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OPTIMAL PRICING BY A RISK-AVERSE SELLER

By

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מרכז פדרמן לחקר הרציונליות

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Optimal Pricing by a Risk-Averse Seller

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Abstract

We consider the basic setup of one seller, one buyer, and one good, where the seller is *risk averse*, and characterize the mechanism that maximizes the seller's expected utility. In contrast to the risk-neutral case, where a single deterministic price is optimal, we show that in the risk averse case the optimal mechanism consists of a continuum of lotteries.

1 Introduction

An agent – the seller – has an object she wants to sell. Other agents – the buyers – want that object. What should the seller do? How should the seller manage the interaction between them? In this scenario, the seller cannot achieve a better expected revenue than via a "second price" auction with reserve price. This famous result, first described in the early eighties by Myerson (1981); Riley and Samuelson (1981), and Riley and Zeckhauser (1983), hinges on several assumptions, one of them being that the agents are risk neutral. Our paper aims to analyze the same situation with a small yet significant change: the seller in our setup is *risk averse*.

We characterize the optimal mechanism when there is risk aversion in the simplest setup, that of only one buyer. While the seller is assumed to be risk averse, the buyer is, as usual, risk neutral (in Section 6.1 we generalize our theorem to the case where the buyer is risk averse). The seller's utility function is thus concave with respect to the payments received from the buyer, and we characterize the optimal mechanism in this specific situation.

Although the case of risk-neutral agents has been studied intensely (see, e.g., Myerson, 1981; Riley and Samuelson, 1981; Riley and Zeckhauser, 1983; Klemperer, 1999), there are surprisingly few results for the case of risk-averse agents. Most of the literature on mechanism design under risk aversion either compares known auctions (e.g., Riley and Samuelson, 1981; Matthews, 1987; McAfee and McMillan, 1987a,b; Hu et al., 2010; Krishna, 2002), or analyzes what happens as the agents become more and more risk averse (e.g., Klemperer, 1999; Hu et al., 2010). Of the

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very few papers that attempt to characterize the optimal mechanism, the most notable is Maskin and Riley (1984). In a recent paper, Baisa (2017) characterizes the optimal mechanism for a setup that assumes risk aversion and a "good product."[1] Another relevant example recently published is that of Evdokia et al. (2018), who characterize the optimal mechanism for a family of risk-loving utility functions. However, to the best of our knowledge, no paper identifies the optimal mechanism in the setup described above, i.e., with one good, one buyer and one risk-averse seller.

A risk-averse seller does not maximize her expected revenue, but rather her expected utility from the revenue. Such a seller will forgo some of her expected revenue in order to be able to sell with higher probability. To illustrate this, assume that $u(x) = \min\{x, 100\}$, i.e., the seller has no need of more than 100 units. Assume that the buyer values the object at either 100 or 300 with equal probability. The revenue-maximizing mechanism will set a price of 300, yielding an expected revenue of $(1/2) \cdot 300 = 150$ and an expected utility of $(1/2) \cdot 100 = 50$. The utility-maximizing mechanism, however, will set a price of 100, thus achieving a lower expected revenue of 100, but a higher expected utility of 100.

Throughout this paper, "optimal" mechanisms refer to mechanisms that maximize the seller's expected utility, while "revenue-maximizing" (RM) mechanisms maximize the seller's expected revenue. Of course, when the seller is risk neutral, revenue-maximizing mechanisms are also optimal.

We also consider the one-price utility-maximizing (OPUM) mechanism, which, out of all one-price mechanisms, yields the seller the highest expected utility. This simple OPUM mechanism can achieve a constant fraction of the optimal utility for the seller (see Proposition 21 for more details). Moreover, when the agents have CARA utility functions, the OPUM mechanism is shown to yield up to 99.5% of the optimal utility (see section 4.2 for more details).

The paper is organized as follows. In Section 2 we define the setup and discuss revenue-maximizing mechanisms. In Section 3 we prove our main result - a characterization of the optimal mechanism. In Section 4 we give an improved characterization of the optimal mechanism under a specific regularity condition. In Section 5 we attempt to determine when the OPUM mechanism is nearly as good as the optimal mechanism. In the last section we generalize some of our results.

2 Preliminaries

Basic setup

As explained in the Introduction, we study the classic problem of the optimal mechanism when there is one seller, one buyer, and one good, with a small, yet significant, change: our seller is risk averse and aims to maximize her expected utility.

[1] As a result, Baisa's mechanism requires the buyers to submit their entire price-demand curve.

In line with the literature, we assume that the seller has no costs and derives no utility from having the good (other than the revenue gained from the sale), that the buyer's value for the good, denoted by x, is drawn from some probability distribution F that is known to the seller, and that this probability distribution F is atomless, has full support on $[a, c]$, and is twice continuously differentiable. We also assume that the buyer's utility from money and possession of the good is quasi-linear. Therefore the buyer's utility function is $V(x, \omega, s) = \omega \cdot x - s$, where x is the buyer's valuation of the good (his type), ω is a binary variable that has a value of 1 if the buyer got the good and 0 if he didn't, and s is the payment from the buyer to the seller.

We diverge from the standard models by letting a concave vNM utility function for money $u(s)$ denote the seller. Thus, we assume that a utility function for money is twice differentiable, strictly increasing, strictly concave, has a finite right derivative at zero, and without loss of generality is null at zero.

Mechanisms

A mechanism is a selling procedure that the seller can choose in order to sell her good. Our goal in this paper is to characterize the mechanism that maximizes the seller's expected utility (henceforth "the optimal mechanism"). As we assume that the buyer is risk neutral, we can use the famous "Revelation Principle" by Myerson (1981; see also the book of Krishna, 2002) and consider only "direct" mechanisms that are incentive compatible (IC) and individually rational (IR). A direct mechanism μ is a pair of functions $\mu = (q, s) : [a, c] \to [0, 1] \times \mathbb{R}$. The *allocation function*, q, determines the buyer's[2] chances of winning the good, $q(x)$, and the *payment function, s,* determines how much he has to pay, $s(x)$. Thus, if the buyer's valuation of the good is x, he will pay $s(x)$ for a lottery in which he can win the good with probability $q(x)$. The mechanism μ satisfies IC if the buyer is (weakly) better off reporting his true value than any other value, i.e., $x \cdot q(x) - s(x) \geq x \cdot q(y) - s(y)$ for any $x, y \in [a, c]$. In our setup, s must be a non-decreasing function in order to satisfy IC. The mechanism μ satisfies IR if the buyer's expected utility from the mechanism is never negative, i.e., $x \cdot q(x) - s(x) \geq 0$. In our setup, when IC is satisfied, IR translates to the requirement that $a \cdot q(a) - s(a) \geq 0$.

Note that in the optimal mechanism it must be that the utility of the smallest buyer type is zero. Furthermore, since the buyer has a quasi-linear utility function, we can use another result by Myerson (1981) by which the optimal payment function s defines, and can be defined by, the optimal allocation function q

$$\begin{aligned} q(x) &= \tfrac{s(a)}{a} + \int_a^x \tfrac{s'_+(t)}{t} dt \\ s(x) &= x \cdot q(x) - \int_a^x q(t)\, dt, \end{aligned} \tag{2.1}$$

where $s'_+(t)$ is the right derivative at t. From here onward, whenever we refer to either q or s as the mechanism itself, we assume that the other function is defined using the above equations.

[2] The buyer here buys a lottery, rather than the good itself. This is in contrast to the risk-neutral case where the discussion can be restricted to deterministic mechanisms.

For every mechanism μ, we define ***a buyer's utility payoff*** function $b(x) = x \cdot q(x) - s(x)$. As is shown in Hart and Nisan (2013, 2017), b must be a convex function that satisfies $0 \leq b'_+ \leq 1$. Moreover, any function $b : [a, c] \to [0, c]$ that is convex and has derivatives between zero and one uniquely defines an IC-IR direct mechanism[3] through $q = b'_+$, $s = x \cdot b'_+ - b$, as illustrated in Figure 2.1.

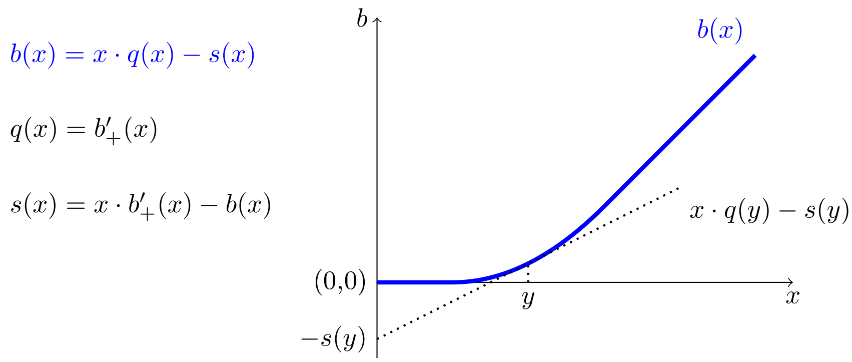

Figure 2.1: Calculating $s(x)$ and $q(x)$ given $b(x)$.

Finally, similarly to Hart and Nisan (2013, 2017), when we maximize the expected utility of the seller, we can assume, w.l.o.g., that $s(a) \geq 0$. Otherwise, it is easy to verify that $\hat{s}(x) \coloneqq s(x) - s(a)$ is an IC-IR mechanism that yields higher expected utility than s, where $\hat{s}(x) \geq 0$.

2.0.1 Revenue-maximizing mechanisms

Myerson and others' (Myerson, 1981; Riley and Zeckhauser, 1983) classic result proves that in our setup there is a revenue-maximizing "posted price" mechanism; i.e., the good is offered for a given price in a "take it or leave it" offer. Clearly, under risk aversion, maximizing the revenue may not coincide with maximizing the utility of the seller. In particular, when the seller is risk averse, a "one-price" mechanism cannot be an optimal mechanism. To see this, consider the following simple example.

