \frac{w_{h_t,(m^1,m^2)}(k)}{w_{h_t}(k)}$$

If $w_{h_t}(k) = 0$ we define $\sigma_t(k, h_t)(m^1)$ arbitrarily. $\sigma_t(k, h_t)(m^1)$ is well defined because $w_{h_t,(m^1,m^2)}$ is constant for all $m^2 \in M$ and $\sum_{m^1 \in M} \mu_{h_t}(m^1) w_{h_t,(m^1,m^2)}(k) = w_{h_t}(k)$. Define for $h_t \in H_t, m^2 \in M$ and $l \in L$ such that $s_{h_t}(l) > 0$

$$\tau_t(l, h_t)(m^2) := \lambda_{h_t}(m^2) \frac{s_{h_t,(m^1,m^2)}(l)}{s_{h_t}(l)}$$

If $s_{h_t}(l) = 0$ we define $\tau_t(l, h_t)(m^2)$ arbitrarily. Again, $\tau_t(l, h_t)(m^2)$ is well defined. For $p' \in \Delta(K)$ and $b' \in W^2_{p'}$ let $\alpha_{b',p'} \in (\Delta(I))^K$ be a strategy of player 1 in $G(p', q')$, guaranteeing that for every $l \in L$ player 2 will not get more than b'^l (It is immediate from the definition of $W^2_{p'}$ that such a strategy exists). For $q' \in \Delta(L)$ and $a' \in W^1_{q'}$ let $\beta_{a',q'} \in (\Delta(J))^L$ be a strategy of player 2 in $G(p', q')$, guaranteeing that for every $k \in K$ player 1 will not get more than a'^k. For $(a', b', p', q') \in EQ$ let $\gamma_{a',b',p',q'} \in (\Delta(I))^K$ and $\delta_{a',b',p',q'} \in (\Delta(J))^L$ be equilibrium strategies for players 1 and player 2 respectively, with expected payoff vectors a' and b'. Choose arbitrary $\alpha' \in (\Delta(I))^K$ and $\beta' \in (\Delta(J))^L$. Define: $c_\infty := \lim_{t\to\infty} c_t$, $d_\infty := \lim_{t\to\infty} d_t$, $w_\infty := \lim_{t\to\infty} w_t$ and $s_\infty := \lim_{t\to\infty} s_t$.

If $(c_\infty, d_\infty, w_\infty, s_\infty)$ exists and $(c_\infty, d_\infty, w_\infty, s_\infty) \in EQ$ then define:

$$\sigma_\infty := \gamma_{c_\infty, d_\infty, w_\infty, s_\infty} \quad \textit{and} \quad \tau_\infty := \delta_{c_\infty, d_\infty, w_\infty, s_\infty}$$

otherwise define:

$$\sigma_\infty := \begin{cases} \alpha_{d_\infty, w_\infty} & \text{if } d_\infty \text{ and } w_\infty \text{ exist} \\ \alpha' & \text{otherwise} \end{cases} \qquad \tau_\infty := \begin{cases} \beta_{c_\infty, s_\infty} & \text{if } c_\infty \text{ and } s_\infty \text{ exist} \\ \beta' & \text{otherwise} \end{cases}$$

Remark: we will prove later (lemma 3.30) that σ_∞ and τ_∞ are well defined.

Lemma 3.28:

1. $P_{\sigma,\tau,p,q}(h_t) = P(f_{h_t}) > 0$ *for all* $h_t \in H_t$.
2. $w_{h_t} = p_{h_t}$ *for all* $h_t \in H_t$. *(recall that* $p_{h_t}(k) := P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t)$*).*
3. $s_{h_t} = q_{h_t}$ *for all* $h_t \in H_t$. *(recall that* $q_{h_t}(l) := P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h_t)$*).*

Proof: We will prove the lemma by induction. For $t = 0 : h_0 = \phi$ and $P_{\sigma,\tau,p,q}(h_0) = P(f_{h_0}) = 1$. $w_{h_0} = p = p_{h_0}$ and $s_{h_0} = q = q_{h_0}$. Now we assume that 1, 2 and 3 are correct for h_t and prove for $(h_t, (m^1, m^2))$. The proof of 3 is similar to the proof of 2, so we will just prove 1 and 2.

