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REPRESENTATIONS OF POLITICAL POWER STRUCTURES BY STRATEGICALLY STABLE GAME FORMS: A SURVEY

By

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Representations of Political Power Structures by Strategically Stable Game Forms: A Survey

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Abstract: We survey the results on representations of committees and constitutions by game forms that possess some kind of equilibrium strategies for each profile of preferences of the players. The survey is restricted to discrete models, that is, we deal with finitely many players and alternatives. No prior knowledge of social choice is assumed: As far as definitions are concerned, the paper is self-contained. Section 2 supplies the necessary general tools for the rest of the paper. Each definition is followed by a simple (but nontrivial) example. In Section 3 we give a complete account of representations of committees (proper and monotonic simple games), by exactly and strongly consistent social choice functions. We start with Peleg's representations of weak games, and then provide a complete and detailed account of Holzman's solution of the representation problem for simple games without veto players. In Section 4 we deal with representations of constitutions by game forms. Following Gärdenfors we model a constitution by a monotonic and super additive effectivity function. We fully characterize the representations for three kinds of equilibrium: Nash equilibrium; acceptable equilibrium (Pareto optimal Nash equilibrium); and strong Nash equilibrium. We conclude in Section 5 with a report on two recent works on representations of constitutions under incomplete information.

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1. Introduction

In this paper we survey results on two kinds of power distributions: committees and constitutions. Committees are well known and appear in many applications (constitutions will be discussed later). For example, every town council is a committee, the UN Security council is a committee, etc. For a recent and comprehensive study of committees the reader is referred to Taylor and Zwicker [1]. The founder of the modern theory of committees is L. S. Shapley (see,

e. g., [2]). Formally, a committee is a proper and monotonic simple game. The usual task of the members of a committee G is to choose one alternative out of a set A of alternatives. Thus we deal with choice problems (G, A). The working of a committee may be quite complex and include several stages. We focus on the voting stage which occurs after the members of G formed their preferences on A. Our idea is to construct a social choice function that associates with every profile of (linear) preferences of the voters an alternative in A such that the following properties are satisfied. (1) The power structure induced by the social choice function coincides with the original G. (2) The outcome of the social choice function belongs to the set of outcomes of strong Nash equilibria of the voting game (for every profile of linear preferences). Condition (2) implies, by well-known results, that the result of the voting, according to our social choice function, is in the (beta) *core* of the voting game (for each profile). Thus, our voting rules enjoy the strongest possible type of stability.

In Section 3 we survey our construction of social choice functions for all committees. The first step was taken by Peleg [3] who dealt (mainly) with weak games (games with veto players). Our methods can handle any number of alternatives for such games. Holzman in two lucid but highly technical papers [4,5] succeeded to give a complete solution to the strong representation problem of committees without vetoers (a strong representation of a committee is, roughly, a voting rule satisfying (1) and (2)). He determined the entire interval of orders of possible strong representations of a committee and, in particular, its maximum, the *capacity* of the committee.

The first (modern) model of a constitution is, as far as we know, Arrow's social welfare function [6]. However, because of Arrow's Impossibility Theorem, we are left only with dictatorial social welfare functions. Thus, as we insist on democratic constitutions, we have used Gärdenfors's [7] definition of a constitution. As we explain in Section 4 Gärdenfors's model is, essentially, a monotonic and super additive effectivity function (as defined independently by Moulin and Peleg [8]). Given a (monotonic and super additive) effectivity function we enquire whether there exists a game form such that (1) Representation: The effectivity function of the game form coincides with the given effectivity function; and (2) Stability: For every profile of the preferences of the citizens the resulting game has a (Pareto optimal) equilibrium point (of a pre-specified type of equilibrium). (1) guarantees that the

members of society (and also groups of members) can exercise their rights simultaneously. (2) guarantees that society is in equilibrium (of some kind).

The first question we address in Subsection 4.1 is the above question with respect to Nash equilibrium. If the effectivity function is monotonic and super additive then it has a representation by a game form. However, the game form may have no Nash equilibrium for certain profiles of preferences (think about the Gibbard Paradox [9]). The (necessary and sufficient) condition for stability is quite delicate. It is formulated with respect to the dual effectivity function and it says that rights for individuals should be "weak" (that is, exclude only few social states). For the exact condition see Theorem 8.

In Subsection 4.2 we consider effectivity functions that may be represented with Pareto optimal Nash equilibria. A game form is acceptable (Hurwicz and Schmeidler [10]) if it is Nash consistent and all its Nash equilibria are Pareto optimal (for every profile of preferences). An effectivity function is acceptable if it may be represented by an acceptable game form. Acceptable effectivity function are characterized by the two conditions of Theorem 8 and the following additional condition: Two disjoint coalitions cannot veto the same alternative.

We conclude Section 4 with representation by strong Nash equilibria. An effectivity function is representable by a strongly consistent game form if and only if it is convex and maximal.

In the last section, Section 5, we report two recent results on representation of constitutions under incomplete information. Also, we devote the final subsection to concluding remarks. The reader may find it interesting to read our remarks on the relationship between our results and the major paradoxes of social choice theory.

We close the introduction with the following important remark. This survey covers only finite problems of representations: Finitely many players and social alternatives. This allows the use of discrete mathematics. There are quite a few studies with a topological or measure space of alternatives. These studies may be relevant in particular cases. The interested reader is referred to Peleg and Peters [11].

2. Preliminaries

Let N be a set of n players (also voters or agents), and let A be a set of m outcomes (also alternatives or social states). We shall assume that N and A are finite and $m, n \geq 2$.

Definition 1. *A* game form (GF) *is a list* $\Gamma = (N; S^1, ..., S^n; \pi; A)$, *where N and A are as above;* S^i *is a non empty set, the (finite) set of strategies of player* $i \varepsilon$ *N; and* $\pi: S^1 \times ... \times S^n \to A$ *is the (surjective) outcome function.*

We shall now give an example of a GF.

Example 1. *Let* $N = \{1, 2, 3\}$ *and let* $A = \{a, b, c\}$. *Let further* $S^1 = \{2, 3\}$ *and* $S^2 = S^3 = A$. *Finally, we define the outcome function* π *as follows:* $\pi(2, x, y) = x$, *and* $\pi(3, x, y) = y$, *for all x, y* ε *A. This is a kingmaker GF: Player 1, the kingmaker, chooses the king of the day from* $\{2, 3\}$, *and the chosen king may pick any alternative from A. As we shall see this example has some nice properties. It is due to Hurwicz and Schmeidler [10].*

Let $\Gamma = (N; S^1, ..., S^n; \pi; A)$ be a GF. Usually the players have some preferences over the outcomes, that is, the members of A. The GF together with the preferences of the players define an (ordinal) n-person game in strategic form. We shall now make these remarks precise.

Definition 2. *Let A be a set of alternatives and let R be a binary relation on A. R is* complete *if for all x, y* ε *A, x R y or y R x. R is* transitive *if for all x, y, z* ε *A, if x R y and y R z, then x R z.R is a* weak order *if R is complete and transitive.*

Let A be a set of alternatives. We denote by $K(A) = K$ the set of all weak orders on A. If X and Y are finite sets, then we denote by X^Y the set of all functions from Y to X. Finally, if Γ is a GF and $R^N \varepsilon K^N$, then $g(\Gamma, R^N)$ denotes the n-person ordinal game in strategic form induced by the pair (Γ, R^N). We are now able to define Nash equilibrium of ordinal strategic games.

Definition 3. *Let* $\Gamma = (N; S^1, ..., S^n; \pi; A)$ *be a GF and let* $R^N \varepsilon K^N$ *be a profile of preferences (i.e., weak orders) of the players.* $s^N \varepsilon S^N$ *is a* Nash equilibrium (NE) *of the game* $g(\Gamma, R^N)$ *if for every player i, we have* $\pi(s^N) R^i \pi(t^i, s^{N \setminus i})$, *for all* $t^i \varepsilon S^i$.

