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RISK OF MONETARY GAMBLES: AN AXIOMATIC APPROACH

By

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ABSTRACT. In this work we present five axioms for a risk-order relation defined over (monetary) gambles. We then characterize an index that satisfies all these axioms – the probability of losing money in a gamble multiplied by the expected value of such an outcome – and prove its uniqueness. We propose to use this function as the risk of a gamble. This index is continuous, homogeneous, monotonic with respect to first- and second-order stochastic dominance, and simple to calculate. We also compare our index with some other risk indices mentioned in the literature.

1. INTRODUCTION

The concept of risk is widely used in economics and is one of the main issues one considers when choosing between alternatives. Today, the most common risk indices in economics, especially in finance, are variance, Value at Risk (VaR), and their variants. However, over the years many other risk indices were developed. Although many of them were not given a theoretical foundation, recently a few papers characterized risk measures using the axiomatic approach (see [1],[2],[7]). In this paper we will use a similar axiomatic approach to give a theoretical basis to a certain risk index.

We are looking for a risk index that will reflect the way a lay person understands and perceives risk. That is, we want our index to grasp the concept of risk as it is understood by, say, a private investor or by a lay person reading an economic article in the newspaper. The risk index our axioms lead us to is the expected value of the possible losses in the gamble (i.e., $|\mathbb{E}[min(A,0)]|$, namely, the absolute value of the expectation of the gamble when all the positive prizes are nullified). Of course, we are not the first to offer this index. In fact, already in 1841 Laplace wrote in his "A Philosophical Essay on Probabilities" [5]:

> "In a series of probable events of which the ones produce a benefit and the others a loss, we shall have the advantage which results from it by making a sum of the products of the probability of each favorable event by the benefit which it procures, and subtracting from this sum that of the products of the probability of each unfavorable event by the loss which is attached to it. If the second

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> sum is greater than the first, the benefit becomes a loss and hope is changed to fear."

Laplace is obviously referring to the expectation of the gamble. However, the way he presents it indicates that what one should fear in a gamble is exactly our index, namely, the loss expectation. More recently, Brachinger and Weber[4] mention this measure in the category of "naive measures for which no strict theoretical foundations have been given" and write that "two dimensions which appear to determine perceived risk have been identified: amount of potential loss and the probability of occurrence of loss" Our measure – i.e., their product – combines these two dimensions in a simple way.

As mentioned, the method we use is the axiomatic approach. We analyze situations of a choice between alternative gambles, and try to pinpoint the risk involved. The situations we analyze are such that we have a strong intuition regarding the risk order of the alternatives, an intuition that we believe is common to all. We formalize our conclusions as mathematical requirements that our risk index must satisfy.

For example, imagine you are standing on one side of a highway and you realize that there is a gold bar on its other side. What affects the risk in this "gamble" of crossing the highway to get the gold? In our opinion, the risk is determined by the width of the highway, the traffic, etc., but not by the size of the gold bar. In other words, we believe that two such gambles with different gold bars – all other things being equal – have the same risk. This example leads us to conclude that in fact risk is based solely on the negative prizes and the positive ones do not affect it. Of course, the positive prizes greatly affect the attractiveness of the gamble and the decision whether to take one gamble or another, but they do not change a gamble's risk.

We are searching for an objective measure of riskiness, i.e., an index that is derived directly from the gamble's distribution (prizes and probabilities), and needs no other information such as the agent's utility function, his wealth status, etc. Naturally, it is likely that different individuals will evaluate a given risk differently: one might say that a "$100 risk" is a considerable risk, and another that it is insignificant. However, this does not mean that they disagree on the magnitude of the risk. To take an example from another realm, not everyone will agree that Mt. Fuji is a high mountain, but any reasonable person will agree that it is higher than Mt. Rushmore.[1] Thus, "risky" can be subjective (like "high"), and yet "riskier" can be objective (like "higher").

Our concept of risk is different from the one we would call uncertainty, even if we don't formally define what uncertainty means. Shifting a gamble (i.e., adding a guaranteed amount of money) might change the risk of the gamble, but intuitively seems not to change the uncertainty involved. Similarly, enlarging only the positive prizes seems to affect the uncertainty of the gamble, but not its risk. In addition, when one is offered a lottery ticket as a gift – i.e., one either wins a prize or nothing happens – no rational person would consider it risky, although the uncertainty might be great. On the other hand, when considering jumping from a flying airplane, – we think it will be widely accepted that the higher the airplane is, the greater the risk and the lower the uncertainty involved. These examples convince us that risk

[1]Mt. Fuji is 3,776 meters high; Mt. Rushmore is 1,745 meters high.

and uncertainty are two separate and distinct concepts and should be dealt with independently.

The paper is organized as follows. Chapter 2 defines various terms such as gamble, risk order, etc. Chapter 3 introduces the axioms and their basic logic. Chapter 4 discusses the main result, namely, that loss expectation is the only index that satisfies all of the axioms. Chapter 5 gives a survey of the related literature, Chapter 6 concludes with a discussion, and Chapter 7 holds all the proofs to the various claims made throughout the paper.

2. Basic definitions

A **gamble** (or a lottery) is a probability distribution over $\mathbb{R}$. We call the possible outcomes "prizes," and for simplicity we restrict our discussion to gambles with finitely many prizes. We denote by $[a_1, p_1; a_2, p_2; ...; a_n, p_n]$ the gamble A where we get the prize a_i with a probability of p_i.

A **risk relation** ("**risk**" for short) is a complete and transitive relation on the space of gambles, denoted by $\succeq$; $A \succeq B$ will be interpreted as "A is at least as risky as B."

We say that a function R from the space of gambles to the real line represents the risk relation $\succ$, and call R a **risk measure** or a **risk index** ("**riskiness**" for short), if R represents $\succ$ in the usual sense, i.e., $A \succ B \iff R(A) > R(B)$.