Example 1. Let F be the uniform distribution over $[0, 1]$, and let the seller's utility function, u, be a strictly concave utility function. For tractability reasons, we express mechanisms as their associated buyer's expected utility payoff. Thus, a one-price mechanism with a price of z (the solid green line in Figure 2.2) is denoted by the buyer's utility payoff function

$$b(x) = \begin{cases} 0 & 0 \leq x \leq z \\ x - z & z \leq x \leq 1. \end{cases}$$

As proven in (Rochet, 1985; Manelli and Vincent, 2007) $b(z) = z \cdot q(z) - s(z)$, $b'_+(z) = q(z) = 1$, and at $(0, -s(z))$ the y-axis meets the line that goes through z with a slope of $b'_+(z)$ (the dotted green line in Figure 2.2).

[3] We restrict ourselves to seller-favorable mechanisms, as defined in (Hart and Reny, 2015).

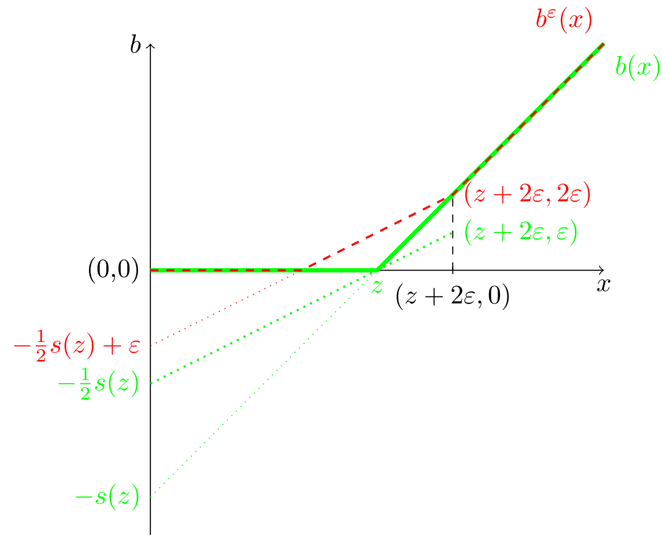

Figure 2.2: Under risk aversion, a one-price mechanism cannot, in general, be an optimal-mechanism.

Let us now analyze the mechanism b^ε (the dashed red line in Figure 2.2). Note that it is parallel to the line that goes through $(z, 0)$ and $(0, -s(z)/2)$ in the small interval $[z-2\varepsilon, z+2\varepsilon]$, and it is equal to b outside this interval. Under the original mechanism b, if the buyer type is in $(z-2\varepsilon, z)$ then the seller is paid nothing, while if the buyer type is in $(z, z+2z)$, the seller receives $s(z)$. In other words, the seller faces a lottery on this interval: receive no pay or $s(z)$ with equal probability. As in Figure 2.2, b^ε replaces this lottery with a guaranteed payment of $s(z)/2-\varepsilon$. Since u is concave, there is a small enough ε such that the seller prefers the constant payment over the lottery, even if this means losing some money on expectation.

To summarize, a "corner" in the function b means the seller faces a lottery. By "cropping the corner" we replace this lottery with a constant payment, albeit at the cost of diminishing the seller's expected revenue. For a risk-neutral seller, that would spell bad news. Our seller, however, is risk averse, and so, if done correctly, this cropping improves her expected utility.

There is nothing unique about the corner in Example 1. Whenever $u'' < 0$, any corner in b can be cropped (see Section 3.3.1). Hence, when u'' is always negative, the buyer's utility payoff function from the optimal mechanism has no corners, making it continuously differentiable. To put it differently, when the seller's utility function is strictly concave, as in our setup, we expect the optimal mechanism (i.e., the functions q and s) to be continuous.

Note that when the smallest buyer type possible is itself a revenue-maximizing price, i.e., the buyer types are distributed on $[a, c]$ and the price a maximizes the expected revenue, there is no corner. In such a case, the optimal mechanism is a one-price mechanism.

Notations and conventions. First, throughout the paper, derivatives of monotone functions (such as q, s, b) are right derivatives, which always exist for monotone functions.

Second, we restrict the class of mechanisms to seller-favorable mechanisms. This means that for every buyer type x, $s(x)$ will be as high as possible without violating IC. Formally, it means that q and s are right-continuous. Moreover, as shown in Hart and Reny (2015), this is done without loss of generality, since we are looking for the optimal mechanism.

Third, we denote by $\mathbb{M}$ the set of IC and IR seller-favorable mechanisms $\mu = (q, s)$ that satisfy $s(a) \geq 0$. We let $U(\mu, F) = \mathbb{E}_{F(x)}[u(s(x))]$ and we let $URev(F) = sup_{\mu \in \mathbb{M}} U(\mu, F)$ be the supremum on the expected utility the seller may achieve by using IC-IR mechanisms.

3 Optimal Mechanisms

In this paper we characterize the unique mechanism that yields the seller the maximal expected utility among all IC and IR mechanisms, given her utility function for money u and the distribution $x \sim f[a, c]$ of the buyer types. Thus, an optimal mechanism is a solution to the maximization problem:

$$argmax_{(s,q) \in \mathbb{M}} \int_a^c u(s(t)) f(t)\, dt.$$

Remark 2. Note that under the optimal mechanism, it must be that $q(c) = 1$. Otherwise $\hat{q} := q/q(c)$ will be an IC-IR mechanism that yields higher expected utility to the seller.

3.0.1 A useful decomposition of an optimal mechanism

In our proofs we will make use of a decomposition of the optimal mechanism μ into mechanisms that have only one price.[4] For this decomposition, let $\mu = (q, s)$ be an optimal mechanism. Then q may be viewed as a cumulative distribution function[5] and we have

$$\begin{array}{rclcl} q(x) & = & \int_a^x dq(t) & = & \int_a^c \mathbf{1}_{x \geq t} dq(t) \\ s(x) & = & \int_a^x t dq(t) & = & \int_a^c t \mathbf{1}_{x \geq t} dq(t) \\ b(x) & = & \int_a^x (x - t)\, dq(t) & = & \int_a^c [x - t]_+ dq(t). \end{array}$$

For more on this decomposition see Hart and Reny (2017).

We will now prove that a unique optimal mechanism always exists and characterize it.

3.1 Existence and uniqueness of the optimal mechanism

Proposition 3. *In our setup an optimal mechanism always exists.*

The proof, being quite standard, is relegated to Appendix 7.1.

[4] Sometimes called "one-price mechanisms" or "posted-price mechanisms."

[5] Indeed, q is non-negative by definition, non-decreasing by IC, and $q(c) = 1$ by remark 2 above.

Proposition 4. *The optimal IC-IR direct (seller-favorable) mechanism is unique.*

Due to the strict concavity of the seller's utility function, a convex combination of any two mechanisms that differ on a non-empty interval yields higher expected utility to the seller. Hence, any two optimal mechanisms must be equal almost everywhere. The seller-favorable requirement ensures that our mechanisms are right-continuous. Therefore, any two optimal mechanisms must be identical. It is a standard argument, and so the formal proof is relegated to Appendix 7.2.

3.2 Characterization of the optimal mechanism

Before we can characterize the optimal mechanism, we need the following concept of monotonicity:

Definition 5. A non-decreasing (non-increasing) function h is said to be *strictly increasing (decreasing) around* x if for any $x' < x$ and $x'' > x$ it holds that $h(x'') > h(x')$ $(h(x'') < h(x'))$.

Note that the notion of the function h being *strictly increasing around* x generalizes the notion of h being strictly increasing in a small neighborhood of x. Our notion allows h to be constant either right before or right after x, or even on both sides if h jumps at x.

We can now state our main theorem.

Theorem 6. *In our setup, the unique optimal IC-IR mechanism $\mu = (q, s)$ satisfies the following conditions:*

1. *The functions q and s are continuous.*

2. *There is a constant $\lambda \geq 0$ s.t. $x \int_x^c u'(s(t)) f(t)\, dt \leq \lambda$ for every x, with equality when q is strictly increasing around x.*

3. *$q(x) = 1$ if and only if $x \geq r$, where r is the minimal RM-price.*

Remark 7. Property 1 uses the strict concavity of the utility function. However, the property can be somewhat generalized to weakly concave utility functions (see Section 6.2).

Remark 8. Property 2 is essentially a continuous version of the Kuhn–Tucker theorem, tailored to our maximization problem.

Remark 9. Property 3 means that the buyer types that would have bought the good in an RM-mechanism are exactly those who get the good with probability one under the optimal mechanism. It also means that $s(r)$ is the maximum of s, and hence that $URev(F) \leq u(s(r)) \leq u(r)$.

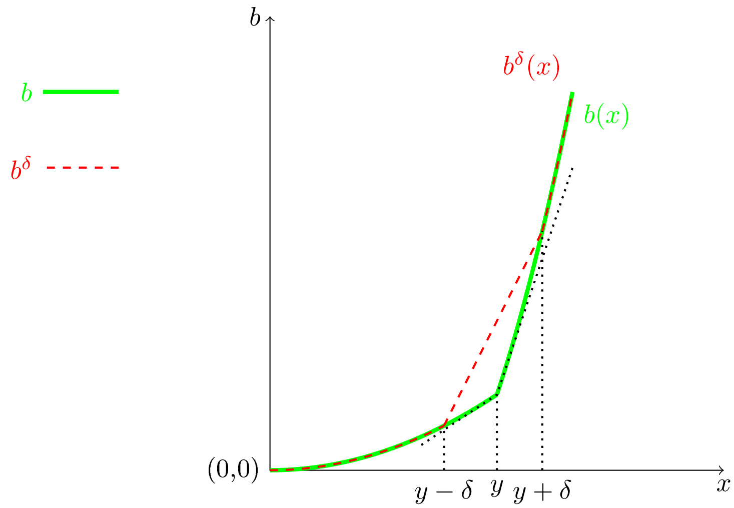

Figure 3.1: $q = b'$ jumps at y, and q^δ (which gives rise to b^δ) yields higher expected utility to the seller.

3.3 Proof of Theorem 6

3.3.1 Proof of Property 1: Continuity of q and s

In order to prove the continuity of q and s, we assume a discontinuity in the payment function for some buyer type y, and show that we can improve the mechanism. For the seller, any buyer type x where the payment function changes is essentially a lottery between the payments made by the different buyer types in a small neighborhood of x. However, our seller is risk averse and strictly prefers the expectation of the lottery – which is not offered to her when there is a discontinuity in the payment function. Thus, similar to example 1, we can improve the mechanism by replacing the lottery with a constant payment that is close to the lottery's expectation. In other words, if q jumps at y, we replace the mechanism, in a small interval around y, with a constant payment that approaches $\left(q\left(y^-\right)+q\left(y^+\right)\right)/2$ (see Figure 3.1).