Remark 1. *Let* Γ_0 *be the GF of Example 1. Then the game* $g(\Gamma_0, R^N)$ *has an NE for every* $R^N \varepsilon K^N$. *Indeed, let* $R^N \varepsilon K^N$ *and let each of the players 2 and 3 choose a best alternative. Then player 1 can choose a player whose chosen alternative maximizes his preference. The resulting triple of strategies is an NE.*

Let A be a set of alternatives, let $R \varepsilon K(A)$, and let $x, y \varepsilon A$. We denote

$$x \, I \, y \quad \text{if } x \, R \, y \text{ and } y \, R \, x \, ; \text{ and} \tag{1}$$

$$x \, P \, y \quad \text{if } x \, R \, y \text{ and not } y \, R \, x. \tag{2}$$

The foregoing notations enable us to proceed with the following definition.

Definition 4. *Let A be a set of alternatives, let N be a set of players, and let $R^N \varepsilon K^N(A)$. An alternative $x \, \varepsilon$ A is* Pareto optimal (PO)*with respect to R^Nif there exists no $y \, \varepsilon$ A such that $y \, P^i \, x$ for all $i \varepsilon$ N.*

Remark 2. *Every NE outcome for Γ_0 is PO (see Example 1 and Remark 1).*

In most political games communication is possible; consider for example parliaments and committees. This implies that coordination of strategies among members of coalitions is possible. In such situations NE's can be upset by coalitions of players. This leads us to consider more robust concepts of equilibrium. We proceed with the following definition.

Definition 5.*Let $\Gamma = (N; S^1, ..., S^n; \pi; A)$ be a GF and let $R^N \varepsilon \, K^N$. An n-tuple of strategiess^N is a strong NE (SNE)ofthe ordinal game $g(\Gamma, R^N)$ if for every nonempty subset T of N, and for every $q^T \varepsilon \, S^T$, there exists $h \, \varepsilon$ T such that $\pi(s^N) \, R^h \pi(q^T, s^{N \setminus T})$.*

Example 2. *Let Γ_0 be the GF of Example 1 and let $R^1 = (a, b, c)$, $R^2 = (b, c, a)$, and $R^3 = (c, a, b)$. Then, as the reader may easily check, the game $g(\Gamma_0, R^N)$ has no SNE. This is the Condorcet Paradox.*

We shall now proceed to construct a GF that has an SNE for every profile of preferences. We start with a definition.

Definition 6. *Let A be a set of alternatives and let R be a weak order. R is a* linear order *if for all $x, y \, \varepsilon$ A, $x \, I \, y$ if and only if $x = y$.*

Let A be a set of alternatives. We denote by $L = L(A)$ the set of all linear orders of A.

Definition 7. *Let A be a set of alternatives and let N be a set of voters. A* social choice function (SCF) *is a function $F: L^N \rightarrow A$.*

Clearly, every SCF is a GF. We shall now give an example of an SCF that has an SNE for each profile of linear preferences.

Example 3.(Sequential sincere vetoing) *Let A be a set of m alternatives and let N be a set of n voters, m = n + 1. Let $R^N \varepsilon L^N$. We shall define $F(R^N)$ in n steps. In the first step player 1 vetoes his last (i. e., worst) alternative. Denote the vetoed alternative by x_1. Now R^1 and x_1 are removed. We are left with the profile $R^{N \setminus 1} \big| (A \setminus \{x_1\})$. At this point player 2 vetoes his worst alternative x_2 and the first step is repeated with respect to the second (restricted) profile, and so on. There is precisely one alternative x that is not vetoed, and we define $F(R^N) = x$. We claim that x is an SNE outcome of the game $g(F, R^N)$. Let $Q^N \varepsilon L^N$ satisfy that the bottom alternative of Q^i is x_i for i = 1, ..., n. Obviously, $F(Q^N) = x$. We claim that Q^N is an SNE in g(F, R^N). Assume, on the contrary, that there exist a nonempty subset T of N and $P^T \varepsilon L^T$ such that $F(Q^{N \setminus T}, P^T)$ = x_j for some j εN, and $x_j R^i x$ for all iε T. Then j is not in T because x $R^i x_j$. Hence x_j is blocked at $(Q^{N \setminus T}, P^T)$ which is impossible. The reader may prove that also games $g(F, R^N) = ((N; L, ..., L; F; A), R^N)$, where $R^N \varepsilon K^N(A)$, have SNE's (see also Section 7.4 in Peleg [12]).*

von Neumann and Morgenstern [13] associated with every strategic game with side payments a coalitional game (also with side payments). They relied on the maxmin principle for correlated strategies (for coalitions) in their definitions. A large part of their book is devoted to the analysis of coalitional games. An axiomatic theory of coalitional games without side payments (NTU games) was introduced by Aumann and Peleg [14]. We are now going to follow this tradition and define coalitional functions for GF's. We start with the following definition.

Definition 8. *Let Γ = (N; S^1, ..., S^n;π; A) be a GF, let T be a non empty subset of N, and let B be a non empty subset of A. T is* effective *for B if there exists $s^T \varepsilon S^T$ such that$\pi(s^T, q^{N \setminus T}) \varepsilon$ B for all $q^{N \setminus T} \varepsilon S^{N \setminus T}$.*

For the continuation of our discussion of effectiveness we need the following notations. Let D be a non empty finite set. We denote: P(D) = {D': D' is a subset of D}; and $P_0(D)$ = {D': D' is a non empty subset of D}. Also, |D| denotes the number of elements of D. Finally, if B is a subset of D, then B^+ = {B': B' contains B and is contained in D}.

Definition 9. *Let Γ = (N; S^1, ..., S^n; π; A) be a GF. The* effectivity function (EF)*of Γ, E^Γ: $P_0(N) \rightarrow P(P_0(A))$, is defined by $E^\Gamma(T)$ = {B: T is effective for B}, for all T $\varepsilon P_0(N)$.*

Definition 9 is due to Moulin and Peleg [8]. As the reader may check, the EF E_0 of the GF Γ_0 of Example 1 is given by: $E_0(T)$ = {A} if $|T| \le 1$, and $E_0(T) = P_0(A)$ if $|T| \ge 2$. The reader may also compute the EF E_1 of Example 3. Indeed, if T is a non empty subset of N, then B $\varepsilon E_1(T)$ if and only if $|B| \ge m - |T|$.

Definition 9 leads us to consider general EF's, that is EF's that are not necessarily derived from GF's. For example, a TU coalitional game is super additive if and only if it is derived

from a (TU) game in strategic form. Thus, as we consider TU games that are not super additive, we shall also consider general EF's.

Definition 10. *Let N be a set of players and let A be a set of alternatives. An* effectivity function (EF) *is a function $E: P_0(N) \rightarrow P(P_0(A))$ that satisfies (1) $A \varepsilon E(T)$ for all $T \varepsilon P_0(N)$; and (2) $E(N) = P_0(A)$.*

Let Γ be a GF. Then E^Γ satisfies (*1*) of the last definition. It satisfies (*2*) if and only if its outcome function is surjective. However, E^Γ is monotonic and super additive. That is:

If $B \varepsilon E^\Gamma(T)$ and B' contains B, then $B' \varepsilon E^\Gamma(T)$. *(Monotonicity)* (3)

If $B_j \varepsilon E^\Gamma(T_j)$, $j = 1, 2$, and $T_1 \cap T_2 = \Phi$, then $B_1 \cap B_2 \varepsilon E^\Gamma(T_1 \cup T_2)$. *(Superadditivity)* (4)

(3) and (4) follow from Definition 9. One of the central solutions of coalitional games is the core. We shall now define the core, as was done in [8], for EF's.

Definition 11. *Let E be an EF and let $R^N \varepsilon K^N$. Let further $T \subset N$, let $B \varepsilon E(T)$, and let $x \varepsilon A \setminus B$. B* dominates *x via T at R^N if b $P^h x$ for all $h \varepsilon T$ and $b \varepsilon B$. x is* dominated *at R^N if there exist $T \subset N$ and $B \varepsilon E(T)$ such that B dominates x via T at R^N. The* core *of E and R^N, $C(E, R^N)$, is the set of all undominated alternatives at R^N.*

The core of an EF may be empty for some preference profiles. (The reader may consider, for example, Example 2.) However, there is an interesting connection between cores of EF's and SNE's of GF's.