First, for convenience, we denote by $[a, p]$ the gamble where we get a with probability p or we get 0 with probability $1 - p$, i.e., $A = [a, p; 0, 1 - p]$. Next, we use A_- to refer to the "losing" part (the non-positive part) of the gamble A, i.e.,

$$A_- = [min(a_1, 0), p_1; min(a_2, 0), p_2; ...; min(a_n, 0), p_n]$$

Last, $[A, p; B, 1 - p]$ is the simple gamble with the same probability distribution as the gamble in which gamble A occurs with probability p and gamble B occurs with probability $1 - p$, i.e, if $A = [a_1, p_{A1}; a_2, p_{A2}; ...; a_n, p_{An}]$ and $B = [b_1, p_{B1}; b_2, p_{B2}; ...; b_m, p_{Bm}]$, then

$$[A, p; B, 1-p] = [a_1, p \cdot p_{A1}; a_2, p \cdot p_{A2}; ...; a_n, p \cdot p_{An}; b_1, (1-p) \cdot p_{B1}; b_2, (1-p) \cdot p_{B2}; ...; b_m, (1-p) \cdot p_{Bm}]$$

3. The axioms

We propose a list of axioms that a **risk relation** should satisfy.

Axiom 1 (Positive Response to Loss [PRL])**.** *Let A, B be two gambles such that $A = [-a, p]$ and $B = [-b, p]$, where $a, b \geq 0$ and $p \in (0, 1]$. Then $a > b \Longrightarrow A \succ B$.*

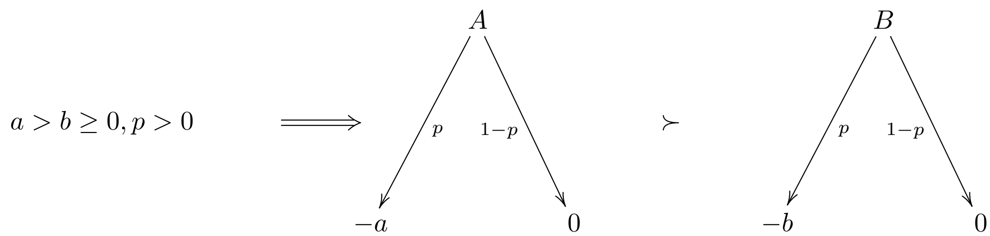

FIGURE 3.1. Illustration of Axiom 1 (Positive Response to Loss)

This axiom embodies the idea that when taking a gamble (as defined in our model), losing money is a risk. Specifically, it means that given two gambles with an equal probability of losing, the higher the loss, the higher the risk.

Axiom 2 (Independence [In])**.** *Let* A, B, C *be gambles, and* $p \in [0, 1)$*. Then* $A \succeq B \iff [A, p; C, 1-p] \succeq [B, p; C, 1-p]$.

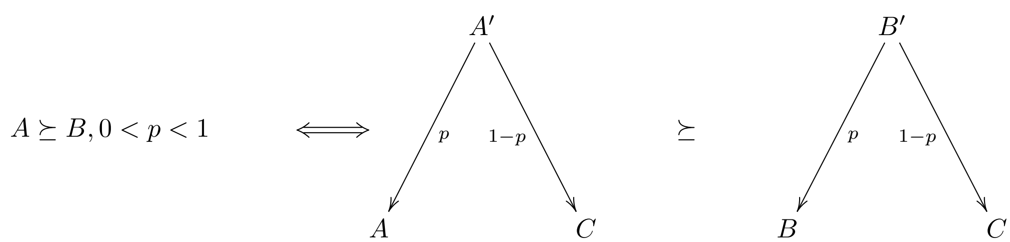

FIGURE 3.2. Illustration of Axiom 2: Independence

Von Neumann and Morgenstern's expected utility theorem is based on this axiom. In our opinion, the same argument that justifies this axiom in the utility context applies here as well: the risk order of two compound gambles should not depend upon the identical parts of those gambles. Consider the following compound gambles $\bar{A}$ and $\bar{B}$, where in the first stage we toss a coin. If in both gambles we play the same after "head," then the order of $\bar{A}$ and $\bar{B}$ (risk-wise) should follow the order of the gambles played if the coin comes up "tail."

Axiom 3 (Continuity [Co])**.** *For any three gambles A,B,C such that* $A \succ B \succ C$*, there exists* $p \in (0, 1)$ *s.t.* $[A, p; C, 1-p] \sim B$*.*

This axiom is also borrowed from Von Neumann and Morgenstern. It is called the Continuity axiom for it means that given any two gambles $A \succ C$, and every

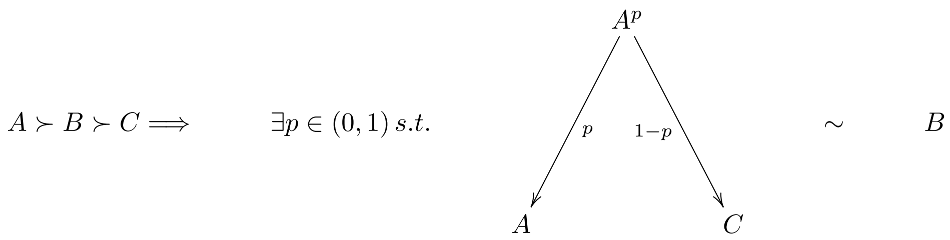

FIGURE 3.3. Illustration of Axiom 3: Continuity

gamble B whose risk lies between them (i.e., $A \succ B \succ C$), we can build a compound gamble of A and C that will be just as risky as B. Accepting Independence and Continuity, we can apply Von Neumann and Morgenstern's proof of the expected utility theorem to get a representation of the risk of a gamble as the expected risk of its prizes (see, for example, [9]). This means that if R is a risk index defined on the space of gambles, there is a risk function r defined on the real line where $R([a_1, p_1; a_2, p_2; ...; a_n, p_n]) = \sum_{i=1}^{n} p_i r(a_i)$.

Axiom 4 (Invariance to Possible Gains [IPG])**.** *Let A,B be two gambles; then* $A_- = B_- \Longrightarrow A \sim B$.

Briefly, this axiom states that the essence of the risk (in monetary gambles) is the possible loss of money, and positive prizes have no effect on the risk of the gamble. According to Brachinger and Weber [4], "It has been conventional wisdom ... that risk is the chance of something bad happening." Aumann and Serrano [2] claim that a gamble that has no losses has no risk. To emphasize the idea underlying this axiom, consider the "gamble" mentioned in the introduction: the gamble one faces when wishing to cross a busy road to get to a gold bar lying on the other side. In our mind, the risk has to do with the width of the road, the number of cars, their speed, etc, but the size of the gold bar has nothing to do with the risk involved in crossing the road.