Proof. Let $\mu = (q, s)$ be a mechanism, where b is the buyer's utility payoff function, and assume that q has a discontinuity in y, i.e., $q(y^-) < q(y^+)$. By eq. (2.1), $s\left(y^+\right) - s\left(y^-\right) = y\left(q\left(y^+\right) - q\left(y^-\right)\right) > 0$.

For every $\delta > 0$, let $\mu^\delta = (q^\delta, s^\delta)$ be the mechanism obtained from μ by linearly interpolating b in the interval $(y-\delta, y+\delta)$ (see Figure 3.1). Since b^δ, the buyer's utility payoff function, is convex and has derivatives between zero and one, it follows that μ^δ is an IC-IR mechanism. Since q^δ and s^δ are constant on $(y-\delta, y+\delta)$, we may denote those constants by $\overline{q}$ and $\overline{s}$, respectively. We have

$$\begin{aligned}\bar{q} &= \frac{b\left(y+\delta\right)-b\left(y-\delta\right)}{2\delta} = \frac{b\left(y+\delta\right)-b\left(y\right)}{2\delta} + \frac{b\left(y\right)-b\left(y-\delta\right)}{2\delta} \\ &\underset{\delta\to 0^+}{\longrightarrow} \frac{q(y^+)+q(y^-)}{2}\end{aligned}$$

and,

$$\begin{aligned} \bar{s} &= \bar{q}\cdot(y+\delta) - b(y+\delta) \\ &\underset{\delta\to 0^+}{\longrightarrow} \frac{q(y^+)+q(y^-)}{2}\cdot y - b(y) = \frac{s(y^+)+s(y^-)}{2}. \end{aligned} \tag{3.1}$$

Recall that $U(\mu, F) = \mathbb{E}_{F(x)}[u(s(x))]$. It follows that

$$\Delta^\delta := U\left(\mu^\delta, F\right) - U(\mu, F) = \int_{y-\delta}^{y+\delta} (u(\bar{s}) - u(s(t)))\, f(t)\, dt.$$

Let us split the integral at y, and estimate the two parts separately:

$$\begin{aligned} \frac{1}{\delta}\int_{y-\delta}^{y} (u(\bar{s}) - u(s(t)))\, f(t)\, dt &\geq \frac{1}{\delta}\int_{y-\delta}^{y} \left(u(\bar{s}) - u\left(s\left(y^-\right)\right)\right) f(t)\, dt \\ &\underset{\delta\to 0^+}{\longrightarrow} \left(u\left(\frac{s(y^+)+s(y^-)}{2}\right) - u\left(s\left(y^-\right)\right)\right) f(y), \end{aligned}$$

where the inequality holds because $s(t) \leq s(y^-)$ for $t < y$ and u is monotonic and for the limit we use eq. (3.1) and the continuity of $f(x)$. Similarly,

$$\begin{aligned} \frac{1}{\delta}\int_{y}^{y+\delta} (u(\bar{s}) - u(s(t)))\, f(t)\, dt &\geq \frac{}{\delta}\int_{y}^{y+\delta} (u(\bar{s}) - u(s(y+\delta)))\, f(t)\, dt \\ &\underset{\delta\to 0^+}{\longrightarrow} \left(u\left(\frac{s(y^+)+s(y^-)}{2}\right) - u\left(s\left(y^+\right)\right)\right) f(y), \end{aligned}$$

where the limit holds as above and for the inequality we use the monotonicity of s and u. Therefore,

$$\underset{\delta\to 0^+}{liminf}\frac{1}{\delta}\Delta^\delta \geq \left(2u\left(\frac{s(y^+)+s(y^-)}{2}\right) - u\left(s\left(y^+\right)\right) - u\left(s\left(y^-\right)\right)\right) f(y) > 0,$$

which is strictly positive by the strict concavity of u, and so $\Delta^\delta > 0$ for $\delta > 0$ small enough. Hence, μ cannot be an optimal mechanism. Naturally, this means that the optimal q must be continuous, and in turn s must also be continuous. □

Remark 10. Note that for our proof to hold, it is enough that $u''(\hat{s}) < 0$ for some $\hat{s} \in (s(y^-), s(y^+))$. This implies that $2u((s(y^+) + s(y^-)/2)) > u(s(y^+)) + u(s(y^-))$, which in turn implies that $\underset{\delta\to 0^+}{liminf}\,\Delta^\delta/\delta$ is strictly positive and that μ cannot be optimal. Therefore, even if we assume weak concavity, if $\hat{s} \in (s(a), s(c))$ and $u''(\hat{s}) < 0$ then there must be some $\hat{x} \in (a, c)$ such that $s(\hat{x}) = \hat{s}$.

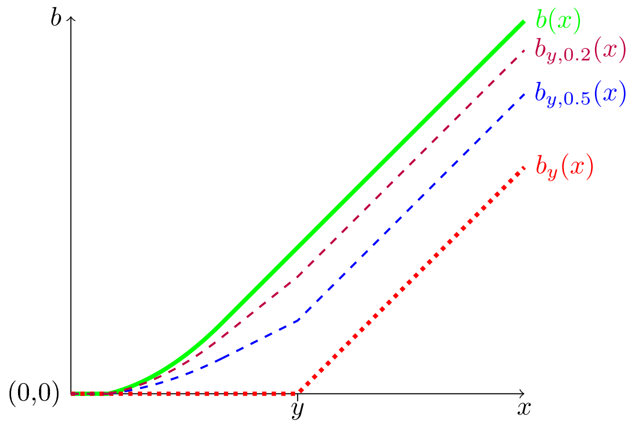

Figure 3.2: Defining $b_{y,\varepsilon}$

3.3.2 Proof of Property 2: There is a constant λ s.t. $x\int_x^c u'(s(t)) f(t)\,dt \leq \lambda$ with equality whenever q is strictly increasing around x

Proof. Let $\lambda = \int_a^c s(t)\,u'(s(t))\,f(t)\,dt$. Then, by Claim 11 below, $x\int_x^c u'(s(t))\,f(t)\,dt \leq \lambda$ and, by Claim 12, there is an equality whenever q is strictly increasing around x. $\square$

Claim 11. $y\int_y^c u'(s(t))\,f(t)\,dt \leq \int_a^c s(t)\,u'(s(t))\,f(t)\,dt$ for any $y \in [a, c]$.

Proof. Let μ be an optimal mechanism with a buyer's utility payoff function b, and let $y \in [a, c]$. We define

$$b_y(x) = (x-y)_+ = \begin{cases} 0 & 0 \leq x < y \\ x-y & y \leq x \leq c \end{cases},$$

$b_{y,\varepsilon} = (1-\varepsilon)\,b + \varepsilon b_y$ (see Figure 3.2). Recall that we restrict ourselves to mechanisms in which $s(x) \geq 0$ and thus it's easy to see that $b_{y,\varepsilon}$ defines an IC-IR mechanism $\mu_{y,\varepsilon}$ with the payment function

$$s_{y,\varepsilon}(x) = \begin{cases} (1-\varepsilon)\,s(x) & a \leq x < y \\ (1-\varepsilon)\,s(x) + \varepsilon y & y \leq x \leq c. \end{cases}$$

The optimality of μ means that its expected utility cannot be lower than that of $\mu_{y,\varepsilon}$, and hence

$$\int_a^c [u(s_{y,\varepsilon}(t)) - u(s(t))]\,f(t)\,dt \leq 0.$$

We next split the integral at y

$$\int_a^y \left[u\left(s_{y,\varepsilon}(t)\right) - u\left(s(t)\right)\right] f(t)\,dt + \int_y^c \left[u\left(s_{y,\varepsilon}(t)\right) - u\left(s(t)\right)\right] f(t)\,dt \le 0$$

and using the definition of $s_{y,\varepsilon}$, we have

$$\int_a^y \left[u\left((1-\varepsilon)\,s(t)\right) - u\left(s(t)\right)\right] f(t)\,dt + \int_y^c \left[u\left((1-\varepsilon)\,s(t) + \varepsilon y\right) - u\left(s(t)\right)\right] f(t)\,dt \le 0. \tag{3.2}$$

Since u is concave, we know that $u(s+h) - u(s) \ge h \cdot u'(s+h)$ for any h, and hence

$$u\left((1-\varepsilon)\,s(t)\right) - u\left(s(t)\right) \ge -\varepsilon s(t) \cdot u'\left((1-\varepsilon)\,s(t)\right)$$
$$u\left((1-\varepsilon)\,s(t) + \varepsilon y\right) - u\left(s(t)\right) \ge \varepsilon\left(y - s(t)\right) \cdot u'\left((1-\varepsilon)\,s(t) + \varepsilon y\right).$$

Applying these two inequalities to Equation (3.2), we get

$$-\varepsilon \int_a^y s(t) \cdot u'\left((1-\varepsilon)\,s(t)\right) f(t)\,dt + \varepsilon \int_y^c \left(y - s(t)\right) \cdot u'\left((1-\varepsilon)\,s(t) + \varepsilon y\right) f(t)\,dt \le 0.$$

Dividing by ε and rearranging, we have

$$y \int_y^c u'\left((1-\varepsilon)\,s(t) + \varepsilon y\right) f(t)\,dt \le \int_a^y s(t) \cdot u'\left((1-\varepsilon)\,s(t)\right) f(t)\,dt + \int_y^c s(t) \cdot u'\left((1-\varepsilon)\,s(t) + \varepsilon y\right) f(t)\,dt.$$

We can then take ε to zero to get

$$y \int_y^c u'\left(s(t)\right) f(t)\,dt \le \int_a^c s(t)\, u'\left(s(t)\right) f(t)\,dt.$$

Note that the right-hand side of the last equation is a constant. □

Claim 12. If $\mu = (q, s)$ is an optimal mechanism and q is strictly increasing around y, then $y \int_y^c u'\left(s(t)\right) f(t)\,dt = \int_a^c s(t)\, u'\left(s(t)\right) f(t)\,dt$.