Theorem 1. *Let $\Gamma = (N; S^1, ..., S^n; \pi; A)$ be a GF, and let $R^N \varepsilon K^N$ be a profile of preferences. If s^N is an SNE of the game $g(\Gamma, R^N)$, then $\pi(s^N) \varepsilon C(E^\Gamma, R^N)$.*

See Aumann [15] for a discussion of this connection in another context, and Section 4.1 in Peleg [12] for a proof of a similar result. Theorem 1 has multiple applications in Sections 3-4.

3. Representations of Simple Games

Let N be a set of n players, $n \geq 2$, and let A be a set of m alternatives, $m \geq 2$. We assume in the sequel that all SCF's are surjective.

Definition 12. *An SCF $F: L^N \rightarrow A$ is* exactly and strongly consistent (ESC) *if for every $R^N \varepsilon L^N$ there exists an SNE Q^N of the game $g(F, R^N)$ such that $F(R^N) = F(Q^N)$.*

The SCF of Example 3 is an ESC SCF. In this part we shall look for ESC SCF's that may serve as voting procedures to committees. We shall soon make the connection to committees, that is, (monotonic and proper) simple games. However, we first remark the following fact.

Remark 3. *Let F be an ESC SCF. Then, for every $R^N \varepsilon L^N, F(R^N)$ is in the core $C(E^F, R^N)$.*

Thus, if *F* is exactly and strongly consistent, then *sincere voting* itself leads to a (coalitionally) stable outcome! (See Theorem 1.) We now connect simple games and SCF's.

Definition 13. *A* simple game *is a pair G = (N, W) where N is a set of players and W is a (nonempty) set of coalitions (that is, nonempty subsets of N). The members of W are called* winning *coalitions. We always assume that a superset of a winning set is winning* (monotonicity*), and that the complement of a winning set is* losing *(that is, not winning)* (properness*). G = (N, W) is* symmetric *if winning depends only on the number of players: T ε W if and only if $|T| \geq q$. In this case we write G = (n, q).*

We now associate with every SCF a simple game $G^*(F) = (N, W^*(F))$ by defining: $W^*(F) = \{T: E^F(T) = P_0(A)\}$. For example, the simple game in Example 3 is the unanimity game $(n, n) = (N, \{N\})$. We are now ready for the central definition of this section.

Definition 14. *Let G = (N, W) be a (proper and monotonic) simple game and let A be a set of (at least two) alternatives. An SCF F: $L^N \rightarrow A$ is a* strong representation *of G if*

$$G^*(F) = G; \text{ and} \tag{5}$$

$$F \text{ is exactly and strongly consistent.} \tag{6}$$

$|A|$ *is called the* order *of the strong representation F.*

There are some simple desirable properties of strong representations. The first that we shall consider is the monotonicity of the representing SCF.

Definition 15. *An SCF F: $L^N \rightarrow A$ is* monotonic *if for all $R^N \varepsilon L^N$, h ε N, and x ε A, if $F(R^N) = x$ and Q^N is obtained from R^N by moving x one place up in R^h and leaving all other preferences unchanged, then $F(Q^N) = x$.*

For example, the SCF of Example 3 is monotonic. Another natural assumption is that the simple game and the representing SCF have the same symmetries. We shall soon make this idea precise, however we first reconsider the SCF *F* of Example 3. For $G^*(F) = (n, n)$ every permutation of the players is a symmetry, whereas *F* itself has no symmetries except the identity.

Definition 16. *Let F: $L^N \rightarrow A$ be an SCF, let G = (N, W) be a simple game, and let t be a permutation of N. t is a* symmetry *of F if $F(R^1, ..., R^n) = F(R^{t(1)}, ..., R^{t(n)})$ for all $R^N \varepsilon L^N$. t is a* symmetry *of G if for all U ε W,*

$t(U) = \{t(h): h \in U\} \in W$. We denote by sym(F) (sym(G)) the set of all symmetries of F (G). F is faithful if sym(F) = sym(G*(F)).

We need one more definition in order to formulate our first representation theorem.

Definition 17. *Let G = (N, W) be a (proper and monotonic) simple game. G is* weak *if V = ∩{S: S ∈ W} ≠Φ. V is the set of* veto *players of G.*

Theorem 2.*Every weak game has a monotonic and faithful strong representation of every order greater than or equal to 2.*

We shall now illustrate Theorem 2 for a "simple" weak game. First we need a definition.

Definition 18. *A simple game G = (N, W) is a* weighted majority game*if there exist non negative numbers $w^1, ..., w^n$and a positive number q such that for all $T \in P_0(N)$, $T \in W$ if and only if $\sum_{h \in T} w^h \geq q$. (Notation: G = [q; $w^1, ..., w^n$].)*

Example 4. *Let G = [3; 2, 1, 1]. Then player 1 is a veto player and G is weak. Let A be a set of m alternatives, m ≥ 2, and let b ∈ A. We now define an SCF F that will strongly represent G. Let $R^N \in L^N$ where N = {1, 2, 3}. We denote $R^1 = (x(1), x(2), ..., x(m))$. Our construction is given by the following rules.*

If $b \neq x(1)$, then $F(R^N) = x(1)$. (7)

If $b = x(1)$ and {h: b R^h x(2)} is winning (in G), then $F(R^N) = b$. (8)

If $b = x(1)$ and x(2) R^2 b and x(2) R^3 b, then $F(R^N) = x(2)$. (9)

The reader is invited to check that G(F) = G and that the transposition (2, 3) is a symmetry of F. We introduce now an additional notation. For $R \in L$ we denote by $t_k(R)$ the k-th alternative in the order R. Thus, $t_1(R)$ is the top alternative in the order R; $t_2(R)$ is the second alternative in the order R, and so on.*

   Let now $R^N \in L^N$ and denote as above $R^1 = (x(1), x(2), ..., x(m))$. In cases (7) or (8), R^N itself is an SNE of g(F, R^N). In case (9) let $Q^N \in L^N$ satisfy $t_1(Q^h) = x(2)$ and $t_m(Q^h) = x(1)$ for all h ∈ N. Then $F(Q^N) = x(2)$ and Q^N is an SNE (only b might dominate x(2); however, this is blocked by {2, 3}).

From now on, we consider a (proper and monotonic) simple game G = (N, W) which is not weak, that is, V = ∩{S: S ∈ W} = Φ. While a monotonic and faithful strong representation of order 2 still exists, strong representations of higher orders need not exist. A simple way to see that the orders of strong representations are bounded in this case is based on the following definition and remark.

Definition 19. *Let G = (N, W) be a non-weak simple game. The* Nakamura number*of G, denoted ν(G), is the least k for which there exist k winning coalitions $S_1, ..., S_k$ with an empty intersection.*

Remark 4. *Let E^G be the EF associated with the simple game G = (N, W), i.e., $E^G(T) = P_0(A)$ if T ∈ W and $E^G(T) = \{A\}$ otherwise. If |A| ≥ ν(G) then one can construct $R^N \in L^N$for which the core C(E^G, R^N) is empty. Hence, no strong representation of G of order m ≥ ν(G) can exist; indeed, if F were such a representation we*

would have $E^G(T) \subset E^F(T)$ for all $T \, \varepsilon \, P_0(N)$, and therefore $C(E^F, R^N) \subset C(E^G, R^N)$ for all $R^N \, \varepsilon \, L^N$, contradicting Remark 3 for the profile R^N found above.

Thus, for a given simple game without veto players, the orders of its strong representations are bounded. This means that such a committee can only handle relatively small sets of alternatives in an exactly and strongly consistent manner. In order to quantify this phenomenon, we introduce the following definition.