1.

$$\begin{aligned} P_{\sigma,\tau,p,q}(h_t, (m^1, m^2)) &= P_{\sigma,\tau,p,q}(h_t) \sum_{k \in K} p_{h_t}(k)\sigma_t(k, h_t)(m^1) \sum_{l \in L} q_{h_t}(l)\tau_t(l, h_t)(m^2) \\ &= P(f_{h_t}) \sum_{k \in K} w_{h_t}(k)\sigma_t(k, h_t)(m^1) \sum_{l \in L} s_{h_t}(l)\tau_t(l, h_t)(m^2) \end{aligned}$$

and from the definition of σ and τ

$$= P(f_{h_t}) \sum_{k \in K} \mu_{h_t}(m^1) w_{h_t,(m^1,m^2)}(k) \sum_{l \in L} \lambda_{h_t}(m^2) s_{h_t,(m^1,m^2)}(l)$$

$$= P(f_{h_t})\mu_{h_t}(m^1)\lambda_{h_t}(m^2) = P(f_{h_t,(m^1,m^2)})$$

$\mu_{h_t}(m^1) > 0$, $\lambda_{h_t}(m^2) > 0$ (the martingale is exact) and $P(f_{h_t}) > 0$ (the induction hypothesis), thus we have $P(f_{h_t,(m^1,m^2)}) > 0$.

2.

$$\begin{aligned} p_{h_t,(m^1,m^2)}(k) &= \frac{p_{h_t}(k)\sigma_t(k, h_t)(m^1)}{\sum_{k' \in K} p_{h_t}(k')\sigma_t(k', h_t)(m^1)} = \frac{w_{h_t}(k)\sigma_t(k, h_t)(m^1)}{\sum_{k' \in K} w_{h_t}(k')\sigma_t(k', h_t)(m^1)} \\ &= \frac{\mu_{h_t}(m^1) w_{h_t,(m^1,m^2)}(k)}{\sum_{k' \in K} \mu_{h_t}(m^1) w_{h_t,(m^1,m^2)}(k')} = w_{h_t,(m^1,m^2)}(k) \end{aligned}$$

∎

Lemma 3.29:
$x_t = (c_t, d_t, p_t, q_t) \in IR$ *for all* t.

Proof: Fix $h_t \in H_t$. Denote $P_{h_t}(\cdot) := P(\cdot \mid h_t)$. Let E_{h_t} be the expectation with respect to P_{h_t}. $x_\infty \in EQ$ P-a.s. (3.6.4), $EQ \subset IR$ and $P(h_t) > 0$ (lemma 3.28), hence $x_\infty \in IR$ P_{h_t}-a.s. $x_{h_t} = E_{h_t}(x_\infty)$ and IR is convex (corollary 3.17) hence $x_{h_t} \in IR$. ∎

Lemma 3.30:

1. *If c_∞ and s_∞ exist then $c_\infty \in W^1_{s_\infty}$.*
2. *If d_∞ and w_∞ exist then $d_\infty \in W^2_{w_\infty}$.*

Proof: We will prove 1; The proof of 2 is similar. Fix h_∞ such that c_{h_∞} and s_{h_∞} exist. $c_{(h_\infty)^t} \in W^1_{s_{(h_\infty)^t}}$ for all t (lemma 3.29). W^1_q is upper semi continuous with respect to q, therefore $c_{h_\infty} \in W^1_{s_{h_\infty}}$. ∎

Lemma 3.31: *c^k_t is a martingale with respect to $P^{k\cdot}$.*

Proof:

$$E^{k\cdot}(c^k_{t+1} \mid h_t) = \sum_{m^1,m^2 \in M} P^{k\cdot}((h_t,(m^1,m^2)) \mid h_t) c^k_{h_t,(m^1,m^2)}$$

$$= \sum_{m^1 \in M} \sum_{m^2 \in M} \sigma(k,h_t)(m^1)\lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)} = \sum_{m^1 \in M} \sigma(k,h_t)(m^1) \sum_{m^2 \in M} \lambda_{h_t}(m^2) c^k_{h_t,(m^1,m^2)}$$

$$= \sum_{m^1 \in M} \sigma(k,h_t)(m^1) c^k_{h_t} = c^k_{h_t} = E^{k\cdot}(c^k_t \mid h_t)$$