To prove this claim, we assume that q is an optimal mechanism that is strictly increasing around y. Using the decomposition introduced in Section 3.0.1, we take a small neighborhood I of y, and uniformly distribute the increase in q inside I over $[a, c]$, keeping q constant on I. Comparing the perturbed mechanism to the optimal mechanism proves our claim.

Proof. Let $\mu = (q, s)$ be an optimal mechanism, and let y be such that q is strictly increasing around y. Then for every $\delta > 0$ we have $\rho := q(y+\delta) - q(y-\delta) > 0$. Define a new mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$ by

$$\tilde{q}(x) := \begin{cases} \frac{1}{1-\rho} q\left(x\right) & a \le x \le y - \delta \\ \frac{1}{1-\rho} q(y-\delta) & y - \delta \le x \le y + \delta \\ \frac{1}{1-\rho}\left(q(t) - \rho\right) & y + \delta \le x \le c. \end{cases} \tag{3.3}$$

Using the decomposition introduced in Section 3.0.1, we get

$$\begin{aligned} \tilde{s}(x) &= \int_a^c t \mathbf{1}_{x \ge t} d\tilde{q}(t) = \frac{1}{1-\rho}\left(\int_a^c t\mathbf{1}_{x\ge t} dq(t) - \int_{y-\delta}^{y+\delta} t\mathbf{1}_{x\ge t} dq(t)\right) \\ &= \frac{1}{1-\rho} s(x) - \frac{1}{1-\rho}\int_{y-\delta}^{y+\delta} t\mathbf{1}_{x\ge t} dq(t). \end{aligned}$$

When $x \le y - \delta$, the last integral equals 0, and when $x \ge y - \delta$ it is non-negative and at most $\int_{y-\delta}^{y+\delta} (y+\delta) dq(t) = (y+\delta)\rho$. Therefore,

$$\frac{1}{1-\rho} s(x) \ge \tilde{s}(x) \ge \frac{1}{1-\rho} s(x) - \mathbf{1}_{x \ge y-\delta} \frac{\rho}{1-\rho}(y+\delta)$$

and

$$\frac{\rho}{1-\rho}\left(s(x) - \mathbf{1}_{x\ge y-\delta}(y+\delta)\right) \le \tilde{s}(x) - s(x) \le \frac{\rho}{1-\rho} s(x).$$

Remember that q, being optimal, is continuous[6] (see proof in Section 3.3.1 above). Therefore, as $\delta \to 0$ we have that $\rho \longrightarrow 0$, which implies that $\tilde{s} \to s$ by use of the last equation.

Using now the concavity of u, we have

$$\begin{aligned} u(\tilde{s}(x)) - u\left(s(x)\right) &\ge u'(\tilde{s}(x)) \cdot (\tilde{s}(x) - s(x)). \\ &\ge \frac{\rho}{1-\rho} u'(\tilde{s}(x)) \cdot \left(s(x) - \mathbf{1}_{x \ge y-\delta}(y+\delta)\right). \end{aligned} \tag{3.4}$$

By the optimality of μ, it follows that $U\left(\tilde{\mu}, F\right) \le U\left(\mu, F\right)$, i.e.,

$$0 \ge \int_a^c \left[u\left(\tilde{s}\left(t\right)\right) - u\left(s\left(t\right)\right)\right] f\left(t\right) dt.$$

By Eq. (3.4), we have

[6] We need the continuity of q here, and hence this proof only holds when u is strictly concave.

$$\begin{aligned} 0 &\geq \frac{\rho}{1-\rho}\int_a^c u'\left(\tilde{s}\left(t\right)\right)s\left(t\right)f\left(t\right)dt \\ &\quad -\frac{\rho}{1-\rho}(y+\delta)\int_{y-\delta}^c u'\left(\tilde{s}\left(t\right)\right)f\left(t\right)dt. \end{aligned}$$

Multiplying by $(1-\rho)/\rho$ and letting $\delta \to 0$ yields, using the bounded convergence theorem[7],

$$0 \geq \int_a^c u'\left(s\left(t\right)\right)s\left(t\right)f\left(t\right)dt - y\int_y^c u'\left(s\left(t\right)\right)f\left(t\right)dt,$$

which, together with claim 11, concludes the proof. □

3.3.3 Proof of Property 3: $q(x) = 1$ if and only if $x \geq r$, where r is the minimal RM-price

Let $y = inf\left\{x|q\left(x\right)=1\right\}$, and so Property 3 can be rewritten as $r = y$. The proof proceeds in two steps. We first prove that $r \leq y$ by way of contradiction and then use Property 2 of Theorem 6 to prove that $r \geq y$, thus completing the proof.

But first, recall that by Property 1 of Theorem 6, the optimal mechanism is continuous, and thus $q(y) = 1$ and $y = min\left\{x|q\left(x\right)=1\right\}$.

Claim 13. $r \leq y$.

Proof. First, let μ be an optimal mechanism and assume that $y < r$. We now show by way of contradiction that we can improve the optimal mechanism μ. We lower the price paid by buyers whose type is in $(y-\varepsilon, r)$, and raise the price paid by buyers whose type is in (r, c) (see Figure 3.3). Since r is an RM-price, we have increased the expected revenue. When this change is small enough, the increase in expected revenue also guarantees higher expected utility to the seller, despite the concavity of the seller's utility function.

Next, using the decomposition introduced in Section 3.0.1, we define

$$q^{\varepsilon}\left(x\right) = \begin{cases} q\left(x\right) & a \leq x \leq y-\varepsilon \\ q\left(y-\varepsilon\right) & y-\varepsilon \leq x < r \\ 1 & r \leq x \leq c, \end{cases}$$

$$s^{\varepsilon}\left(x\right) = \int_a^c t1_{x\geq t}dq^{\varepsilon}\left(t\right) = \begin{cases} s\left(x\right) & a \leq x \leq y-\varepsilon \\ s\left(y-\varepsilon\right) & y-\varepsilon \leq x \leq r \\ s\left(y-\varepsilon\right)+r\left(1-q\left(y-\varepsilon\right)\right) & r \leq x \leq c, \end{cases}$$

[7]We use the requirement that u' is bounded. Hence, this result does not necessarily hold for utility functions with an unbounded derivative, such as CRRA utility functions.

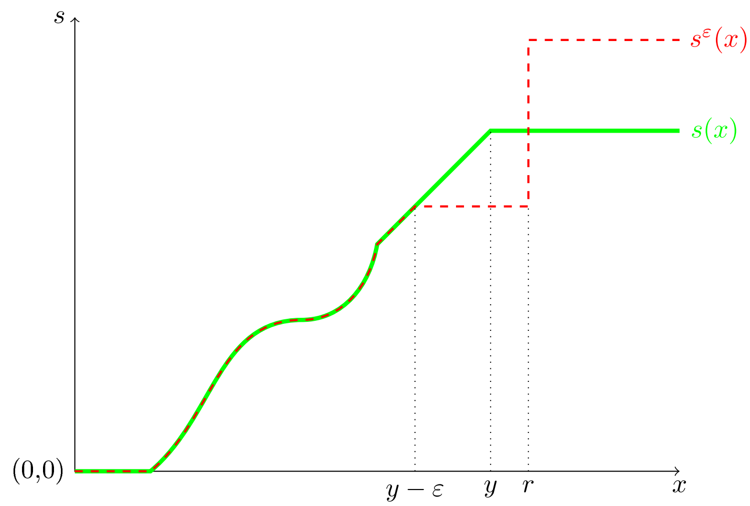

Figure 3.3: Building s^ε from s under the assumption that $y < r$

and so

$$\begin{aligned} U(s^\varepsilon, F) - U(s, F) = & \int_{y-\varepsilon}^{r} [u(s(y-\varepsilon)) - u(s(t))] f(t)\, dt \\ & + \int_{r}^{c} [u(s^\varepsilon(r)) - u(s(y))] f(t)\, dt. \end{aligned}$$

By the concavity of u as well as its monotonicity, we have

$$\begin{aligned} U(s^\varepsilon, F) - U(s, F) \geq & \quad u'(s(y-\varepsilon))[s(y-\varepsilon) - s(y)](F(r) - F(y-\varepsilon)) \\ & + u'(s^\varepsilon(r))[s(y-\varepsilon) + r(1 - q(y-\varepsilon)) - s(y)](1 - F(r)) \\ \geq & \quad u'(s(y-\varepsilon))[s(y-\varepsilon) - s(y)](1 - F(y-\varepsilon)) \\ & + u'(s^\varepsilon(r))\, r(1 - q(y-\varepsilon))(1 - F(r)). \end{aligned}$$

Recall that $s(y) - s(y-\varepsilon) = \int_{y-\varepsilon}^{y} t dq(t) \leq y(1 - q(y-\varepsilon))$. Thus, we can divide the last equation by $(1 - q(y-\varepsilon))$, which is greater than zero by the definition of y, to get

$$\frac{U(s^\varepsilon, F) - U(s, F)}{1 - q(y-\varepsilon)} \geq -u'(s(y-\varepsilon))\, y(1 - F(y-\varepsilon)) + u'(s^\varepsilon(r))\, r(1 - F(r)).$$

As ε goes to zero, both $u'(s(y-\varepsilon))$ and $u'(s^\varepsilon(r))$ go to[8] $u'(s(y))$, and so the right-hand side of the last inequality becomes

$$u'(s(y))[r(1 - F(r)) - y(1 - F(y))].$$

[8]We rely here on the continuity of s. Hence our proof of claim (13) does not hold when u is weakly concave.