Definition 20. *Let $G = (N, W)$ be a non-weak simple game. The* capacity *of G, denoted $\mu(G)$, is the largest m for which G has a strong representation of order m.*

By Remark 4 we always have $\mu(G) < \nu(G)$. Since $\nu(G) \le n$ (in a non-weak game, one can find for each player a winning coalition not containing him), we conclude that $\mu(G) \le n - 1$. In some natural cases, the limitation is much stricter, as shown in the next remark.

Remark 5. *Assume that $G = (N, W)$ is not weak, and that the complement of any losing coalition is winning (such games are called* strong; *they include simple majority games with an odd number of players). Then the reader can check that $\nu(G) = 3$, and therefore $\mu(G) = 2$.*

On the positive side, there is a way to generalize the idea of sequential vetoing (Example 3 above), so as to obtain a class of exactly and strongly consistent SCF's which allows some flexibility in the allocation of power to coalitions. We proceed to define these voting procedures, in order to use them later as strong representations of simple games, with orders lying in a certain range depending on the game.

Definition 21. *Assume that $m \le n + 1$, and let $\beta: A \rightarrow \{1, \dots, n\}$ satisfy $\sum_{x \, \varepsilon \, A} \beta(x) = n + 1$. The number $\beta(x)$ is called the* blocking coefficient *of x, intended as the number of voters required to block the alternative x (in Example 3 we had $\beta(x) = 1$ for all $x \, \varepsilon \, A$). The* effectivity function *E_β associated with β is defined by: $B \, \varepsilon \, E_\beta(T)$ if and only if $|T| \ge \sum_{x \, \varepsilon \, A \setminus B} \beta(x)$. An SCF $F: L^N \rightarrow A$ is an E_β – core selection if $F(R^N) \, \varepsilon \, C(E_\beta, R^N)$ for all $R^N \, \varepsilon \, L^N$.*

Remark 6. *A different but equivalent way to obtain the SCF's in the above definition is to consider feasible elimination procedures with respect to β, as introduced by Peleg [16]. Roughly speaking, given $R^N \, \varepsilon \, L^N$, one allows sequential vetoing as in Example 3, but at each stage an alternative x_i is vetoed by $\beta(x_i)$ voters who consider it the worst among the remaining alternatives, ending with one surviving alternative. It can be proved that the set of surviving alternatives corresponding to all possible orders of elimination is precisely $C(E_\beta, R^N)$. This shows in particular that $C(E_\beta, R^N) \ne \Phi$ for all $R^N \, \varepsilon \, L^N$, and thus guarantees the existence of an E_β – core selection.*

The following facts about E_β – core selections are based mostly on Peleg [16], Oren [17], and Polishchuk [18].

Theorem 3. *Let $\beta: A \to \{1, \ldots, n\}$ satisfy $\sum_{x \,\varepsilon\, A} \beta(x) = n + 1$. Then:*

(a) *For every E_β – core selection F we have $E^F = E_\beta$.*

(b) *Every E_β - core selection F is exactly and strongly consistent.*

(c) *There exists an E_β – core selection F which is monotonic and anonymous (i.e., invariant under any permutation of the voters).*

Example 5. *Consider the symmetric simple game G = (5, 4), in other words, a committee of 5 members with full power allocated to coalitions of 4 or more members. If the committee has to choose among 3 alternatives a, b, and c, we can assign the blocking coefficients $\beta(a) = \beta(b) = \beta(c) = 2$. Then $B \,\varepsilon\, E_\beta(T)$ if and only if $|T| \geq 2|A \setminus B|$, meaning that any two players can block an alternative, but it takes four players to enforce an alternative. The latter agrees with the initially given allocation of power in G = (5, 4).*

    Let $R^N \,\varepsilon\, L^N$ be given by: $R^1 = (a, b, c)$, $R^2 = (a, c, b)$, $R^3 = R^4 = (b, c, a)$, and $R^5 = (c, b, a)$. Then $\{b, c\}$ dominates a via $\{3, 4, 5\}$, while b and c are undominated, and so $C(E_\beta, R^N) = \{b, c\}$. Thus, an E_β – core selection F gives either $F(R^N) = b$ or $F(R^N) = c$. In the former case, a Q^N having $t_3(Q^1) = t_3(Q^3) = c$ and $t_3(Q^4) = t_3(Q^5) = a$ is an SNE of $g(F, R^N)$ with $F(Q^N) = F(R^N)$. In the latter case, a Q^N having $t_3(Q^2) = t_3(Q^5) = b$ and $t_3(Q^3) = t_3(Q^4) = a$ is an SNE of $g(F, R^N)$ with $F(Q^N) = F(R^N)$. Similar constructions yield exact SNE's for any $R^N \,\varepsilon\, L^N$, confirming that F is ESC. Thus, F provides a strong representation of G = (5, 4) of order 3.

We are now ready to state the following theorem, which fully describes the faithful strong representations of symmetric, non-weak simple games.

Theorem 4. *Let G = (n, q) be a (proper) symmetric, non-weak simple game, i.e., $\frac{n}{2} < q < n$. Then:*

(a) *$\mu(G) = \lfloor \frac{n+1}{n-q+1} \rfloor$, where $\lfloor \ \rfloor$ denotes the integer part.*

(b) *For every $2 \leq m \leq \lfloor \frac{n+1}{n-q+1} \rfloor$, there exists a monotonic and faithful strong representation of G of order m.*

(c) *For every $2 \leq m \leq \lfloor \frac{n+1}{n-q+1} \rfloor$, an SCF $F: L^N \to A$ is a faithful strong representation of G of order m if and only if F is an anonymous E_β – core selection for some $\beta: A \to \{n-q+1, \ldots, n\}$ satisfying $\sum_{x \,\varepsilon\, A} \beta(x) = n + 1$ and $\min_{x \,\varepsilon\, A} \beta(x) = n - q + 1$.*

One direction of part (c), namely the fact that E_β – core selections with the appropriate blocking coefficients $\beta(\cdot)$ yield strong representations of G, follows directly from Theorem 3. This also confirms part (b) and the inequality $\mu(G) \geq \lfloor \frac{n+1}{n-q+1} \rfloor$ in part (a), because the condition $2 \leq m \leq \lfloor \frac{n+1}{n-q+1} \rfloor$ allows us to choose the m blocking coefficients so that their sum is $n + 1$ and their minimum is $n - q + 1$. However, the opposite direction of part (c), and the inequality $\mu(G) \leq \lfloor \frac{n+1}{n-q+1} \rfloor$ in part (a), are based on a system of necessary conditions for strong representations of non-weak games established in Holzman [4].

To illustrate the behavior of $\mu(G)$ for symmetric simple games, fix the size n of the committee. If the power structure corresponds to simple majority rule, i.e., $q = \lfloor \frac{n}{2} \rfloor + 1$, then by

Theorem 4(a) the capacity $\mu(G)$ equals 2 (this is in agreement with Remark 5). Now let us gradually increase q, the size of the special majority required in the committee. This will initially have no effect on the capacity, until q is around $\frac{2n}{3}$, at which point the capacity $\mu(G)$ increases to 3. Then the capacity stays constant until q is around $\frac{3n}{4}$, where it increases to 4, and so on. For $q = n - 1$ we obtain a capacity of $\lfloor\frac{n+1}{2}\rfloor$, which is the largest we can get for a non-weak symmetric game. (This reflects the fact that the existence of a strong representation of G of order m requires more than just the non emptiness of the core $C(E^G, \cdot)$. As discussed in Remark 4, the latter only requires $m < v(G)$, which for $G = (n, n - 1)$ gives $m < n$.) Note that raising q to become n would result in the weak game $G = (n, n)$, for which the capacity is infinite by Theorem 2.

The capacity of a general (not necessarily symmetric) non-weak simple game was fully determined in Holzman [5]. The determination depends on the combinatorial structure of the family W of winning coalitions, and is quite complex (see below). Roughly speaking, the capacity is determined not only by the relative size of winning coalitions (corresponding to $\frac{q}{n}$ in the symmetric case), but also by the rigidity of the allocation of power. By this we mean, intuitively, that the aggregate power of a system of disjoint coalitions is not affected by exchanging players among the coalitions, and is only minimally affected by adding a single new player. In order to state the general result, we need the following definitions. In all of them, a simple game $G = (N, W)$ is given.