∎

$a^k_\infty = c^k_\infty$ $P^{k\cdot}$-a.s. (because $(c_\infty, d_\infty, w_\infty, s_\infty) \in EQ$ $P_{\sigma,\tau,p,q}$-a.s., hence also $P^{k\cdot}$-a.s. and σ_∞ and τ_∞ are equilibrium strategies in G with payoffs c_∞ and d_∞ whenever $(c_\infty, d_\infty, p_\infty, q_\infty) \in EQ$), therefore $E^{k\cdot}(\mathbf{a}) = a^k$ (lemma 3.31). Similarly $E^{\cdot l}(\mathbf{b}) = b^l$. We have to show that no player can get more by using a different strategy. We will prove it for player 1. Assume that player 2 plays τ and player 1 plays σ'. Denote by $\mu'_{h_t}(m^1)$ the probability of player 1 sending the message m^1 after history h_t, using σ'. $\lambda_{h_t}(m^2)$ is the probability of player 2 sending m^2 after h_t.

Lemma 3.32: *$\{c_t\}_{t=0}^\infty$ and $\{s_t\}_{t=0}^\infty$ are martingales with respect to $P^{k\cdot}_{\sigma',\tau}$.*

Proof: We will prove this only for $\{c_t\}_{t=0}^\infty$. The proof for $\{s_t\}_{t=0}^\infty$ is similar.

$$E^{k\cdot}_{\sigma',\tau}(c_{t+1} \mid h_t) = \sum_{m^1 \in M} \sum_{m^2 \in M} \sigma'_t(k,h_t)(m^1)\lambda_{h_t}(m^2) c_{h_t,(m^1,m^2)}$$

$$= \sum_{m^1 \in M} \sigma'_t(k,h_t)(m^1) \sum_{m^2 \in M} \lambda_{h_t}(m^2) c_{h_t,(m^1,m^2)} = \sum_{m^1 \in M} \sigma'_t(k,h_t)(m^1) c_{h_t} = c_{h_t} = E^{k\cdot}_{\sigma',\tau}(c_t \mid h_t)$$

∎

$\{c_t\}_{t=0}^\infty$ and $\{s_t\}_{t=0}^\infty$ are bounded martingales with respect to $P^{k\cdot}_{\sigma',\tau}$, hence they converge $P^{k\cdot}_{\sigma',\tau}$-a.s. to c_∞ and s_∞ respectively, and $c_t = E^{k\cdot}_{\sigma',\tau}(c_\infty \mid h_t)$. Therefore

$$a^k = c^k_0 = E^{k\cdot}_{\sigma',\tau}(c^k_\infty \mid h_0)$$

Define $a'^k_\infty := \sum_{l \in L} q_\infty(l) A^{k,l}(\sigma'(k,h_\infty), \tau(l,h_\infty))$.

Lemma 3.33:
$c_\infty^k \geq a_\infty'^k$ $P_{\sigma',\tau}^{k\cdot}$*-a.s.*

Proof: $\lim_{t\to\infty}(c_t, s_t)$ exists $P_{\sigma',\tau}^{k\cdot}$-a.s. and equals (c_∞, s_∞). From lemma 3.28 $w_\infty = p_\infty$ and $s_\infty = q_\infty$. There are 2 cases to consider (when $(c_\infty, q_\infty) = (c_\infty, s_\infty)$ exists):

1. d_∞ and p_∞ exist and (c_∞, d_∞) is an equilibrium in $G(p_\infty, q_\infty)$: In this case player 2 plays $\delta_{c_\infty, d_\infty, p_\infty, q_\infty}$, hence player 1 can get no more than c_∞. Hence $c_\infty^k \geq a_\infty'^k$.

2. d_∞ or q_∞ do not exist or (c_∞, d_∞) is not an equilibrium in $G(p_\infty, q_\infty)$: In this case player 2 plays $\beta_{c_\infty, q_\infty}$ guaranteeing that player 1 will not get more than c_∞ and again $c_\infty^k \geq a_\infty'^k$.