Note that $x\left(1-F\left(x\right)\right)$ is the expected revenue one gets when the good is assigned a price of x. Since r is the minimal RM-price, it must be that $r\left(1-F\left(r\right)\right)-y\left(1-F\left(y\right)\right)>0$. Thus, when ε is small enough, $U\left(s^{\varepsilon},F\right)-U\left(s,F\right)>0$, in contradiction to the optimality of s, which, in turn, refutes the assumption that $y<r$. □

Claim 14. $y\leq r$

Proof. First, let $G\left(x\right)=x\int_{x}^{c}u'\left(s\left(t\right)\right)f\left(t\right)dt=x\left(1-F\left(x\right)\right)h\left(x\right)$, where

$$h\left(x\right):=\int_{x}^{c}\frac{f\left(t\right)}{1-F\left(x\right)}u'\left(s\left(t\right)\right)dt=\mathbb{E}_{f\left(x\right)}\left[u'\left(s\left(t\right)\right)|t\geq x\right].$$

By the definition of y, q is strictly increasing around y. Thus, by property 2 of Theorem[9] (6), G is maximal at y and, in particular, $G\left(y\right)\geq G\left(r\right)$. Hence,

$$y\left(1-F\left(y\right)\right)h\left(y\right)\geq r\left(1-F\left(r\right)\right)h\left(r\right).$$

Recall that r, being an RM-price, is a maximum point of $x\left(1-F\left(x\right)\right)$. Hence, $y\left(1-F\left(y\right)\right)\leq r\left(1-F\left(r\right)\right)$, which implies that $h\left(y\right)\geq h\left(r\right)$.

Finally, note that $u^{'}\left(s\right)$ is a non-decreasing function and is strictly decreasing[10] on $[a,y]$. Therefore, h, being the expectation of $u^{'}\left(s\right)$ taken on $[x,c]$, is strictly decreasing on $[a,y]$ and non-increasing on $[y,z]$. Consequently, $h\left(y\right)\geq h\left(r\right)$ implies that $y\leq r$. □

4 The Optimal Mechanism under some Regularity Condition

Theorem 6 gives us a characterization of the optimal mechanism. This characterization, however, does not perfectly identify the optimal mechanism, as there may be mechanisms that satisfy this property and are not optimal[11]. In this chapter we show that under a certain regularity condition, our characterization enables us to find the optimal mechanism by optimizing one parameter only. Moreover, if the smallest buyer type has no value for the good, i.e., $a=0$, and this regularity condition holds, there is only one mechanism that satisfies Theorem 6. Since an optimal mechanism always exists, this mechanism must be the optimal mechanism.

4.1 Refining the characterization by assuming some regularity condition

Let us look at Property 2 of Theorem 6:

$$x\int_{x}^{c}u'\left(s\left(t\right)\right)f\left(t\right)dt\leq\lambda.$$

[9]Section 3.3.2 above, where we prove property 2, relies on the continuity of s as well as on the unboundedness of $u^{'}$. Hence, our proof that $y\leq r$ only holds when u is strictly concave and $u^{'}$ is bounded.

[10]By the definition of y, if $x<y$ then $q\left(x\right)<q\left(y\right)$, and so $s\left(x\right)<s\left(y\right)$ which means that $u^{'}\left(s\left(x\right)\right)>u^{'}\left(s\left(y\right)\right)$.

[11]Since Theorem (6) is only a necessary condition for a mechanism to be optimal.

Note that since u' is strictly decreasing, the higher $s(t)$ is, the lower $u'(s(t))$ is. This suggests that in the optimal mechanism we want λ to be as low as possible. Property 2, however, only holds on the integral level. The following corollary will enable us to make point wise arguments by switching from an integral equation to a differential equation.

Corollary 15. *If s is strictly increasing around y, then $y^2 u'(s(y)) f(y) = \lambda$, where λ is the constant from Property 2 of Theorem 6.*

Proof. First, let us recall that $G(x) := x \int_x^c u'(s(t)) f(t)\, dt$ and is differentiable with

$$G'(x) = \frac{G(x)}{x} - xu'(s(x)) f(x). \tag{4.1}$$

If s is strictly increasing around y, then IC requires that q be strictly increasing around y. By Theorem 6, both $G(y) = \lambda$, and λ is the maximum of G, which implies that $G'(y) = 0$. Plugging both $G(y) = \lambda$ and $G'(y) = 0$ into eq. (4.1), we get

$$\lambda = u'(s(y)) y^2 f(y). \tag{4.2}$$

□

Remark 16. Note that eq. (4.2) is the Euler–Lagrange equation. Indeed, our goal is to maximize the functional $U(s(t)) = \int_a^c u(s(t)) f(t)\, dt$. Thus, using the buyer's utility payoff function b, we can denote $L(t, b(t), b'(t)) = u(t \cdot b'(t) - b(t)) f(t)$. The Euler–Lagrange equation states that[12] $L_2' = {}^{d}\!/_{dt} L_3'$ or, in our case, $2w(t) + tw'(t) = 0$, where $w(t) = u'(s(t)) f(t)$. The solution to this differential equation is $w(t) = \lambda / t^2$, which gives us $\lambda = u'(s(t)) t^2 f(t)$. Note, however, that we maximize over convex functions only, which is probably why in our case the Euler–Lagrange equation holds only when q is strictly increasing around y.

Note that when $y^2 u'(s(y)) f(y) = \lambda$, or $u'(s(y)) = \lambda / y^2 f(y)$, it follows that the lower λ is, the higher $s(y)$ is.[13] Unfortunately, this equation holds only when s is strictly increasing around y. Thus, in general, higher λ might give higher expected utility to the seller, by shifting the places where s changes. This is why we need to introduce our regularity condition, and for the rest of this section we will assume that $x^2 f(x)$ is an increasing function.

Next, recall that in our setup, u is a strictly concave function, and so we can define $v(x) = u'^{-1}(x)$ to be the inverse function of the derivative of u. Define $z = inf\{x | s(x) > s(a)\}$, and by continuity $s(z) = s(a)$. We are now ready to prove that in order to find the optimal mechanism, we only need to optimize two parameters: $s(a)$ and z.

[12] We let L_n' denote the derivative of L with respect to its n-th argument.
[13] Since $u'(s)$ is a non-increasing function.

Corollary 17. *In our setup, if $g(x) \coloneqq x^2 f(x)$ is an increasing function, then the optimal mechanism has the following structure:*

$$s(x) = \begin{cases} s(a) & a \le x \le z \\ \upsilon\left(\frac{u'(s(a))g(z)}{g(x)}\right) & z \le x \le r \\ \upsilon\left(\frac{u'(s(a))g(z)}{g(r)}\right) & r \le x \le c, \end{cases}$$

for some z and $s(a)$.

Remark. Clearly, if $x \in [a, z]$ then $s(x) = s(a)$ (by the definition of z) and $x \in [r, c]$ implies that $s(x) = s(r)$ (Property 3 of Theorem 6). Thus, we only need to prove what happens when $x \in [z, r]$.

Proof. We start by proving that if g is a strictly increasing function, then on $[z, r]$ s is strictly increasing and, by Corollary 15, $u'(s(x))g(x) = \lambda$.

First, assume that $s(\hat{x}_1) = s(\hat{x}_2)$, $\hat{x}_1 < \hat{x}_2$ and $(\hat{x}_1, \hat{x}_2) \subset [z, r]$ and let $x_1 = inf\{x | s(x) = s(\hat{x}_1)\}$ and $x_2 = sup\{x | s(x) = s(\hat{x}_1)\}$. Since s is continuous, it follows that $s(x_1) = s(x_2)$ with s strictly increasing around x_1 and around x_2. By Corollary 15, $u'(s(x_1))g(x_1) = \lambda = u'(s(x_2))g(x_2)$. This equality, together with[14] $g(x_1) < g(x_2)$, gives us that $u'(s(x_1)) > u'(s(x_2))$, and hence $s(x_1) < s(x_2)$, a contradiction.

Thus, on $[z, r]$, s is a strictly increasing function, and, by Corollary 15, $u'(s(x))g(x) = \lambda$,or alternatively, $s(x) = \upsilon(\lambda/g(x))$. Evaluating λ at z implies that $\lambda = u'(s(a))g(z)$, which implies that $s(x) = \upsilon(u'(s(a))g(z)/g(x))$. Thus, the optimal mechanism must have the following form:

$$s(x) = \begin{cases} s(a) & x \le z \\ \upsilon\left(\frac{u'(s(a))g(z)}{g(x)}\right) & z \le x \le r \\ \upsilon\left(\frac{u'(s(a))g(z)}{g(r)}\right) & r \le x. \end{cases}$$

□

Thus, if $x^2 f(x)$ is an increasing function, finding the optimal mechanism reduces to optimizing over two variables: $s(a)$ and z. We can, however, do better than that. Theorem 6 gives us another equation: $q(r) = 1$. Combining this equation with $s'(x) = xq'(x)$ (which holds for the optimal mechanism

[15]), we have

$$1 = q(r) = q(a) + \int_z^r q'(t)\,dt = \frac{s(a)}{a} + \int_z^r \frac{s'(t)}{t}dt.$$

[14] Since $x_1 < x_2$ and g is strictly increasing, it follows that $g(x_1) < g(x_2)$.
[15] It can easily be derived from eq. (2.1).

Next, $s(x) = \upsilon(u'(s(a))\,g(z)/g(x))$ on $[z,r]$ and hence[16]

$$s'(x) = -\upsilon'\left(\frac{u'(s(a))\,g(z)}{g(x)}\right)\frac{u'(s(a))\,g(z)\,g'(x)}{g(x)^2}$$

Combining the last two equations, we get

$$1 = \frac{s(a)}{a} - u'(s(a))\,g(z)\int_z^r \upsilon'\left(\frac{u'(s(a))\,g(z)}{g(t)}\right)\frac{g'(t)}{tg(t)^2}dt,$$

which gives us a connection between $s(a)$ and z, thus decreasing the level of freedom of our optimization problem.

We can do still better, however. It turns out that fixing one parameter determines the optimal mechanism completely, as our next theorem demonstrates.

Theorem 18. *In our setup, we assume that* $g(x) = x^2 f(x)$ *is an increasing function, and denote* $m(\sigma) = min\left\{z|\sigma/a + \int_z^r \frac{s'_\sigma(t)}{t}dt \le 1\right\}$ *where* $s_\sigma(t) = \upsilon(u'(\sigma)\,g(z)/g(t))$. *In this case, the optimal mechanism has the following structure:*

$$s(x) = \begin{cases} \sigma & a \le x \le m(\sigma) \\ \upsilon\left(\frac{u'(\sigma)g(m(\sigma))}{g(x)}\right) & m(\sigma) \le x \le r \\ \upsilon\left(\frac{u'(\sigma)g(m(\sigma))}{g(r)}\right) & r \le x \le c, \end{cases}$$

where $\sigma \in [0,a]$.