Axiom 5 (Invariance to Translation [IT])**.** *Let* $A = [-a, q_1]$, $B = [-b, q_2]$, $a, b \geq 0$, $c < min(a, b)$, $p < min(q_1, q_2)$. *Then* $A \succ B \Longrightarrow [-a, q_1 - p; -a + c, p; 0, 1 - q_1] \succ [-b, q_2 - p; -b + c, p; 0, 1 - q_2]$.

Axiom 5 says that when comparing two binary gambles, adding the same loss/gain (amount and probability) to both should not change the risk order of the two: the riskier gamble is still riskier. Of course, following our logic (and axiom IPG), Axiom 5 should hold as long as it does not change a negative prize into a positive one.

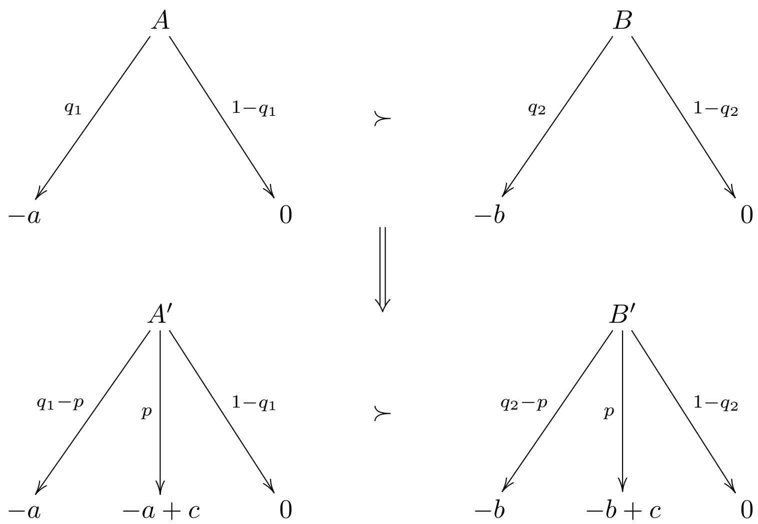

FIGURE 3.4. Illustration of Axiom 5: Invariance to Translation

To emphasize the reasoning behind this axiom, compare the risk in the following two (unattractive) gambles. In the first gamble, if Djokovic wins the next Australian Open (which he won five times in the past) you either lose \$25 or nothing happens. In the second gamble you either lose \$45 if he wins the next Wimbledon (which he won twice) or nothing happens. Now, say that after you decide the risk order of the two gambles, you are told that if he wins both Wimbledon and the Australian Open you lose an extra \$5. Does it change the risk ordering of the bets? In our mind, the answer is no; It should not change the risk order of the two "gambles."

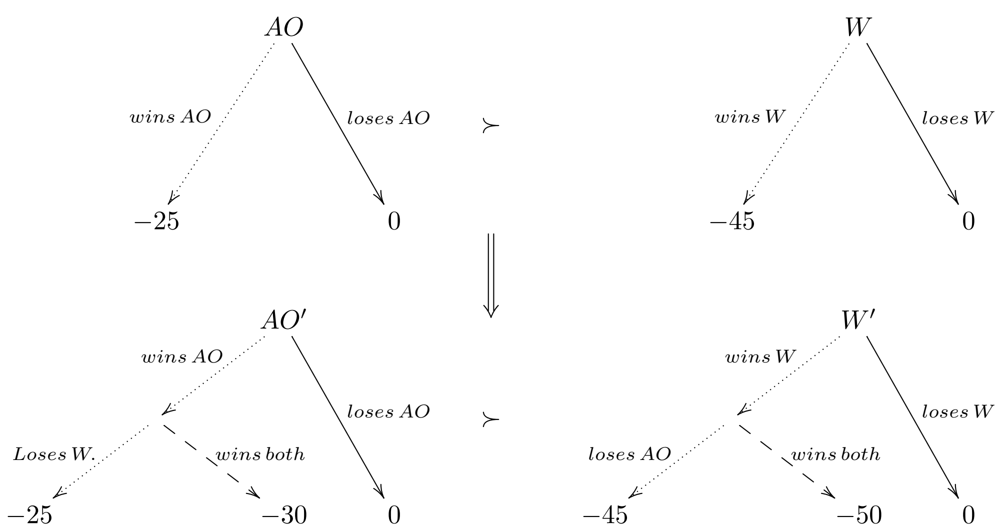

FIGURE 3.5. example 1 of Axiom 5

In the previous example, the setup was such that the probabilities were subjective. However, the logic behind the axiom is valid with objective probabilities as well. Let us look at the following two (again, unattractive) gambles. In gamble A, you throw a die and if the result is even (probability of $3/6$) you either lose \$9 or you lose nothing. In gamble B you either lose \$6 if the result is larger than 2 (probability of $4/6$) or you lose nothing. Now, which is riskier, gamble A or gamble B? Whatever the answer is, we believe that adding the condition of losing an extra \$1 when the result of both gambles is 6 shouldn't change the risk order of the two gambles, i.e., if A is riskier, then so is $\bar{A}$, where you might lose \$9 (with probability $2/6$) or \$10 (with probability of $1/6$) or lose nothing.

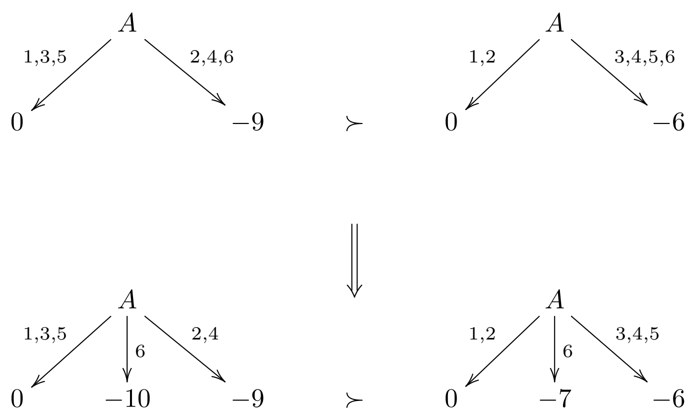

Figure 3.6. Example 2 of Axiom 5

This axiom may seem related to the Independence axiom. Indeed, the reasoning behind the axioms is somewhat similar as both claim that the "identical part" of two gambles should not affect their risk order. However, the "identical part" is determined differently in each axiom. Moreover, each axiom is irrelevant in the setup in which the other axiom is defined. Furthermore, as can be seen in the next chapter, there are indices that satisfy one axiom and violate the other.