Definition 22. *A coalition S is* blocking*if $N \setminus S \notin W$. Such a coalition is* minimal blocking*if none of its proper subcoalitions is blocking.*

Definition 23. *Let $p \geq r$ be positive integers. A coalition S is p-*blocking*if S can be written as the disjoint union of p blocking coalitions. A coalition S is of* type *(p, r) if S can be written as the union of p coalitions, at least r of which are non-blocking and the rest are minimal blocking.*

Definition 24.*G is*singular*if there exists a 2-blocking coalition of type (2, 1). Otherwise, G is* regular.

Definition 25. *Let $p \geq 2$ be an integer. A coalition S is p-*critical*if S is of type (p, 2), and $S \cup \{i\}$ is p-blocking for every $i \in N \setminus S$.*

We denote by $\kappa(G)$ the least p for which there exists a p-critical coalition in G. This is well defined, because N is of type $(p, 2)$ for p large enough, and the second requirement of Definition 25 vacuously holds for $S = N$.

Theorem 5. *Let $G = (N, W)$ be a (proper) non-weak simple game. Then:*
  (a) *The capacity of G is given by:*

$$\text{If } G \text{ is singular then } \mu(G) = 2. \tag{10}$$

$$\text{If } G \text{ is regular and } N \text{ is of type } (\kappa(G), 2) \text{ then } \mu(G) = \kappa(G) - 1. \tag{11}$$

$$\text{If } G \text{ is regular and } N \text{ is not of type } (\kappa(G), 2) \text{ then } \mu(G) = \kappa(G). \tag{12}$$

  (b) *For every $2 \leq m \leq \mu(G)$, there exists a monotonic and faithful strong representation of G of order m.*

Comparing to Theorem 4 for the symmetric case, we note that in the general case we still have a precise (though more complicated) formula for the capacity (part (a)), and the existence of monotonic and faithful strong representations of every order up to the capacity (part (b)). We do not, however, have an analog of part (c): a full description of the class of all (faithful) strong representations is not known in the general case.

To illustrate the application of Theorem 5 in concrete examples, we consider three weighted majority games.

Example 6. *Let $G = [7; 2, 2, 2, 1, 1, 1]$. Then the weight needed for blocking is 3. The coalition $S = \{1, 2, 4, 5\}$ can be written as $\{1, 4\} \cup \{2, 5\}$ and also as $\{1, 2\} \cup \{4, 5\}$, showing that it is both 2-blocking and of type $(2, 1)$. Thus, G is singular, and so by (10) we have $\mu(G) = 2$.*

Example 7. *Let $G = [9; 2, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Again, the weight needed for blocking is 3. Here, any minimal blocking coalition has weight exactly 3, which guarantees that G is regular. Next, we look for p-critical coalitions. Let us write $w(T) = \sum_{h \varepsilon T} w^h$. If S is p-critical then, on the one hand, $w(S) \leq 3p - 2$ and, on the other hand, $w(S \cup \{i\}) \geq 3p$ for every $i \varepsilon N \setminus S$. This is impossible for $p = 2, 3$ because $w(S) \leq 7$ implies that there is $i \varepsilon N \setminus S$ with $w^i = 1$, and also for $p = 4$ because $w(S) \leq 10$ implies that $S \neq N$ and $w(S \cup \{i\}) \leq w(N) < 12$. Thus, $\kappa(G) \geq 5$, and in fact $\kappa(G) = 5$ because N is of type $(5, 2)$. It follows from (11) that $\mu(G) = \kappa(G) - 1 = 4$.*

Example 8. *Let $G = [9; 2, 2, 1, 1, 1, 1, 1, 1, 1, 1]$. Here, the weight needed for blocking is 4, and any minimal blocking coalition has weight exactly 4, hence G is regular. We check that there are no 2-critical coalitions. Indeed, if S is of type $(2, 2)$ then $w(S) \leq 6$, hence there is $i \varepsilon N \setminus S$ with $w^i = 1$, so that $w(S \cup \{i\}) \leq 7$. There do exist 3-critical coalitions. Indeed, $S = N \setminus \{1\}$ can be written as $\{2, 3, 4\} \cup \{5, 6, 7\} \cup \{8, 9,$*

10}, hence is of type (3, 2), while $S \cup \{1\}$ can be written as $\{1, 2\} \cup \{3, 4, 5, 6\} \cup \{7, 8, 9, 10\}$, hence is 3-blocking. Thus $\kappa(G) = 3$. Since N is not of type (3, 2), we conclude from (12) that $\mu(G) = \kappa(G) = 3$.

4. Representations of Effectivity Functions

Let N be a set of players and A be a set of (possible) social states as above. Gärdenfors [7] defines a *right* of a coalition S (a non-empty subset of N) as a non-empty subset B of A. Intuitively, this means that S is legally entitled to have the final social state in B. Thus a *rights system*, or a *constitution*, is the collection of all pairs (S, B) such that B is a right of S. Hence, a constitution is a function $E: P_0(N) \rightarrow P(P_0(A))$. It is obvious, by definition, that the constitution is an EF (see Definition 10). Gärdenfors also assumes monotonicity (see (3)), and coherence. (An EF E is *coherent* if for all $T_h \varepsilon P_0(N)$, $B_h \varepsilon E(T_h)$, $h = 1, 2$, such that $T_1 \cap T_2 = \Phi$, it follows that $B_1 \cap B_2 \neq \Phi$.) We shall assume throughout the stronger assumption of superadditivity (see (4)). The reason for this will become immediately clear. First we present the following example of Gibbard [9].

Example 9. *Consider a society $N = \{1, 2\}$ of two individuals. Each citizen has two shirts, white and blue, and has the right to choose the color of his shirt freely. The set of alternatives that is open to the society is $A = \{w, w; w, b; b, w; b, b\}$, where (w, w) denotes the event that both individuals wear a white shirt, etc. The constitution E is given by: $E(1) = \{w, w; w, b\}^+ \cup \{b, w; b, b\}^+$; $E(2) = \{w, w; b, w\}^+ \cup \{w, b; b, b\}^+$; and $E(N) = P_0(A)$.*

We now arrive at one of the central definitions of this section.

Definition 26. *Let E be an EF. A GF Γ is a* representation *of E if $E^\Gamma = E$.*

Example 10. *We shall construct a representation for the constitution of Example 9. Let $N = \{1, 2\}$, let $S^1 = S^2 = \{w, b\}$, and let $\pi: S^1 \times S^2 \rightarrow A$ be the identity function; then $\Gamma = (N; S^1, S^2; \pi; A)$ is a representation of the foregoing constitution.*

Clearly, if an EF has a representation, then it is monotonic and superadditive (see (3) and (4)). Fortunately, the converse result, due to Peleg [19], is also true:

Theorem 6. *Let E be an EF. Then E has a representation by a GF Γ if and only if it is monotonic and superadditive.*

There are several proofs of Theorem 6. We shall always rely on the proof (of Theorem 2.4.7) in Peleg and Peters [11] which can be modified to yield representations that possess various types of equilibria (for all profiles of preferences).

4.1. Nash Consistent Representations of EF's

We need the following notation in the sequel.

Notation 1. *Let A be a set of alternatives and let R $\varepsilon K(A)$. For x ε A we denote: L(x, R) = {y ε A:x R y}.*

Let $\Gamma = (N; S^1, ..., S^n ; \pi; A)$ be a GF and let $R^N \varepsilon K^N$ be a profile of preferences. Assume that s^N is an NE of the game $g(\Gamma, R^N)$ and let $\pi(s^N) = x$. As x is an NE outcome we have: $\pi(s^{N \setminus h}, q^h) \varepsilon L(x, R^h)$ for all $q^h \varepsilon S^h$ and all $h \varepsilon N$. Hence, $L(x, R^h) \varepsilon E^\Gamma(N \setminus h)$ for all $h \varepsilon N$. Assume further that Γ is *Nash consistent*, that is, the game $g(\Gamma, R^N)$ has an NE for every $R^N \varepsilon K^N$. Then we know that E^Γ satisfies the following condition.