Proof. We already know that if $z = inf\{x|s(x) > s(a)\}$, and $x \in [z,r]$, then $s(x) = \upsilon(u'(\sigma)\,g(z)/g(x))$. We also know that the lower z is, the lower $g(z)$ is, and sothe higher $s(x)$ is (since υ is a decreasing function). Consequently, the optimal mechanism will have the smallest possible z.[17] □

Remark 19. Similarly, we can show that if g is an increasing function, and s is the optimal mechanism, then given z, $s(a) = max\left\{\sigma|\sigma/a + \int_z^r (s'(t)/t)\,dt \le 1\right\}$, where $s(x) = \upsilon(u'(\sigma)\,g(z)/g(x))$.

Thus, given $s(a)$, the optimal mechanism is completely determined. It can be seen that we reduced the optimization problem to an optimization over one parameter only. Next, in the special case where $a = 0$, we don't need to optimize at all.

[16] υ is a decreasing function, hence $s'(x) > 0$, as expected.
[17] A smaller z would result in $q(c) > 1$, which implies that μ is not a viable mechanism.

Corollary 20. *In our setup, if $g(x) := x^2 f(x)$ is an increasing function and $a = 0$, then the optimal mechanism is*

$$s(x) = \begin{cases} 0 & a \le x \le m \\ \upsilon\left(\frac{u'(0)g(m)}{g(x)}\right) & m \le x \le r \\ \upsilon\left(\frac{u'(0)g(m)}{g(r)}\right) & r \le x \le c, \end{cases}$$

where $m = min\left\{z | \int_z^r (s'(t)/t)\,dt \le 1\right\}$ and $s(x) = \upsilon(u'(0)g(z)/g(x))$ on $[z, r]$.

Proof. Plug $a = 0$ into Theorem 18. □

4.2 Finding the optimal mechanism: An example using CARA functions

In this example we will demonstrate how, in certain cases, we can use our characterization to calculate the optimal mechanism. We assume that the seller has a constant absolute risk aversion (CARA) utility function, i.e., $u(s) = 1 - e^{-\alpha s}$, and that the buyer types are distributed uniformly over $[0, 1]$. We then solve for the optimal mechanism, and analyze it using comparative statics.

We start by noting that since F is uniform over $[0, 1]$, it follows that $f(x) = 1$, $g(x) = x^2$ (which is an increasing function), and that the minimal RM-price is $r = 1/2$. We also note that $u'(s) = \alpha e^{-\alpha s}$, and hence[18] $\upsilon(y) = \alpha^{-1} \cdot ln(\alpha/y)$ and $\upsilon'(y) = -1/\alpha y$. Lastly, $u'(0) = \alpha$, and hence

$$\upsilon\left(\frac{u'(0)g(m)}{g(x)}\right) = \upsilon\left(\frac{\alpha m^2}{x^2}\right) = \alpha^{-1} ln\left(\frac{x^2}{m^2}\right).$$

Plugging the last equation into Corollary 20, we get that the optimal mechanism is

$$s(x) = \begin{cases} 0 & x \le m \\ \frac{2}{\alpha} ln\left(\frac{x}{m}\right) & m \le x \le r \\ \frac{2}{\alpha} ln\left(\frac{r}{m}\right) & r \le x, \end{cases}$$

where $m = min\left\{z | \int_z^r \left[2/\alpha t^2\right] dt \le 1\right\}$[19]

Next, solving $\int_z^r [s'(t)/t]\,dt = 1$ with $s'(x) = 2/\alpha x$ gives us

$$1 = \int_z^r \frac{2}{\alpha t^2} dt = -\frac{2}{\alpha} \frac{1}{t}\Big|_z^r = \frac{2}{\alpha}\left(\frac{1}{z} - \frac{1}{r}\right),$$

[18] Recall that $\upsilon = u'^{-1}$.
[19] On the interval $[z, r]$, $s(t) = 2\alpha^{-1} ln(t/z)$ and so $s'(t) = 2/\alpha t$.

Expected utility	α=1	α=5	α=10	α=50	α=100
Optimal mechanism	0.2	0.56	0.71	0.93	0.96
RM mechanism	0.197	0.46	0.5	0.5	0.5
RM/optimal ratio	0.985	0.82	0.7	0.54	0.52

Table 4.1: Comparison of the optimal mechanism to the RM-mechanism

which implies $z = 2r/(\alpha r + 2)$, i.e., $z = 2/(\alpha + 4)$ (since $r = 1/2$). Hence, the optimal mechanism is

$$s(x) = \begin{cases} 0 & 0 \leq x \leq \frac{2}{\alpha+4} \\ \frac{2}{\alpha} ln\left(\frac{\alpha+4}{2}x\right) & \frac{2}{\alpha+4} \leq x \leq \frac{1}{2} \\ \frac{2}{\alpha} ln\left(\frac{\alpha+4}{4}\right) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

4.2.1 Comparative statics

Let us now compare the seller's expected utility from the optimal mechanism vs. the RM-mechanism in the above example, i.e., the seller has a CARA utility function and the buyer type x is distributed uniformly over $[0,1]$. Let us also compare the two mechanisms when the risk aversion coefficient α changes.

We first compute the seller's expected utility from the optimal mechanism: $U(s) = \int_a^c u(s(t))\, dt$. Let

$$u(s(x)) = 1 - e^{-\alpha s(x)} = \begin{cases} 0 & 0 \leq x \leq 2/(\alpha+4) \\ 1 - \frac{4}{(\alpha+4)^2 x^2} & 2/(\alpha+4) \leq x \leq 1/2 \\ 1 - \frac{16}{(\alpha+4)^2} & 1/2 \leq x \leq 1 \end{cases}$$

and

$$U(s) = \int_{2/(\alpha+4)}^{1/2} \left[1 - \frac{4}{(\alpha+4)^2 x^2}\right] dx + \frac{1}{2}\left(1 - \frac{16}{(\alpha+4)^2}\right) =$$

$$= \frac{1}{2} + \frac{8}{(\alpha+4)^2} - \frac{2}{\alpha+4} - \frac{2}{\alpha+4} + \frac{1}{2} - \frac{8}{(\alpha+4)^2} = 1 - \frac{4}{\alpha+4}.$$

As for the RM-mechanism, i.e., if the seller sells the good for a price of $1/2$, the seller's expected utility is $\frac{1}{2}u(1/2) = \left(1 - e^{-\alpha/2}\right)/2$.

Table (4.1) evaluates the optimal mechanism for different values of α. Note that the larger α is, the more risk-averse the seller is. As expected, the optimal mechanism can yield the seller significantly more expected utility than the RM-mechanism can. It also appears that the gap between the two mechanisms becomes more significant as the risk aversion of the seller increases. On the other hand, in our example, when we compare the OPUM mechanisms[20] to the optimal mechanisms, the loss of expected utility appears to be insignificant (Table 4.2). Furthermore, the

[20] Recall that an OPUM mechanism is a one-price utility-maximizing mechanism

Expected utility	α=1	α=5	α=10	α=50	α=100
Optimal mechanism	0.2	0.56	0.71	0.93	0.96
OPUM mechanism	0.199	0.54	0.69	0.9	0.94
OPUM/optimal ratio	0.995	0.964	0.972	0.968	0.98

Table 4.2: Comparison of the optimal mechanism to the one-price utility maximizing (OPUM) mechanism

OPUM[21] to optimal ratio doesn't necessarily decrease as α grows large. Section 5 contains a detailed analysis of simple mechanisms and specifically the OPUM mechanisms.

5 Simple Mechanisms

We call a mechanism with a single price a *simple mechanism.* For example, there is always an RM-mechanism that is a simple mechanism. The OPUM mechanism, which guarantees the seller the highest expected utility achievable by simple mechanisms, is another example of a simple mechanism.

As seen in Table 4.2, in terms of the expected utility for the seller, the OPUM mechanism may sometimes be almost as good as the optimal mechanism. In this section we address this point. In particular, we show that the OPUM mechanism can guarantee at least a constant fraction of the expected utility gained by the optimal mechanism. Moreover, we show that this constant depends solely on the probability distribution of the buyers. Our analysis is far from tight, however, and the question remains open: how well does the OPUM mechanism fare compared to the optimal mechanism, in terms of expected utility to the seller?

Before we proceed, let us note that in this section s is an optimal mechanism, and ρ is the price of the OPUM mechanism, i.e., $u(\rho)(1-F(\rho)) \geq u(s)(1-F(s))$ for any other price s. We now define $OPT = \int_a^c u(s(t)) f(t)\, dt$ to be the expected utility from the optimal mechanism, and $OPUM = (1-F(\rho))\, u(\rho)$ to be the expected utility from the OPUM mechanism.

Let us also note that in this section we assume that the OPUM mechanism exists, which is not necessarily true outside our specified domain. Take for example the case where $u(s) = ln(s)$ and $x \sim U[0,1]$, which lies outside our domain. In this case, the expected utility from setting a price of p is $p \cdot ln(0) + (1-p) \cdot ln(p)$, which, of course, has no maximal point.

We can now prove the following proposition.

Proposition 21. *In our setup, the OPUM mechanism attains at least* $(1-F(r))$ *of the optimal expected utility, i.e.,* $OPUM/OPT \geq 1-F(r)$.

[21] Recall that an OPUM mechanism is a one-price utility-maximizing mechanism

Proof. Let s be an optimal mechanism. Since s is non-decreasing and $s(r) = s(c)$ (as $q(r) = 1$), it follows that $s(t) \leq s(r)$ for all t. Since s is also IR, it follows that $s(r) \leq r$ and

$$OPT = \int_a^c u(s(t)) f(t)\, dt \leq u(s(r)) \leq u(r).$$

The simple mechanism with price r yields the seller an expected utility of $(1 - F(r))\, u(r)$, and hence

$$OPUM \geq (1 - F(r))\, u(r).$$

Combining the last two inequalities gives us $OPUM \geq (1 - F(r))\, u(r) \geq (1 - F(r))\, OPT$ or

$$\frac{OPUM}{OPT} \geq 1 - F(r).$$

□

Remark 22. This proof holds in a much broader domain than our setup, and so the OPUM mechanism can guarantee at least $1 - F(r)$ of the optimal expected utility even when f and u don't meet our requirements.