∎

From lemma 3.33 follows:

$$a^k = E_{\sigma',\tau}^{k\cdot}(c_\infty^k) \geq E_{\sigma',\tau}^{k\cdot}(a'_\infty) = E_{\sigma',\tau}^{k\cdot}(\mathbf{a})$$

∎

4 On the number of possible messages

In this section we provide an example of a game, for which there exists a pair of payoff vectors, (c, d), such that (c, d) is an equilibrium when there are at least 3 possible messages, and is not an equilibrium when there are only 2 possible messages (Moreover, with 2 messages one can not obtain (c, d), neither any equilibrium payoffs $\geq (c, d)$). Analogous examples can be built showing that for every natural number n, there exist a game and a pair of payoff vectors, (c, d), such that (c, d) is an equilibrium when there are at least n possible messages, and is not an equilibrium when there are less than n possible messages.

We define a game $\Gamma(p, q, M)$ with independent incomplete information on both sides. Let $K = L = \{1, 2, 3\}$ and $p = q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let

$$A^{1,1} = \begin{pmatrix} 1 & 3 & 3 \\ -3 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix} \quad A^{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A^{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^{2,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A^{2,2} = \begin{pmatrix} 0 & -3 & 0 \\ 3 & 1 & 3 \\ 0 & -3 & 0 \end{pmatrix} \quad A^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^{3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A^{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad A^{3,3} = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$B^{1,1} = \begin{pmatrix} 1 & -3 & -3 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \quad B^{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B^{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B^{2,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B^{2,2} = \begin{pmatrix} 0 & 3 & 0 \\ -3 & 1 & -3 \\ 0 & 3 & 0 \end{pmatrix} \quad B^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B^{3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B^{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad B^{3,3} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ -3 & -3 & 1 \end{pmatrix}$$

Proposition 4.1: *Let $c = d = (1,1,1)$.*

1. *If $|M| \geq 3$ then (c,d) are equilibrium payoffs in $\Gamma(p,q,M)$.*
2. *If $|M| = 2$ then (c,d) are not equilibrium payoffs in $\Gamma(p,q,M)$.*

Proof: Note that for every $k \in K$ and $l \in L$ the following holds:

$$A^{k,l}(i,j) + B^{k,l}(i,j) = \begin{cases} 2 & if\ j = k \text{ and } i = l \\ 0 & \text{otherwise} \end{cases}$$

Therefore (c,d) will be achieved if and only if in the action phase of the game, player 1 plays $i = \mathbf{l}$ and player 2 plays $j = \mathbf{k}$. This can be done only after complete revelation of the private information of the two players ($\mathbf{k}$ and $\mathbf{l}$).

If $|M| \geq 3$ the two players can reveal $\mathbf{k}$ and $\mathbf{l}$ simultaneously in the first period of the talk phase of the game. Then in the action phase player 1 plays $\mathbf{l}$ and player 2 plays $\mathbf{k}$. These are clearly equilibrium strategies of $\Gamma(p,q,M)$ with payoffs (c,d). Formally, let $M = \{z_1, z_2, z_3, ..., z_{|M|}\}$. Denote by m_t^1 and m_t^2 the t^{th} messages sent by player 1 and player 2 respectively. The equilibrium strategies are as follows: In the first period of the talk phase player 1 sends the massage $m_1^1 = z_{\mathbf{k}}$ and player 2 sends the massage $m_1^2 = z_{\mathbf{l}}$ (i.e, [9] $\sigma(k,h_0) = z_k$ for all $k \in K$ and $\tau(l,h_0) = z_l$ for all $l \in L$). For $t > 0$, $\sigma(k,h_0)$ and $\tau(l,h_0)$ can be chosen arbitrarily, as they have no influence on the action phase. In the action phase player 1 plays according to m_1^2 (which is according to $\mathbf{l}$), i.e, he will play the action 1 if $m_1^2 = z_1$, 2 if $m_1^2 = z_2$ and 3 if $m_1^2 = z_3$. similarly, player 2 plays according to m_1^1. The payoffs obtained by this pair of strategies are clearly (c,d). To see that the strategies defined above are equilibrium strategies, note that no player can gain more than 1 by deviating in the action phase. By deviating in the talk phase a player can make the other player change his action in the action phase of the game. This may bring him a payoff of 3 instead of 1 in one of the payoff matrices (and with probability $\frac{1}{3}$), but this can not be made without receiving payoff 0 with probability $\frac{2}{3}$, and the expected payoff can not exceed 1.