(*)For every $R^N \varepsilon K^N$ there exists x (=$x(R^N)$) such that $L(x, R^h) \varepsilon E^\Gamma(N \setminus h)$ for all $h \varepsilon N$.

Let now E be an EF and let Γ be a representation of E. Assume further that Γ is Nash consistent. Then we know that E satisfies the following conditions.

E is monotonic and superadditive. (13)

For every $R^N \varepsilon K^N$ there exists x (=$x(R^N)$) such that $L(x, R^h) \varepsilon E(N \setminus h)$ for all $h \varepsilon N$. (14)

Thus, (13) and (14) are necessary conditions for the existence of a Nash consistent representation for E. Fortunately, they are also sufficient!

Theorem 7. *An EF E has a Nash consistent representation by a GF if and only if it satisfies Conditions (13) and (14).*

For a proof of Theorem 7 see Section 3.2 of [11]. Condition (14) is very laborious to determine: We have to make a non simple check for every $R^N \varepsilon K^N$. So we shall now try to find a condition that involves only E itself. Our first step is to define the polar EF of an EF.

Definition 27. *Let E be an EF. The* polar *of E, E*, is given by:*
E*(T) = {B': B' ∩ B ≠ Φ for all B $\varepsilon E(N \setminus T)$}, for all T $\varepsilon P_0(N)$.

Intuitively, $B \varepsilon E^*(T)$ if $N \setminus T$ cannot *prevent* T from reaching (some point in) B. (At this point we define $E(\Phi) = \{A\}$. Then E^* is, indeed, an EF.) Furthermore, E^* is monotonic. However, E^* may fail to be superadditive even when E is monotonic and superadditive.

Example 11. *Indeed, consider the polar E* of the EF E of Example 9:*

E*(1) = {w, w; w, b}⁺ ∪ {w ,w; b, b}⁺ ∪ {b ,w; w ,b}⁺ ∪ {b, w; b, b}⁺; and
E*(2) = {w, w; b, w}⁺ ∪ {w, w; b, b}⁺ ∪ {w, b; b, w}⁺ ∪ {w, b; b ,b}⁺.
Thus {w, w; b, b} ε E*(1) and {w, b; b, w} ε E*(2). As {w, w; b, b} ∩ {w, b; b, w} = Φ,E* is not superadditive.

We shall now use the polar of an EF E in order to determine whether it has a Nash consistent representation by a GF. Thus let E be a monotonic and superadditive EF. E has a Nash consistent representation if (14) is satisfied. Let R^N be a profile of preferences. For each h ε N denote $B(h) = \{y\ ε\ A: L(y, R^h)\ ε\ E(N \setminus h)\}$. We claim that $B(h)\ ε\ E*(h)$. Indeed, let $C\ εE(N \setminus h)$ and let z be a maximum member of C with respect to R^h. Then $L(z, R^h) ⊃ C$, and therefore $L(z, R^h)\ ε\ E(N \setminus h)$. Thus, $z\ εB(h) ∩ C$. Since C is arbitrary, $B(h)\ ε\ E*(h)$. It is clear that if $x\ ε\{B(h): h\ ε\ N\}$, then x satisfies (14) for R^N. Hence we have proved one direction of the following important theorem, due to Peleg, Peters, and Storcken [20].

Theorem 8. *Let E be a monotonic and superadditive EF and let E* be the polar of E. Then E has a Nash consistent representation if and only if the following condition holds:*

$$[B(h)\ ε\ E*(h)\ for\ h = 1, ..., n] →∩\{B(h): h = 1, ..., n\} ≠ Φ. \tag{15}$$

The reader may find the proof of the necessity part of Theorem 8 in Section 3.3 of [11]. Example 11 and Theorem 8 imply that the EF E of Example 9 has no Nash consistent representation. Indeed, we can prove this independently. Let Γ be any representation of E, let $R^1 = (w, w; b, b; w, b; b, w)$, and let $R^2 = (w, b; b, w; w, w; b, b)$. Then the (ordinal) game $g(\Gamma, R^N)$ has no NE! (Player 1 cannot prevent player 2 from reaching $\{w, b; b, w\}\ ε\ E*(2)$, and player 2 cannot prevent player 1 from reaching $\{w, w; b, b\}\ ε\ E*(1)$. This game is due to [9] and is known as "Gibbard's First Paradox".) However, many EF's have Nash consistent representations. Indeed, call an EF *E maximal* if E is superadditive and is equal to its polar, that is, $E = E*$. Then a maximal E has a Nash consistent representation by Theorem 8 (one can prove that monotonicity follows from maximality). We shall now discuss a concrete example of an EF that has a Nash consistent representation.

Example 12. *Let N = {m₁, m₂, f} be a society of two men (m₁ and m₂) and a woman (f). The rules of marriage are the orthodox ones: Only the couples (m₁, f) and (m₂, f) are legal. In addition everybody has the right to remain single. The set of social states is A = {w₁, w₂, s}, where wₕ is the state with the couple (mₕ, f), h = 1, 2, and s is the state when f is single. Let E be the constitution. E({m₁, f}) ={ s}⁺ ∪ {w₁}⁺. Hence E*(m₂) = {s, w₁}⁺. Similarly, E*(m₁) = {s, w₂}⁺. Finally, E({m₁, m₂}) = {s}⁺; hence E*(f) = {s}⁺ . It is now obvious that E* satisfies(15). Thus E has a Nash consistent representation.*

When investigating Nash consistent representations of constitutions we encounter the following pleasant surprise: For particular constructions of Nash consistent representations, for every profile of preferences each NE is *Pareto dominated by aPareto optimal NE* (of the same profile). More precisely, we have the following result.

Theorem 9.*Let the EF E be monotonic and superadditive and let it satisfy (15). Then there exists a Nash consistent representation Γ of E with the following property: For every $R^N \varepsilon K^N$ and for every Nash equilibrium outcome x of g(Γ, R^N), there exists a Pareto optimal Nash equilibrium outcome y (of the same game), such that y R^h x for all h ε N.*

We are unable to discuss the proof of Theorem 9 because it depends on technical results which we have avoided, mainly the proof of Theorem 2.4.7 in [11]. We now introduce the definition of liberalism.

Definition 28. *A constitution E: $P(N) \rightarrow P(P_0(A))$ satisfies* liberalism *if every member of the society has a non-trivial right; that is, for each h ε N there exists x ε A such that A \ x ε E(h).*

See Sen [21] for the origin of liberalism. Example 12 satisfies liberalism (everybody has the right to remain single). Nevertheless it has a Nash consistent representation in the sense of Theorem 9. We may conclude that in our model liberalism allows for some kind of Pareto optimal behavior on the part of the players.

4.2. Acceptable Representations of EF's

We shall now ask whether it is possible to characterize those EF's that have a Nash consistent representation such that all its NE's are Pareto optimal. We start with the following definition.

Definition 29. *Let Γ = (N; S^1, ..., S^n; π; A) be a Nash consistent GF. Γ is* acceptable *if for every $R^N \varepsilon K^N$, every NE of the game g(Γ, R^N) is Pareto optimal. (An NE is Pareto optimal if its outcome is Pareto optimal.)*

Definition 29 is due to Hurwicz and Schmeidler [10]. The GF of Example 1 is acceptable. An EF is *acceptable* if it has an acceptable representation. Our current goal is to characterize acceptable EF's. Let Γ = (N; S^1, ..., S^n; π; A) be an acceptable GF, let Q and T be disjoint nonempty subsets of N, let B ε $E^r(Q)$, and let C ε $E^r(T)$. We claim that (*) B $\cup$ C = A. Indeed, assume on the contrary that there is an alternative x $\varepsilon A \setminus (B \cup C)$. Define a preference relation R on A by x P y for all y $\neq$ x, and y I z for all z, y ε A \ x. Let R^N = (R, ..., R), let s^Qsatisfy $\pi(s^Q, s^{N \setminus}$

Q) ε B for all $s^{N \setminus Q} \varepsilon$ $S^{N \setminus Q}$, and let s^T satisfy $\pi(s^T, s^{N \setminus T})$ ε C for all $s^{N \setminus T} \varepsilon$ $S^{N \setminus T}$. Then $(s^Q, s^T, p^{N \setminus (Q \cup T)}$) is an NE of $g(\Gamma, R^N)$ that is Pareto dominated by x (for any feasible choice of $p^{N \setminus (Q \cup T)}$). Thus, we have proved (*). In summary, if Γ is acceptable, then

$$[Q, T \varepsilon P_0(N), Q \cap T = \Phi, B \varepsilon E^\Gamma(Q), \text{ and } C \varepsilon E^\Gamma(T)] \rightarrow B \cup C = A. \tag{16}$$

We now conclude that every acceptable EF E satisfies

$$[Q, T \varepsilon P_0(N), Q \cap T = \Phi, B \varepsilon E(Q), \text{ and } C \varepsilon E(T)] \rightarrow B \cup C = A. \tag{17}$$

Quite surprisingly, the necessary conditions are, again, also sufficient.