Note that when our assumptions are violated, there may not be an optimal mechanism, and even if one exists, it may be hard to calculate. Thus, depending on our needs and on $F(r)$, and in light of the last remark, the simple OPUM mechanism may sometimes provide a good enough approximation to the optimal mechanism.

Note that the OPUM mechanism can yield much higher expected utility than is suggested by the bound we attained on the OPUM to optimal ratio, namely $1 - F(r)$. See for example Table 5.1, where this bound is equal to 1/2 while the OPUM mechanism yields up to 99.5% of the optimal expected utility. Indeed, when $a = 0$, i.e., when $x \sim F[0, c]$, we can improve our bound, even if it still seems far from tight. The following two propositions give us two more bounds.

Proposition 23. *In our setup, when* $x \sim [0, c]$*, the OPUM/optimal ratio is at least*

$$(1 - F(r)) / \left(1 - F\left(r\sqrt{u'(r) / u'(0)}\right)\right).$$

Proof. Let $x \sim F[0, c]$ and s be the optimal mechanism. Let

$$M(x) = \begin{cases} 0 & x < s(r) \\ s(r) & x \geq s(r). \end{cases}$$

Since $M(x)$ is a simple mechanism, it follows that

$$OPUM \geq \mathbb{E}\left[u\left(M\left(x\right)\right)\right] = \left(1 - F\left(s\left(r\right)\right)\right) u\left(s\left(r\right)\right).$$

Now, letting $z = inf\left\{x | s\left(x\right) > 0\right\}$, and noting that $s\left(x\right) = 0$ on $[a, z]$, we get

$$OPT = \int_z^c u\left(s\left(t\right)\right) f\left(t\right) dt \leq \left(1 - F\left(z\right)\right) u\left(s\left(r\right)\right).$$

Thus, we have

$$\frac{OPUM}{OPT} \geq \frac{1 - F\left(s\left(r\right)\right)}{1 - F\left(z\right)} \geq \frac{1 - F\left(r\right)}{1 - F\left(z\right)}. \tag{5.1}$$

Lastly, by corollary 15, $\lambda = z^2 u'\left(0\right) = r^2 u'\left(s\left(r\right)\right) \Rightarrow z^2 = r^2 u'\left(s\left(r\right)\right) / u'\left(0\right) \geq r^2 u'\left(r\right) / u'\left(0\right)$ and $z \geq r\sqrt{u'\left(r\right) / u'\left(0\right)}$. Hence,

$$\frac{OPUM}{OPT} \geq \frac{1 - F\left(r\right)}{1 - F\left(r\sqrt{\frac{u'(r)}{u'(0)}}\right)}.$$

□

Our next bound is even more specific, and only holds when $x \sim U\left[0, c\right]$, i.e., $F\left(x\right) = x/c$. Also, even when it holds, it might be worse than our second bound, as is seen in Table 5.1.

Proposition 24. *In our setup, when* $x \sim U\left[0, c\right]$*, the OPUM mechanism attains at least* $\left(c - r\right) / \left(c - \left(c - r\right) u'\left(r\right) / u'\left(0\right)\right)$ *of the optimal expected utility.*

Proof. Substituting $F\left(x\right) = x/c$ in eq. (5.1), we have

$$\frac{OPUM}{OPT} \geq \frac{1 - F\left(r\right)}{1 - F\left(z\right)} = \frac{c - r}{c - z}$$

Now, our main theorem implies that

$$z \int_z^c u'\left(s\left(t\right)\right) f\left(t\right) dt = \lambda \geq r \int_r^c u'\left(s\left(t\right)\right) f\left(t\right) dt$$

and, since $s\left(x\right) = s\left(r\right)$ on $[r, c]$, we have

$$z \int_z^r u'\left(s\left(t\right)\right) f\left(t\right) dt + z u'\left(s\left(r\right)\right) \left(1 - F\left(r\right)\right) \geq r u'\left(s\left(r\right)\right) \left(1 - F\left(r\right)\right)$$

and, using the concavity of $u\left(s\right)$ and rearranging, it becomes

$$z u'\left(s\left(z\right)\right) \left(F\left(r\right) - F\left(z\right)\right) \geq \left(r - z\right) u'\left(s\left(r\right)\right) \left(1 - F\left(r\right)\right).$$

Expected utility	α=1	α=5	α=10	α=50	α=100
OPUM/optimal ratio	0.995	0.964	0.972	0.968	0.98
First bound	0.5	0.5	0.5	0.5	0.5
Second bound	0.819	0.584	0.521	0.5	0.5
Third bound	0.718	0.521	0.502	0.5	0.5

Table 5.1: Comparing the three bounds to the actual OPT/OPUM ratio when $x \sim U[0,1]$ and $u(s) = 1 - e^{-\alpha s}$

Next, using $F(x) = x/c$, we get

$$zu'(s(z))\frac{r-z}{c} \geq (r-z)\,u'(s(r))\frac{c-r}{c},$$

which can be rewritten as

$$z \geq \frac{u'(s(r))}{u'(s(z))}(c-r).$$

Lastly, $s(z) \geq 0$ and $s(r) \leq r$, hence $z \geq (c-r)\,u'(r)/u'(0)$, and we have our bound, namely,

$$\frac{OPUM}{OPT} \geq \frac{c-r}{c-(c-r)\frac{u'(r)}{u'(0)}}.$$

□

Note that when $u'(0) \gg u'(r)$, the two latter bounds converge to our first bound, namely, $1 - F(r)$.

Table 5.1 shows the three bounds we derived for the OPUM/optimal ratio. The bounds are calculated for the example given in the previous section, where $x \sim U[0,1]$ and $u(s) = 1 - e^{-\alpha s}$, and hence $r = 0.5$ and $u'(s) = \alpha e^{-\alpha s}$. Clearly, in that example the real ratio is much higher than the bounds suggest.

6 Extending Our Setup

6.1 Introducing risk aversion on the buyer's side

So far we have assumed risk aversion on the seller's side, and the buyer was assumed to be risk neutral. In this section we assume that the buyer is risk averse. Thus, the buyer's utility function is $V(x, \omega, s) = x\omega + v(-s)$, where x is the buyer type (the buyer's valuation of the good), ω is a binary variable that has a value of 1 if the buyer gained the good and 0 if he didn't, s is the payment from the buyer to the seller (in dollars), and v is the buyer's utility function for money.[22] It turns out that our results are easily generalized to the case of a risk-averse buyer.

Building on the scenario where the buyer is risk averse (original scenario), we substitute the utility units (utils) of the buyer for money (transformed scenario). Thus, the payment function determines how many utils each buyer type loses, namely, $\beta = -v(-s)$. The buyer is linear in his utils, and we show that the seller's utility function is strictly concave in β. Consequently, all our results, and in particular our characterization of the optimal mechanism,

[22] As defined in section 2, v is strictly concave in s, twice differential, strictly increasing, has a finite derivative, and w.l.o.g. we let $v(0) = 0$.

hold in the transformed scenario, and can be used to characterize the optimal mechanism in the original scenario of risk-averse buyer.

Let B denote the scenario where the buyer's utility is $V(x,\omega,s) = x\omega + v(-s)$ and the seller's utility is u. Both u and v are utility functions for money as in Section 2, except that now we allow u to be weakly concave or even linear. We define $\tilde{B}$ as the scenario derived from B by assigning $\beta = -v(-s)$. Thus, we substitute $s = -v^{-1}(\beta)$ and the utilities in $\tilde{B}$ are $\tilde{V}(x,\omega,\beta) = x\omega - \beta$ and $\tilde{u}(\beta) = u\left(-v^{-1}(-\beta)\right)$. Note that $\tilde{B}$ is in fact our basic setup, as the buyer is risk neutral and $\tilde{u}$ is strictly concave in β, twice differential, strictly increasing, has a finite derivative, and $\tilde{u}(0) = 0$ (see claim 29 below). Hence, there is an optimal mechanism $\tilde{\mu} = (\tilde{q}, \tilde{s})$ that satisfies Theorem 6 and we can use it to characterize the optimal mechanism under scenario B.

We start by proving that an optimal mechanism exists under scenario B.

Proposition 25. *A unique optimal IC-IR direct mechanism always exists under scenario B.*

Proof. Let $\tilde{\mu} = (\tilde{q}, \tilde{s})$ be the optimal mechanism that satisfies Theorem 6 and let $\hat{q} = \tilde{q}$ and $\hat{s} = -v^{-1}(-\tilde{s})$. Clearly $\hat{\mu} = (\hat{q}, \hat{s})$ is an IC-IR direct mechanism.[23] In order to see that $\hat{s}$ is also optimal in B, let s be any IC-IR direct mechanism. Then $\overline{s} = -v(-s)$ is an IC-IR direct mechanism in $\tilde{B}$ (again, see claim 30), and we have

$$\begin{aligned}\int_a^c u(s(t))\, f(t)\, dt &= \int_a^c u\left(-v^{-1}(-\overline{s}(t))\right) f(t)\, dt = \int_a^c \tilde{u}(\overline{s}(t))\, f(t)\, dt \\ &\leq \int_a^c \tilde{u}(\tilde{s}(t))\, f(t)\, dt = \int_a^c u(\hat{s}(t))\, f(t)\, dt,\end{aligned}$$

where the inequality holds due to the optimality of $\tilde{s}$, and the three equalities follow from the definitions of $\overline{s}$, $\tilde{u}$, and $\hat{s}$, respectively. Hence, $\hat{s}$ is an optimal mechanism in B.

Finally, to see that the optimal mechanism under the original scenario is unique, recall the uniqueness proof from Section 7.2. That proof did not make assumptions about the buyer preferences towards risk, and for this reason holds when the buyer is risk-averse. To summarize, $\hat{\mu}$ is a unique, optimal, IC-IR, direct mechanism that always exists under scenario B. □

Remark 26. Note that in fact we proved the stronger claim that the unique optimal IC-IR direct mechanism under scenario B is $\hat{\mu} = (\hat{q}, \hat{s})$, as defined above.

We can now prove a generalized version of our main theorem.