4. Our risk index

We are now ready to state our main result.

Theorem 6. *A risk relation $\succ$ over the space of gambles satisfies our five axioms (Positive Response to Losses, Independence, Continuity, Invariance to Possible Gains, and Invariance to Translation), if and only if it can be represented by* $R([a_1, p_1; a_2, p_2; ...; a_n, p_n]) = -\sum_{a_i<0} p_i a_i$.

Every other risk measure satisfying our five axioms is ordinally equivalent to R. Moreover, all these axioms are essential for the uniqueness of our index. Eliminating any of the axioms would lead to other possible risk relations that are not ordinally equivalent to R. Specifically, let $A = [a_1, p_1; a_2, p_2; ...; a_n, p_n]$, where $a_1 < a_2 < ... < a_n$, and define the following:

$$R_1(A) := -R(A) = \sum_{a_i<0} p_i a_i$$

$$R_2(A) = -\sum_{a_i<0} a_i f(i, p_i) \text{ where } f(i, p_i) = \begin{cases} p_i & i = 1, 2 \\ p_i^2 & otherwise \end{cases}$$

$$R_3(A) := max(0, -a_1)$$

$$R_4(A) := -\sum_{a_i} p_i a_i$$

$$R_5(A) := \sum_{a_i<0} p_i a_i^2$$

Thus, as can be seen in Table 1, each of the above indices satisfies all the axioms but one.

Table 1. Independence of the Axioms

axiom \ risk index	R_1	R_2	R_3	R_4	R_5
Positive Response to Loss	X	√	√	√	√
Independence	√	X	√	√	√
Continuity	√	√	X	√	√
Invariance to Possible Gains	√	√	√	X	√
Invariance to Translation	√	√	√	√	X

Our index satisfies a number of useful properties. Following is a partial list. Throughout this section, A is the gamble $[a_1, p_1; a_2, p_2; ...; a_n, p_n]$.

Objectivity: The risk measure thus defined is objective and depends entirely on the gamble itself. It is not necessary to know who the decision maker is, what his utility function is, how wealthy he is, etc.

Dimension: It is measured in money, since it is a weighted mean of the possible losses, with their probabilities as the weights.

Maximal loss: We think that the risk of a gamble depends on possible losses (hence our axiom IPG) and therefore should not exceed the risk of its worst possible outcome, i.e., $R(A) \leq R([min(A), 1])$. Indeed, our risk satisfies that property. If $min(A) \geq 0$ then $R(A) = 0 = R([min(A), 1])$; otherwise, $R(A) = -\sum_{a_i<0} p_i a_i \leq -min_{a_i} a_i = R([min(A), 1])$. This is in contrast to Foster and Hart's [6] riskiness of a gamble that always exceeds its maximal possible loss. Their measure and our index are further discussed below.

Positive homogeneity: For any gamble A and positive constant α, define by αA the gamble where for each i, one gets αa_i with probability p_i. Then we have $R(\alpha A) = -\sum_{\alpha a_i<0} p_i \cdot \alpha a_i = -\alpha \sum_{a_i<0} p_i \cdot a_i = \alpha R(A)$.

Compound gambles: For every two gambles A, B and $p \in (0,1)$: $R([A,p;B,1-p]) = pR(A) + (1-p)R(B)$. Let $C = [A,p;B,1-p]$; then this property follows directly from $R(C) = -\sum_{c_i<0} p_{C i} c_i = -\sum_{a_i<0} p \cdot p_{A i} a_i - \sum_{b_j<0} (1-p) \cdot p_{B j} b_j = pR(A) + (1-p)R(B)$.

Stochastic dominance: This may be the most accepted concept of riskiness. An order Q is said to be first- (second-) order monotonic if $Q(A) < Q(B)$ whenever A first- (second-) order dominates B. We use the notation M-FOD (M-SOD) to indicate this property. If it is a weak inequality, i.e., $Q(A) \leq Q(B)$, then the order is said to be weakly first- (second-) order monotonic with respect to stochastic dominance. This is denoted by WM-FOD and WM-SOD. Our index is both WM-FOD and WM-SOD (for the proof see Section 7.2).

Continuity: Our risk measure is continuous.

5. The literature

The literature mentions many different risk measures (see, for example, [4] and [3]). Our measure, "the expected value of loss," is mentioned there in the category of "naive measures for which no strict theoretical foundations have been given."

As pointed out in [2], many of the known measures are not monotonic with respect to stochastic dominance, i.e., all the risk measures based on the spread of the gamble (variance, standard deviation, etc.), standard deviation divided by mean, etc. It seems that many of the known measures, specifically those used in practice, overlook certain aspects of the concept of risk.

Artzner, Delbaen, Eber, and Heath [1] introduced four axioms characterizing "coherent" measures of risk. Our risk measure satisfies three of these axioms, but not their "Translation" axiom.[2] Our order also satisfies an extra axiom presented in their paper, though it is not required of a measure in order for it to be considered "coherent."

We now discuss in detail two risk measures that are monotonic with respect to stochastic dominance.

5.1. Aumann and Serrano's index.

Aumann and Serrano [2] define two axioms that give rise to a unique number $R(g)$ associated with every gamble and interpreted as its risk. Their duality axiom, the more cardinal axiom, projects the partial order induced by Arrow–Pratt's risk aversion of agents (represented by utility functions) to a risk order over the space of gambles. This is done by considering acceptance or rejection of gambles by different risk-averse agents. We believe that accepting or rejecting a gamble does not necessarily correlate completely with the gamble's riskiness, as, for example, it takes into account the positive prizes that we think do not affect the risk.

In their section "Extending the domain" (p. 821), Aumann and Serrano state that "when there are no negative values, there is no risk." We fully agree with them on this point, and this leads us to our Invariance to Possible Gains axiom, i.e., possible positive outcomes do not affect the gamble's risk. However, their index

[2] Their Translation axiom says that for any gamble A and real number a, $R(A+a) = R(A) - a$. Our measure doesn't satisfy this axiom: if A has only positive prizes, then $R(A) = 0 = R(A+a)$.

does not satisfy this axiom. In our opinion, this is due to their focus on accepting and rejecting gambles. Even if positive outcomes do not affect the risk of a gamble, surely they greatly affect the decision of accepting or rejecting it.