Theorem 10. *An EF E is acceptable if and only if it satisfies (13), (15), and (17).*

The proof of the sufficiency part of the above theorem is very technical and will not be illustrated here (see Peleg [22]). We now return to Sen's [21] concept of liberalism (adapted to EF's).

Definition 30. *A super additive and monotonic EF E satisfies* minimal liberalism (ML) *if there exist distinct players h and k and sets $B(h)$, $B(k)$ ε $P_0(A) \setminus \{A\}$, such that $B(h)$ ε $E(h)$ and $B(k)$ ε $E(k)$.*

The reader may check that ML implies non-dictatorship (a super additive and monotonic EF E is *dictatorial* if there exists a player d ε N such that $E(d) = P_0(A)$). We may conclude now an analog to Sen's [21] Paradox of the Paretian Liberal.

Theorem 11. *If E is an acceptable EF, then E violates minimal liberalism.*

(See also Theorem 4.2.4 of [11].) We shall now prove Theorem 11. We make this exception since: i) the proof is short; and ii) the above theorem is a link, at least conceptually, between the Arrow model and the Gärdenfors model.

Proof of Theorem 11. Assume, on the contrary, that there exist distinct players h and k and sets of alternatives, $B(h)$ ε $E(h) \setminus \{A\}$, and $B(k)$ ε $E(k) \setminus \{A\}$. By (17) $B(h) \cup B(k) = A$. Hence there exist x $\varepsilon B(h) \setminus B(k)$, and y ε $B(k) \setminus B(h)$. Therefore, applying (17) again, x ε B for all B $\varepsilon E(N \setminus k)$. Thus, $\{x\}$ ε $E^*(k)$. Similarly, $\{y\}$ ε $E^*(h)$. However, this contradicts (15) and completes the proof. □

4.3. Strongly Consistent Representations of EF's

A GF is strongly consistent if for each profile of preferences the resulting game has an SNE. Thus each ESC SCF is, in particular, strongly consistent (see Definition 12). Another property of strongly consistent GF's is stability, that is, non emptiness of the core (see Theorem 1). We shall now make these remarks precise.

Definition 31. *A GF $\Gamma = (N; S^1, ..., S^n; \pi; A)$ is* strongly consistent *if for each profile of weak preferences $R^N \varepsilon\ K^N$ the game $g(\Gamma, R^N)$ has an SNE.*

Example 3 is strongly consistent. Also, if a GF Γ is strongly consistent, then $C(E^\Gamma, R^N) \neq \Phi$ for every profile of weak orders. This leads to the following definition.

Definition 32. *Let $E: P_0(N) \rightarrow P(P_0(A))$ be an EF. E is* stable *if $C(E, R^N) \neq \Phi$ for all $R^N \varepsilon\ K^N$.*

Thus, we have identified one (necessary) condition for the existence of a strongly consistent representation.

If an EF has a strongly consistent representation, then it is stable. (18) The missing (necessary) condition is maximality (see the discussion after Theorem 8). We recall that an EF E is maximal if it is super additive and, in addition, $E = E^*$. When a GF $\Gamma = (N; S^1, ..., S^n; \pi; A)$ is strongly consistent, then its EF is maximal. Indeed, assume on the contrary, that there exist $T\ \varepsilon P_0(N)$ and $B\ \varepsilon\ E^{\Gamma*}(T) \setminus E^\Gamma(T)$. Consider now a profile R^N such that 1) $x\ P^h\ y$ for all $x\ \varepsilon B$, $y\varepsilon\ A \setminus B$, and $h\ \varepsilon\ T$; and 2) $x\ P^h\ y$ for all $x\ \varepsilon\ A \setminus B$, $y\ \varepsilon\ B$, and $h\ \varepsilon\ N \setminus T$. We claim that the game $g(\Gamma, R^N)$ has no SNE. Indeed, let q^Nbe an arbitrary n-tuple of strategies. If $\pi(q^N)\ \varepsilon\ B$, then $N \setminus T$ has a strategy $s^{N \setminus T}$ such that $\pi(q^T, s^{N \setminus T})\ \varepsilon\ A \setminus B$ (T is not effective for B). Thus, $N \setminus T$ has a profitable deviation from q^N. Similarly, if $\pi(q^N)\ \varepsilon\ A \setminus B$, then T has a profitable deviation from q^N ($N \setminus T$ is not effective for $A \setminus B$!). Again, fortunately, the necessary conditions are also sufficient.

Theorem 12.*An EF E has a strongly consistent representation if and only if it is stable and maximal.*

Theorem 12 is difficult to apply directly because it is difficult to verify stability of EF's. Fortunately, we can formulate the theorem differently. First we need the following definition.

Definition 33. *An EF $E: P_0(N) \rightarrow P(P_0(A))$ is* convex *if*
$[B_h \varepsilon E(S_h), h = 1, 2] \rightarrow [B_1 \cap B_2 \varepsilon E(S_1 \cup S_2)$ or $B_1 \cup B_2\ \varepsilon\ E(S_1 \cap S_2)]$.

There are two basic results on convex EF's: a) A convex EF is stable; and b) A maximal and stable EF is convex (see Peleg [12], Appendix to Chapter 6)). These results enable us to formulate the following theorem.

Theorem 13.*An EF E has a strongly consistent representation if and only if it is convex and maximal.*

5. Recent Results and Open Problems

We shall report in this section two recent results on representations of constitutions under incomplete information, and also outline some possible future continuations of the theory developed in Section 4. The description of the work on incomplete information models shall be brief as this work is rather complicated.

5.1. Representation under Incomplete Information

Let $E: P_0(N) \to P(P_0(A))$ be an EF, that is, a constitution. E is assumed to be common knowledge among the players. The information available to the players is modeled by an *information structure IS = $(T^1, ..., T^n; p^1, ..., p^n)$.* Here T^h is the (finite) set of *types* of player h, $h \, \varepsilon \, N$, and p^h is a probability distribution on $T = T^1 \times ... \times T^n$. We assume that $p^h(t^h) > 0$ for all $t^h \varepsilon T^h$ and all $h \, \varepsilon \, N$. Thus the conditional probabilities $p^h(t^{-h}|t^h)$ are always well defined. (Here $t^{-h} = t^{N \setminus h}$.) These conditional probabilities are called the *beliefs* of the players. We are interested in obtaining (Bayesian Nash) equilibria in pure strategies. Therefore, we represent the constitution by decision schemes. We now provide a precise definition. Let $\Delta(A)$ be the set of all probability measures on A.