Theorem 27. *Under scenario B there always exists a unique optimal IC-IR direct mechanism $\hat{\mu} = (\hat{q}, \hat{s})$. This optimal mechanism satisfies the following conditions:*

1. *The functions $\hat{q}$ and $\hat{s}$ are continuous.*

[23] See claim 30 below for a formal proof, noting that $\tilde{\mu} = (\tilde{q}, \tilde{s}) = (\tilde{q}, -v(-\hat{s}))$ is an IC-IR direct mechanism in $\tilde{B}$.

2. *There is a constant* $\lambda \geq 0$ *s.t.*

$$x \int_x^c \frac{u'(\hat{s}(t))}{v'(v(-\hat{s}(t)))} f(t)\, dt \leq \lambda$$

for every x*, with equality when* $\hat{q}$ *is strictly increasing around* x*.*

3. $\hat{q}(x) = 1$ *if and only if* $x \geq r$*, where* r *is the minimal RM-price under scenario* $\tilde{B}$*.*

Proof. Let $\tilde{s}$ be the optimal mechanism that satisfies Theorem 6 under scenario $\tilde{B}$ and define $\hat{\mu} = (\hat{q}, \hat{s}) = \left(\tilde{q}, -v^{-1}(-\tilde{s})\right)$. Then $\hat{\mu}$ is the unique optimal mechanism under scenario B (see Remark 26) and it's easy to see that the functions $\hat{s}$ and $\hat{q}$ satisfy Properties 1 and 3 of Theorem 27. As for Property 2, note that $\tilde{u}(\beta) = u\left(-v^{-1}(-\beta)\right)$, hence $\tilde{u}'(\beta) = u'\left(-v^{-1}(-\beta)\right)/v'(-\beta)$, and specifically $\tilde{u}'(\tilde{s}) = u'\left(-v^{-1}(-\tilde{s})\right)/v'(-\tilde{s})$, which, when substituting for $\hat{s}$, becomes $\tilde{u}'(\tilde{s}) = u'(\hat{s})/v'(v(-\hat{s}))$. Since $\tilde{s}$ satisfies Theorem 6, there is a λ such that $x \int_x^c \tilde{u}'(\tilde{s}(t)) f(t)\, dt \leq \lambda$ with equality when $\hat{q}$ is strictly increasing around x. We can now substitute for $\tilde{u}'(\tilde{s})$ and get

$$x \int_x^c \frac{u'(\hat{s}(t))}{v'(v(-\hat{s}(t)))} f(t)\, dt \leq \lambda$$

with equality when $\hat{q}$ is strictly increasing around x. □

Remark 28. Note that if we assume that the seller has a strictly concave utility function and that the buyer is risk-neutral, i.e., if v is linear, then Theorem 27 becomes Theorem 6.

Claim 29. Let the buyer's utility function be $V(x, \omega, s) = x\omega + v(-s)$, where v is strictly concave in s, twice differential, strictly increasing, has a finite derivative, and let $v(0) = 0$. Let the seller's utility function u be concave,[24] twice differential, strictly increasing, have a finite derivative, and let $u(0) = 0$. Then $\tilde{u}(\beta) = u\left(-v^{-1}(-\beta)\right)$ is strictly concave in β, twice differential, strictly increasing, has a finite derivative, and $\tilde{u}(0) = 0$.

Proof. Denote the buyer by $V(x, \omega, s) = x\omega + v(-s)$, as above. Let $\beta = -v(-s)$; i.e., the payment the buyer has to make "costs" him β utility units. Define now $\tilde{u}(\beta) = u\left(-v^{-1}(-\beta)\right) = u(s)$. Since v is a strictly concave function, v^{-1} is a strictly convex function, $v^{-1}(-\beta)$ is strictly convex, and $-v^{-1}(-\beta)$ is a strictly concave function. Finally, since u is concave, $u\left(-v^{-1}(-\beta)\right)$ is a strictly concave function. Similarly we can show that $\tilde{u}$ is also strictly increasing, twice differentiable, and has a finite derivative. We also have that $\tilde{u}(0) = u\left(-v^{-1}(0)\right) = u(0) = 0$. □

Claim 30. The mechanism $\mu = (q, s)$ is an IC-IR direct mechanism under scenario B if and only if $\overline{\mu} = (q, -v(-s))$ is an IC-IR mechanism under scenario $\tilde{B}$.

Proof. First, μ and $\overline{\mu}$ are both direct mechanisms. Second, by the definition of $\overline{\mu}$, a buyer who reports y, has the same utility payoff whether we are using μunder scenario B or using $\overline{\mu}$ under scenario $\tilde{B}$. Consequently, both μ is IR if and only if $\overline{\mu}$ is IR, and μ is IC if and only if $\overline{\mu}$ is IC. □

[24] We allow u to be weakly concave or even linear.

6.2 Weakly concave utility functions

If we drop the assumption that u is strictly concave, and assume instead that u is weakly concave, then the optimal mechanism may not be unique, and our main theorem is transformed as follows.

Theorem 31. *In our setup, modified by permitting u to be a weakly concave utility function, an optimal IC-IR mechanism s must satisfy the following conditions:*

1. Let $\hat{s} \in (s(a), s(c))$. If u' is strictly decreasing around $\hat{s}$, then there exists an $\hat{x} \in (a, c)$ s.t. $s(\hat{x}) = \hat{s}$.

2. There is a constant $\lambda \geq 0$ s.t. $x \int_x^c u'(s(t)) f(t)\, dt \leq \lambda$ for every x.

3. If $x < r$ then $q(x) < 1$.

The proofs are basically the same as in the strictly concave case, and so they are omitted here. For the difference between the three properties here and in our main theorem, see Remark 10, footnote 6, and footnote 8, respectively.

6.3 Unbounded u' (or $u'(x) \underset{x \to 0}{\longrightarrow} \infty$)

If $u^{'}$ is unbounded, our main theorem is as follows.

Proposition 32. *In our setup, modified by permitting u' to be unbounded, the optimal IC-IR mechanism $\mu = (q, s)$ satisfies the following conditions:*

1. The functions q, s are continuous.

2. There is a constant $\lambda \geq 0$ s.t. $x \int_x^c u'(s(t)) f(t)\, dt \leq \lambda$ for every x.

3. If $x < r$ then $q(x) < 1$.

The proof of Proposition 32 follows the proof of Theorem 6, and is therefore omitted here. Note, however, that similar to the weakly concave case, the proof of Claim 12 is no longer valid (see footnote 7). This is why Properties 2 and 3 of Proposition 32 are the same as in Theorem 31 and differ from Properties 2 and 3 of Theorem 6.

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7 Appendix

7.1 Proof of existence of the optimal mechanism

Proposition 33. *In our setup an optimal mechanism always exists.*

Proof. In order to prove that there exists an optimal mechanism, we will show that there exists a mechanism $\hat{\mu} \in \mathbb{M}$ s.t.[25] $URev\,(F) = \int_a^c u\,(\hat{s}\,(t))\,f\,(t)\,dt$.

We begin by noting that if $\mu \in \mathbb{M}$, then $s\,(x) \leq c$, and $URev\,(F) \leq (c-a) \cdot u\,(c)$ must be finite. Thus, if we take $\mathbb{R} = \{r | \exists \mu \in \mathbb{M}\, s.t.\, r = U\,(\mu, F)\}$, there is a sequence $r_i \in \mathbb{R}$ that converges to $URev\,(F)$. Let s_i be the corresponding payment functions, i.e., $r_i = \int_a^c u\,(s_i\,(t))\,f\,(t)\,dt$, and let $b_i = x \cdot q_i - s_i$ be the corresponding buyer's utility payoff functions. These b_i are continuous, uniformly bounded (as $0 \leq b_i\,(x) \leq c$), and equicontinuous (as they are Lipschitz functions with Lipschitz constant 1) and thus there is a sub-sequence that converges uniformly (by the Arzelà–Ascoli theorem), and we denote this limit by $\hat{b}$. It's easy to show that $\hat{b}$ is convex, non-decreasing, and has derivatives between zero and one, and thus that $\hat{\mu} = (\hat{q}, \hat{s}) \in \mathbb{M}$, where $\hat{q} = \hat{b}^{'}$ and $\hat{s} = x\hat{b}^{'} - \hat{b}$.

Next, we note that $u\,(s_i\,(x)) \leq u'\,(0)\,(s_i\,(x)) \leq u'\,(0) \cdot x$, where the first inequality is due to the concavity of u together with the monotonicity of s_i, and the second inequality is due to IR. Applying Lebesgue's dominated convergence theorem,[26] it is straightforward to show that

$$URev\,(F) = lim_{i \to \infty} \int_a^c u\,(s_i\,(t))\,f\,(t)\,dt = \int_a^c u\,(\hat{s}\,(t))\,f\,(t)\,dt.$$

Therefore, in our setup an optimal mechanism always exists. □

[25] Recall that $URev\,(F)$ is the highest expected utility possible, given F.

[26] The proof also holds when u' is unbounded, since when u' is unbounded we can use $u\,(s_i\,(x)) \leq u\,(\varepsilon) + u^{'}\,(\varepsilon) \cdot x - \varepsilon \cdot u'\,(\varepsilon)$ to the same effect.

7.2 Proof of the uniqueness of the optimal mechanism

Proposition 34. *The optimal IC-IR direct mechanism is unique.*

Proof. Assume that $\mu_1, \mu_2 \in \mathbb{M}$ are both optimal mechanisms and let $A = \{x | s_1(x) \neq s_2(x)\}$. If we define $\mu = (\mu_1 + \mu_2)/2$, then it is clearly an IC-IR mechanism, with the payment function $s = (s_1 + s_2)/2$. Consequently, if we let $I = [a, c]$ then $U(\mu, F) = \int_A u(s(t)) f(t) dt + \int_{I \backslash A} u(s(t)) f(t) dt$. Now, by the strict concavity of u it must be that $u(s(x)) > (u(s_1(x)) + u(s_2(x)))/2$ on A and that $\mu = \mu_1 = \mu_2$ on $I \backslash A$. We know, however, that μ_1 and μ_2 are optimal mechanisms, and hence A must be of zero measure. Furthermore, since $s_1 = s_2$ a.e. and they are both right-continuous (remember that w.l.o.g. we only consider seller-favorable mechanisms), it must be that $s_1 = s_2$. Thus, the optimal IC-IR direct mechanism must be unique. $\square$