Now let us consider Aumann and Serrano's results concerning diluted gambles (p. 819), defined as compound gambles in which with probability p one receives the original gamble and with probability $1-p$ one receives 0 (i.e., nothing happens). By our annotation, if A is the original gamble, $[A, p; 0, 1-p]$ is a diluted gamble. According to Aumann and Serrano, a diluted gamble has the same risk as the original. In the gamble $L_1 = [-1, 0.5; 1.1, 0.5]$ mentioned in their paper, $R^{AS}(L_1) =$ \$11.01. Take a series of its dilutions, $L_n = [A, \frac{1}{n}; 0, 1-\frac{1}{n}]$. For example, $L_{1000} = [-1, 0.0005; 0, 0.999; 1.1, 0.0005]$. Then we have that $11.01 = R^{AS}(L_1) = R^{AS}(L_2) = \ldots = R^{AS}(L_n) = \ldots$, even though $L_n \overset{n\to\infty}{\to} 0$. Specifically, it doesn't seem logical to us that $L_1 \sim L_{1000}$ (see Figure 6 for illustration), since playing the latter once seems to have almost no risk. Now, the expected value is lowered by the dilution, and the risk is supposedly the same. So how come, as Aumann and Serrano maintain, any expected utility maximizer will accept or reject both the gamble and its dilution? Isn't it more likely that the risk has changed as well?

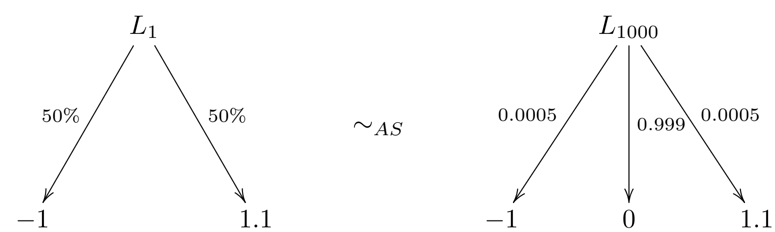

FIGURE 5.1. Diluted gambles example

Our index does not satisfy Aumann and Serrano's duality axiom. Specifically, as mentioned above, duality leads to equal risk of a gamble and its dilution. Our index will attach a lower risk to a diluted gamble, provided there is a possible loss. Let $A = [a_1, p_1; a_2, p_2; \ldots; a_n, p_n]$, and let $q \in (0, 1)$. Then $R([A, q; 0, 1-q]) = -\sum_{a_i<0} q \cdot p_i a_i = -q \sum_{a_i<0} p_i a_i = qR(A)$. Hence, as long as A has some risk (there are possible negative outcomes), the gamble will be riskier than its dilution.

5.2. **Foster and Hart's measure.** Foster and Hart (FH) [6] suggest a measure of risk based on assessing the long-run consequences of accepting series of gambles. In order to determine a gamble's risk, they consider a series of gambles based on it, and find a clear-cut wealth level below which you go bankrupt if you accept this infinite series. They use that wealth level as the risk of the original gamble. Riedel and Hellmann [8] extend it to continuous gambles by reverting to the maximal loss possible whenever the FH riskiness cannot be computed. In our model, a gamble happens once: we choose to accept or reject it, and do not consider it infinite. That distinction in our basic assumptions gives rise to the following differences between their measure and our index.

First, as Aumann and Serrano mention in their paper [2], FH's measure of riskiness is not continuous.

Second, FH's measure does not satisfy our Invariance to Possible Gains axiom. Take the following two gambles: $A = [-45, 0.5; 90, 0.5]$ and $B = [-45, 0.5; 50000, 0.5]$.

Accepting Invariance to Possible Gains means $A \sim B$. However, $R^{FH}(A) \sim 90$ while $R^{FH}(B) \sim 45$. Moreover, let $C = [-1, 0.5; 1.011, 0.5]$; then, according to FH, C is twice as risky as B. If one has \$50, it is supposedly risky to accept C and safe to accept B. In our view, however, although many agents may choose B, accepting C once seems intuitively less risky.

Furthermore, as they examine long series of gambles, they find that the riskiness of a gamble and its dilutions are equal and will lead to the same result (bankruptcy/infinite wealth), but in different time expectancies. However, we examine a one-time gamble and therefore find that its risk can change when it is diluted.

Lastly, if we try to extend their logic beyond their domain of gambles, then any gamble g with a negative expectation is infinitely risky: play the gamble an infinite number of times and bankruptcy is certain. This does not, however, seem to be the case when we consider the gamble as a one-time game. For example, we believe $D = [-0.01, 0.01; 0, 0.99]$ has very little risk and is intuitively less risky than the gamble $E = [-100, 0.5; 105, 0.5]$.

6. Discussion

6.1. **Zero as a special benchmark.** Throughout this work, zero as a possible outcome has a special role. In particular, our fourth axiom – Invariance to Possible Gains – distinguishes between the possible risky prizes (negative ones) and the possible non-risky prizes (positive ones). Zero can be replaced by any real number and our results will still be valid. Let t be the special new threshold; then the risk of a gamble A will be $R(A-t)$ where R is our index. One may well argue that zero plays the role of an arbitrary cutoff.

However, as mentioned above, Aumann and Serrano [2], as well as Foster and Hart[6], restrict their domain to gambles with possible losses due to a similar distinction, namely, that there must be some possible loss in order for a gamble to be risky. Moreover, loss aversion is a well-known and well-documented phenomenon (see, for example, [10]). One can also interpret this special treatment of zero as a comparison of a given gamble to the status quo. In conclusion, zero, though arbitrary in a sense, has some appeal as a special benchmark.