If $|M| = 2$ the private information can not be totally revealed in one period. Assume that there exist equilibrium strategies in which the two players reveal $\mathbf{k}$ (player 1) and $\mathbf{l}$ (player 2) during the talk phase of the game and then play $i = \mathbf{l}$ (player 1) and $j = \mathbf{k}$ (player 2) in the action phase of the game. Assume also that $M = \{z_1, z_2\}$. At least one of the players must receive some of the information of the other player before revealing all of his. Assume w.l.o.g that this is player 1 and assume that t+1 is the first period in which information is being revealed. This means that there exists an history h_t such that $P(h_t) > 0$ and $p_{h_t} = q_{h_t} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $q_{(h_t,(m^1,z_1))} \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (for all m^1). w.l.o.g assume that $q_{(h_t,(m^1,z_1))}(1) > \frac{1}{3}$ and $q_{(h_t,(m^1,z_1))}(2) < \frac{1}{3}$. Given $(h_t,(m^1,z_1))$ and $\mathbf{k} = 1$

[9] despite the abuse of notation z_k also denotes the probability vector in $\Delta(M)$ which gives probability 1 to z_k.

player 1 will get $3 \times q_{(h_t,(m^1,z_1))}(1) > 1$ by making player 2 think that $\mathbf{k} \neq 1$ (i.e, switching to the strategy that he should play given $\mathbf{k} = 2$ or to the strategy that he should play given $\mathbf{k} = 3$. Using this strategy player 1 will get a payoff of 3 when $\mathbf{l} = 1$, i.e, with probability $q_{(h_t,(m^1,z_1))}(1)$, and 0 otherwise, having expected payoff of $3 \times q_{(h_t,(m^1,z_1))}(1) > 1$). This can not be done only when $p_{(h_t,(m^1,z_1))}(1) \in \{0,1\}$. We assumed that these are equilibrium strategies, therefore player 1 can not achieve more than 1, and therefore $p_{(h_t,(m^1,z_1))}(1) \in \{0,1\}$, hence also $p_{(h_t,(m^1,z_2))}(1) \in \{0,1\}$. Now, $q_{(h_t,(m^1,z_2))}(2) > \frac{1}{3}$ (because $q_{(h_t,(m^1,z_1))}(2) < \frac{1}{3}$) and as before we must have $p_{h_t,(m^1,m^2)}(2) \in \{0,1\}$ for all $m^2 \in M$ and therefore also $p_{h_t,(m^1,m^2)}(3) \in \{0,1\}$ for all $m^2 \in M$, which means a complete revelation of $\mathbf{k}$ in one period - a contradiction. ∎

Remark: Analogous examples can be built for every $n \geq 2$, i.e, for every $n \geq 2$ there exists a game and two payoff vectors (c, d) such that

1. If $|M| \geq n$ then (c, d) are equilibrium payoffs in $\Gamma(p, q, M)$.
2. If $|M| < n$ then (c, d) are not equilibrium payoffs in $\Gamma(p, q, M)$.

One way is to define n^2 games, in which the sets of actions are of size n. Let $K = L = \{1, 2, 3, ..., n\}$, $p = q = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$. The structure of the $n \times n$ payoff matrices is similar to the structure of the 3×3 matrices defined above:

$$A^{k,l}(l,k) = B^{k,l}(l,k) = 1 \text{ for all } 1 \leq k, l \leq n$$

$$A^{k,k}(k,j) = n \ , \ A^{k,k}(j,k) = -n \ , \ B^{k,k}(k,j) = -n \ , \ B^{k,k}(j,k) = n \quad \text{for all } k \neq j$$

$$\text{and } A^{k,l}(i,j) = B^{k,l}(i,j) = 0 \text{ otherwise}$$

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