Definition 34. *A* decision scheme(DS)*is a function d: $K^N \times T \to \Delta(A)$.*

A DS defines a social choice correspondence (SCC) $H: K^N \times T \to P_0(A)$ by $H(R^N, t) = \{x: d(x; R^N, t) > 0\}$. With the SCC H we associate an EF E^H in a way that generalizes Definitions 8 and 9: A set $Q \, \varepsilon \, P_0(N)$ is *effective* for $B \, \varepsilon \, P_0(A)$ if there exist $R^Q \varepsilon \, K^Q$ and $t^Q \varepsilon \, T^Q$ such that $H((R^Q, R^{N \setminus Q}), (t^Q, t^{N \setminus Q})) \subset B$ for all $R^{N \setminus Q} \varepsilon \, K^{N \setminus Q}$ and $t^{N \setminus Q} \varepsilon \, T^{N \setminus Q}$. Finally, $E^H(Q) = \{B: Q$ is effective for $B\}$ for all $Q \, \varepsilon \, P_0(N)$. The EF of d, E^d, is defined by $E^d = E^H$. And d is a *representation* of E if $E^d = E$.

The (von Neumann-Morgenstern) payoff functions of the players are given by $u^h: A \times T \to Re$, for all $h \, \varepsilon \, N$. We are now able to describe the game with incomplete information

confronted by the players: $M = (N;K, ..., K; T^1, ..., T^n; p^1, ..., p^n; u^1, ..., u^n; d)$ where all the components have already been defined. Now a (pure) strategy of player h is a function $s^h: T^h \rightarrow K \times T^h$ (for all $h \ \varepsilon \ N$). The payoff to type $t^h \varepsilon T^h$ when an n-tuple of strategies $s = (s^1, ..., s^n)$ is used is

$$U^h(s \mid t^h) = \sum p^h(t^{-h} \mid t^h) \sum_{x \ \varepsilon \ A} u^h(x, t) \ d(x; s^1(t^1), ..., s^n(t^n)),$$

where the outer summation is over all $t^{-h} \ \varepsilon \ T^{N \setminus h}$.

(19)

An n-tuple of strategies $s = (s^1, ..., s^n)$ is a *Bayesian Nash equilibrium (BNE)* of M if for all $h \ \varepsilon \ N$, $t^h \varepsilon T^h$, and $(R^h, t^{*h}) \ \varepsilon \ K \times T^h$ we have

$$U^h(s \mid t^h) \geq \sum p^h(t^{-h} \mid t^h) \sum_{x \ \varepsilon \ A} u^h(x, t) \ d(x; s^{-h}(t^{-h}), (R^h, t^{*h})),$$

where, again, the outer summation is over all $t^{-h} \ \varepsilon \ T^{N \setminus h}$.

(20)

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The main result on representation under incomplete information is the following.

Theorem 14. *Let $E: P_0(N) \rightarrow P(P_0(A))$ be a monotonic and superadditive EF, let $(T^1, ..., T^n; p^1, ..., p^n)$ be an information structure, and let $u^1, ..., u^n$ be vNM utilities for the players. Then there exists a decision scheme $d: K^N \times T \rightarrow \Delta(A)$ such that*
1) d is a representation of E ($E = E^d$); and
2) the Bayesian game $M = (N; K, ..., K; T^1, ..., T^n; p^1, ..., p^n; u^1, ..., u^n; d)$ has a Bayesian Nash equilibrium in pure strategies.

The proof uses the concept of the *uniform core* of an EF. Abdou and Keiding [23] proved that the uniform core of a monotonic and super additive EF is nonempty. Keiding and Peleg [24] showed that the uniform core correspondence is a representation of its EF. For a recent lucid discussion of the uniform core the reader is referred to Section 6.4 of [11]. Theorem 14 is due to Peleg and Zamir [25].

Peters, Schröder, and Vermeulen [26] deal also with representations of constitutions under incomplete information. They make the special assumption of private values (the utility function of a player depends only on his type) and prove existence of ex-post NE's.

5.2. Concluding Remarks

In this survey we do not attempt to give an historical account of (the old branch of science of) social choice. We also do not assume any formal knowledge of social choice. We address ourselves to the reader who knows the details of (the formulation of) Arrow's Impossibility Theorem and the Gibbard-Satterthwaite Theorem (and perhaps also heard about Sen's Liberal Paradox), and is more or less convinced that social choice theory is the land of impossibilities or, at least, of second-best solutions. Proofs of the Arrow Impossibility Theorem (AIT) and the Gibbard-Satterthwaite (G-S) Theorem are nowadays abundant; indeed, almost every new book on game theory contains detailed proofs (see, e. g., Peters [27] and Maschler, Solan, and Zamir [28]). But, of course, we shall give a precise formulation of both AIT and G-S in this section and indicate how the foregoing text may resolve some difficulties that they raise. We know that this sounds very ambitious, and indeed the solutions that we offered are conceptually and mathematically somewhat sophisticated. However, for finitely many alternatives and voters, as in our text, everything is "elementary and constructive". Our main tool is the axiomatic method as has been advanced by Arrow. Characterizing our solutions axiomatically gives the reader the possibility to judge clearly what options are available to remove difficulties.

We start our discussion with the G-S Theorem. We shall discuss its implications for social choice functions (see Definition 7). (We assume that the reader is familiar with the context of the G-S Theorem; otherwise, we direct him to the relevant definitions in Section 2.) Let, as usual, A be a set of at least three alternatives and let N be a set of at least two voters. Let $F: L^N \to A$ be an SCF and let R^N be a profile of linear preferences. F is *nonmanipulable* at R^N if R^N itself is a Nash equilibrium of the voting game $g(F, R^N)$ generated by F and R^N. F is nonmanipulable if it is nonmanipulable at every profile of linear preferences. F is *dictatorial* if there exists a player d, a dictator, such that $F(R^N) \, R^d \, x$ for all R^N and for all x in the range of F. The G-S Theorem [29,30] says that if an SCF is nonmanipulable and its range contains at least three alternatives, then it is dictatorial. We have seen above that there do exist non-dictatorial and even anonymous SCF's, which are exactly and strongly consistent (see Definition 12). In what sense does the use of such SCF's offer a solution to the manipulability problem? Let $F: L^N \to A$ be an ESC SCF. For R^N in L^N let $H(R^N)$ be the set of all SNE's with outcome $F(R^N)$. This set is non empty by the definition of ESC. All the players are indifferent between the points of $H(R^N)$. Let $h(R^N)$ be an arbitrary selection from $H(\cdot)$ known to all players (the games $g(F, \cdot)$ are

"noncooperative" in the sense that communication and correlation of strategies are possible, but not binding agreements). The rule "play $h^t(R^N)$ when R^N is the sincere profile", t in N, is self-enforcing in the sense that no coalition has a profitable deviation. Thus, although the rule "vote sincerely" is manipulable by G-S, we have found a different, more sophisticated rule of voting behavior, which cannot be manipulated and still yields the sincere outcome. Another interpretation of the merits of using an ESC SCF is that it allows to punish manipulators, thus making the sincere profile an equilibrium of threats (see Pattanaik [31] and Section 8.5 in [11]).

We now recall Arrow's Impossibility Theorem (AIT). A *social welfare function (SWF)* is a function $V: L^N \to L$. V is *Paretian* if for all $R^N \varepsilon L^N$ and all $x, y \varepsilon A$, the condition $x R^h y$ for all $h \varepsilon N$ implies that $x V(R^N) y$. V satisfies *Independence of Irrelevant Alternatives (IIA)* if for all R^N, $Q^N \varepsilon L^N$ and all $x, y \varepsilon A$, the condition $x R^h y$ if and only if $x Q^h y$ for all $h \varepsilon N$ implies that $x V(R^N) y$ if and only if $x V(Q^N) y$. V is *dictatorial* if there exists a voter d such that $V(R^N) = R^d$ for all $R^N \varepsilon L^N$. AIT [6,32] says that if V is Paretian and satisfies IIA, then V is dictatorial (we have assumed that there are at least three alternatives). If we follow Arrow and define a constitution as a "well-behaved" SWF, then we are left only with dictatorial constitutions. However, we have followed Gärdenfors [7] in Section 4 and found stable "nice" constitutions. In our model all the citizens may exercise their rights precisely as specified by the (relevant) constitution.

We conclude this subsection with the remark that there are still many open problems in the theory of representations of political power structures by stable GF's. For example, we have not succeeded to examine representations by coalition proof Nash equilibrium. Also, there is urgent need to investigate dynamic models of representation.

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