6.2. **cVaR and our axioms.** In light of the last section, one might feel more comfortable with a cutoff for the probabilities instead of the prizes. For example, instead of defining the risk as stemming from the negative prizes, or from possible outcomes below a certain threshold, we can define it as coming from the worst α-quantile of the gamble for a specific $\alpha \in (0, 1)$. A natural candidate is the "Conditional Value at Risk" (cVaR or Expected Shortfall). Though it fails to satisfy three of our axioms, it is, in a sense, a dual index. It does not fulfill Invariance to Possible Gains but it does satisfy a modified version of this axiom where the cutoff is applied to the probabilities instead of the prizes. Roughly, if two gambles agree on the α-quantile, then they have the same risk. Another axiom cVaR does not satisfy is the Independence axiom. However, it does satisfy the dual independence suggested by Yaari [11]. Lastly, $cVaR_\alpha$ also doesn't satisfy our Invariance to Translation axiom. However, it does hold a slightly modified version of this axiom where p is required to be less than α.

So, it seems that a similar structure of axioms, or dual axioms, where the roles of the probabilities and the prizes are in some sense switched, might lead to a dual index, possibly cVaR.

7. Proofs

Let $\succ$ be a risk order over the space of gambles, that satisfies our five axioms. From (In) and (Co) we get, by means of the Expected Utility Theorem (see, for example, [9]), that $\succ$ can be represented by an index R of the form $R([a_1,p_1;a_2,p_2,...,a_n,p_n]) = \sum_{a_i} p_i r(a_i)$ where r is a function from $\mathbb{R}$ to $\mathbb{R}$, interpreted as the risk of a given prize.

For any $x \in \mathbb{R}$, we denote by x^* the gamble $[x,1]$ where we get x for certain. Now, if $x > 0$ then $x^*_{-} = 0^*_{-}$ and so from (IPG) we have that $x* \sim 0*$. However, $x* \sim 0* \Longrightarrow R(x^*) = R(0^*) \Longrightarrow r(x) = r(0)$. Now, we are trying to characterize the order relation induced by R, and so w.l.o.g. we can set $r(0) = 0$.

Next, if $x < 0$, then (PRL) implies that $x* \succ 0^* \Longrightarrow R(x^*) > R(0^*) \Longrightarrow r(x) > r(0) = 0$. Moreover, if $x < y < 0$, it follows from (PRL) that $r(x) > r(y) > 0$, and so r is monotonic.

We now have an "expected risk" representation: $R(A) = \sum_{a_i} p \cdot r(a_i)$, with $r : \mathbb{R} \to \mathbb{R}$ such that r is 0 on the positive half of the real line and is a strictly monotonic (decreasing) function on the negative half, i.e., $x > 0 \Rightarrow r(x) = 0$ and $x < y < 0 \Rightarrow r(x) > r(y) > 0$.

Next, we want to prove that r is linear on the negative half of the real line. To do so, let $a < b < 0$, $x < y < 0$ s.t. $b - a = y - x$ and define $h = [r(a) - r(b)] - [r(x) - r(y)]$. Note that by what we have seen up to now, we have $r(a) > r(b) > 0$, $r(x) > r(y) > 0$. We will now assume that $h > 0$ and use (IT) to derive a contradiction.

Let $t = min(1,h)$, $\epsilon = \frac{1}{2[1+r(x)]}$. Note that $t, \epsilon \in (0,1)$ and so $0 < t\epsilon < 1$. Further, choose an arbitrary $q_2 \in (0,1)$, set $q_1 = (1-t\epsilon)q_2\frac{r(y)}{r(x)} < q_2$, and define $L_1 = [x,q_1], L_2 = [y,q_2]$.

Then $R(L_1) = q_1 r(x) = (1-t\epsilon)q_2\frac{r(y)}{r(x)}r(x) < q_2 r(y) = R(L_2)$ and so $L_1 \prec L_2$. Applying (IT) with $c = a - x = b - y$[3] (p will be defined later) we get:

$L_1^{'} = [a,p;x,q_1-p;0,1-q_1] \prec L_2^{'} = [b,p;y,q_2-p;0,1-q_2]$. Equivalently, $R(L_1^{'}) - R(L_2^{'}) < 0$.

Now $R(L_1^{'}) - R(L_2^{'}) = pr(a) + (q_1-p)r(x) - [pr(b) + (q_2-p)r(y)] = p \cdot \{[r(a) - r(b)] - [r(x) - r(y)]\} + q_1 r(x) - q_2 r(y) = ph - t\epsilon q_2 r(y)$.

However, if we let $p = \frac{r(y)}{1+r(x)}q_2$[4] we get $R(L_1^{'}) - R(L_2^{'}) = \frac{r(y)}{1+r(x)}q_2 h - \frac{t}{2[1+r(x)]}q_2 r(y) = (h - \frac{t}{2})\frac{q_2 r(y)}{1+r(x)} > 0$, a contradiction.

If $h < 0$ we can define $\bar{h} = -h$ and proceed as above in order to get the contradiction.

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[3] $c = a - x \overbrace{<}^{a<0} -x$, $c = b - y \overbrace{<}^{b<0} -y$, and hence $c < min(-x,-y)$, as required by IT.

[4] $q_1 = (1-t\epsilon)q_2\frac{r(y)}{r(x)} = [1 - \frac{t}{2[1+r(x)]}]q_2\frac{r(y)}{r(x)} = \frac{2+2r(x)-t}{2r(x)}\overbrace{\frac{r(y)}{1+r(x)}q_2}^{p} = \frac{2r(x)+2-t}{2r(x)}p = [1 + \overbrace{\frac{2-t}{2r(x)}}^{>0}]p > p$. Thus $p < q_1 < q_2$ and we can indeed apply IT.

All in all, we proved that given $a < b < 0$, $x < y < 0$ s.t. $b - a = y - x$, we must have that $r(a) - r(b) = r(x) - r(y)$.

Before we continue, note that r is a strictly monotonic (decreasing) function on the negative half of the real line, and so it is continuous almost everywhere, and has only "jump" discontinuities. However, we are actually interested in the preference order represented by r (we only care about the ordinal properties of r) and so we can assume w.l.o.g. that r is in fact continuous on the negative part of the real line. In addition, since $r(0) = 0$ we can assume that $r(x) \underset{x \to 0_-}{\longrightarrow} 0$ as well and so, in fact, r can be assumed to be continuous along the real line.

Conclusion 7. Thus far, we have that:

(1) r is continuous.
(2) For any $a < b < 0$, $x < y < 0$, the following is true: $x - y = a - b \Rightarrow r(x) - r(y) = r(a) - r(b)$.

Now, let $a, x, y < 0$ s.t. $x - y = a$. Define $b_n = -\frac{1}{n}$ and $y_n = y - \frac{1}{n}$ to get $x - y_n = x - (y - \frac{1}{n}) = x - y - (-\frac{1}{n}) = a - b_n$. However, by part 2 of conclusion 7 above, we have that $r(x) - r(y_n) = r(a) - r(b_n)$. Applying part 1 of conclusion 7, we have that $x = y + a \Longrightarrow r(x) = r(y) + r(a)$. That means that for any $n \in \mathbb{N}$, $r(-n) = n \cdot r(-1)$, and so in turn for any $q \in \mathbb{Q}_-$, $r(q) = q \cdot r(-1)$. From the continuity of r we get for any negative real number x that $r(x) = x \cdot r(-1)$. Hence r is linear.

As before, we care only about the ordinality of r and so we can assume w.l.o.g. that $r(-1) = 1$, and since $r(0) = 0$ we get our main result, namely, that every risk order R satisfying our axioms can be represented (ordinally) by the following formula: $R(A) = -\sum_{a_i<0} p_i \cdot a_i$.

7.2. **Proof of WM-FOD and WM-SOD.**

7.2.1. *WM-SOD.* Let A, B be two gambles where A second-order dominates B, or $\forall x : \int_{-\infty}^{x} [F_B(t) - F_A(t)]dt \geq 0$. Note that $F_A(x) = P(A \leq x) = \sum_{a_i \leq x} p_i$ and therefore $\int_{-\infty}^{x} F_A(t)dt = -\sum_{a_i \leq x} a_i p_i$. Hence, for $x = 0$ we get $\int_{-\infty}^{0} F_A(t)dt = -\sum_{a_i \leq 0} a_i p_i = R(A)$. Now, if gamble A second-order dominates gamble B then, specifically, $R(B) - R(A) = \int_{-\infty}^{0} [F_B(t) - F_A(t)]dt \geq 0$. Hence, $R(A) \leq R(B)$ or $A \preceq B$. Thus R is weakly monotonic with respect to second-order stochastic dominance.

7.2.2. *WM-FOD.* If gamble A first-order dominates gamble B, then it also second-order dominates it. Hence, as we saw above, $A \preceq B$, and so R is weakly monotonic with respect to first-order stochastic dominance.

7.3. **Proof of the independence of the axioms.**

7.3.1. *Axiom I: Positive response to loss.* Let A be a gamble and $R_1(A) := -R(A) = \sum\limits_{a_i<0} p_i a_i$.

- PRL does not hold. If $A = [-a, p]$ and $B = [-b, p]$, where $a, b \geq 0$ and $p \in (0, 1]$, then $a > b \Longrightarrow R_1(A) = -ap < -bp = R_1(B)$.

- In holds. Let A, B, C be gambles, and $p \in [0,1)$. Now, our R satisfies In and $R_1(A) = -R(A)$ so $R_1(A) > R_1(B)$ iff $R(A) < R(B)$. However, since R satisfies In, $R(A) < R(B)$ iff $R([A,p;C,1-p]) < R([B,p;C,1-p])$, which means that $R_1([A,p;C,1-p]) > R_1([B,p;C,1-p])$.
- Co holds. Let *A,B,C* be three gambles such that $R_1(A) > R_1(B) > R_1(C)$. Now, $R = -R_1$ so $R(C) > R(B) > R(A)$. We know that R satisfies Continuity, and hence there exists $p \in (0,1)$ s.t. $R([A,p;C,1-p]) = R(B)$. Using $R = -R_1$ again we get $R_1([A,p;C,1-p]) = R_1(B)$.
- IPG holds. Let $A_- = B_-$ with $A = [a_1,p_1;a_2,p_2;...;a_n,p_n]$ and $B = [b_1,q_1;b_2,q_2;...;b_m,q_m]$; then, by the definition of R_1, we have $R_1(A) = \sum_{a_i<0} p_i a_i \overbrace{=}^{A_-=B_-} \sum_{b_i<0} q_i b_i = R_1(B)$.
- IT holds. Let $A = [-a,q_1]$, $B = [-b,q_2]$, $a,b \geq 0$, $c < min(a,b)$, $p < min(q_1,q_2)$, and let $R_1(A) > R_1(B)$. That means that $R(B) > R(A)$, which in turn means that $R([-b+c,p;-b,q_2-p;0,1-q_2]) > R([-a+c,p;-a,q_1-p;0,1-q_1])$. Substituting $-R_1(A)$ for $R(A)$ and multiplying by minus one will yield $R_1([-a+c,p;-a,q_1-p;0,1-q_1]) > R_1([-b+c,p;-b,q_2-p;0,1-q_2])$, as required.

7.3.2. *Axiom II: Independence.*

Let $A = [a_1,p_1;a_2,p_2;...;a_n,p_n]$ be a gamble with $a_1 < a_2 < ... < a_n$ and define $R_2(A) = -\sum_{a_i<0} a_i f(i,p_i)$ where $f(i,p_i) = \begin{cases} p_i & i = 1,2 \\ p_i^2 & otherwise \end{cases}$

- PRL holds. Let $A = [-a,p]$ and $B = [-b,p]$, where $a,b \geq 0$ and $p \in (0,1]$. Now assume that $a > b$. Then $R_2(A) = ap > bp = R_2(B)$.
- In doesn't hold. Let $A = [-10,0.5], B = [-15,0.25;-5,0.25;0,0.5], C = [-10,0.2]$, and $p = 0.5$. Then $R_2(A) = 5 = R_2(B)$. However, $A' = [A,p;C,1-p] = [-10,0.35], B' = [-15,0.125;-10,0.1;-5,0.125;0,0.65]$, and so $R_2(A') = 3.5$ while $R_2(B') = 15 \cdot 0.125 + 10 \cdot 0.1 + 5 \cdot 0.125^2 = \frac{189}{64} < 3$.
- Co holds. Let *A,B,C* be three gambles such that $R_2(A) > R_2(B) > R_2(C)$. Now $R_2([A,q;C,1-q])$ is a continuous function of q, and we have that $lim_{q\to0} R_2([A,q;C,1-q]) = R_2(C)$ and $lim_{q\to1} R_2([A,q;C,1-q]) = R_2(A)$, and so there exists $p \in (0,1)$ s.t. $R_2([A,p;C,1-p]) = R_2(B)$.
- IPG holds. Let $A_- = B_-$. Then by definition of R_2, we have $R_2(A) = -\sum_{a_i<0} a_i f(i,p_i^A) = -\sum_{b_i<0} b_i f(i,p_i^B) = R_2(B)$.
- IT holds. Let $A = [-a,q_1]$, $B = [-b,q_2]$, $a,b \geq 0$, $c < min(a,b)$, $p < min(q_1,q_2)$, and let $R_2(A) > R_2(B)$. That is true iff $aq_1 > bq_2$. Now, let $A' = [-a+c,p;-a,q_1-p;0,1-q_1]$ and $B' = [-b+c,p;-b,q_2-p;0,1-q_2]$. Then $R_2(A') = (a-c)p + a(q_1-p) = aq_1 - cp > bq_2 - cp = (b-c)p + b(q_2-p) = R_2(B')$.

7.3.3. *Axiom III: Continuity.*

In this section, unless defined otherwise, $A = [a_1,p_1;a_2,p_2;...;a_n,p_n]$, $B = [b_1,q_1^B;b_2,q_2^B;...;b_{n_B},q_{n_B}^B]$, and $C = [c_1,q_1^C;c_2,q_2^C;...;c_{n_C},q_{n_C}^C]$, where $a_1 < a_2 < ...$, $b_1 < b_2 < ...$ and $c_1 < c_2 < ...$. We define $R_3(A) = max(0,-a_1)$.

- PRL holds. If $A = [-a,p]$ and $B = [-b,p]$, where $a,b \geq 0$ and $p \in (0,1]$. Then $a > b \Longrightarrow R_3(A) = a > b = R_3(B)$.

- In holds. Let A, B, C be gambles, and $p \in [0,1)$. Define $A' = [A, p; C, 1-p], B' = [B, p; C, 1-p]$ and assume $R_3(A) \geq R_3(B)$ or $max(0, -a_1) \geq max(0, -b_1)$. Now that happens iff $R_3(A') = max(-a_1, -c_1, 0) \geq max(-b_1, -c_1, 0) = R_3(B')$, which means iff $A' \succeq B'$.
- *Co* does not hold. Let $A = [-10, 0.5], B = [-3, 0.2]$, and $C = [1, 0.7]$. Then $R_3(A) = 10 > R_3(B) = 3 > R_3(C) = 0$, but $\forall p \in (0,1) : R_3([A, p; C, 1-p]) = 10 > R_3(B)$.
- IPG holds. Let A and B be gambles with $A_- = B_-$. Then either $a_1, b_1 \geq 0$ or $a_1 = b_1$. In any case, $R_3(A) = max(0, -a_1) = max(0, -b_1) = R_3(B)$.
- IT holds. Let $A = [-a, q_1]$, $B = [-b, q_2]$, $a, b \geq 0$, $c < min(a, b)$, $p < min(q_1, q_2)$, and let $R_3(A) > R_3(B)$ or $a > b$. That means that $R_3([-a + c, p; -a, q_1 - p; 0, 1 - q_1]) = max(a, a - c) > max(b, b - c) = R_3([-b + c, p; -b, q_2 - p; 0, 1 - q_2])$, as required.

7.3.4. *Axiom IV: Invariance to possible gains.*
Let A be a gamble and define $R_4(A) := -\sum_{a_i} p_i a_i$. Note that[5]

$$(*) \quad R_4([A, p; C, 1-p]) = pR_4(A) + (1-p)R_4(C).$$

- PRL holds. If $A = [-a, p]$ and $B = [-b, p]$, where $a, b \geq 0$ and $p \in (0,1]$. Then $a > b \Longrightarrow R_4(A) = ap > bp = R_4(B)$.
- In holds. Let $A = [a_1, p_1^A; a_2, p_2^A; ...; a_{n_A}, p_{n_A}^A], B = [b_1, p_1^B; b_2, p_2^B; ...; b_{n_B}, p_{n_B}^B], C = [c_1, p_1^C; c_2, p_2^C; ...; c_{n_C}, p_{n_C}^C]$ be three gambles, and $p \in [0,1)$. Then, by (*), $R_4(A) \geq R_4(B) \Longleftrightarrow R_4([A, p; C, 1-p]) \geq R_4([B, p; C, 1-p])$.
- Co holds. Let A,B,C be three gambles such that $R_4(A) > R_4(B) > R_4(C)$. Then $f(q) = R_4([A, q; C, 1-q]) = qR_4(A) + (1-q)R_4(C)$ is a continuous function from $[0,1]$ to the real line with $f(0) = R_4(C) < R_4(B) < R_4(A) = f(1)$. Hence, there exists $p \in (0,1)$ s.t. $f(p) = R_4(B)$, and hence $R_4([A, p; C, 1-p]) = R_4(B)$.
- IPG doesn't hold. Let $A = [-1, 0.5; 4, 0.5]$ and $B = [-1, 0.5; 3, 0.5]$. Clearly, $A_- = B_-$ but $R_4(A) = -1.5 < -1 = R_4(B)$.
- IT holds. Let $A = [-a, q_1]$, $B = [-b, q_2]$, $a, b \geq 0$, $c < min(a, b)$, $p < min(q_1, q_2)$, and let $R_4(A) > R_4(B)$. It is easy to see that $R_4([-a + c, p; -a, q_1 - p; 0, 1 - q_1]) = (a - c)p + a(q_1 - p) = aq_1 - cp = R_4(A) - cp > R_4(B) - cp = R_4([-b + c, p; -b, q_2 - p; 0, 1 - q_2])$, as required.

7.3.5. *Axiom V: Invariance to Translation.*
Let A be a gamble and define $R_5(A) := \sum_{a_i < 0} p_i a_i^2$. Note that if
$A = [a_1, p_1^A; a_2, p_2^A; ...; a_{n_A}, p_{n_A}^A]$ and $C = [c_1, p_1^C; c_2, p_2^C; ...; c_{n_C}, p_{n_C}^C]$, then
$[A, p; C, 1-p] =$
$[a_1, pp_1^A; a_2, pp_2^A; ...; a_{n_A}, pp_{n_A}^A; c_1, (1-p)p_1^C; c_2, (1-p)p_2^C; ...; c_{n_C}, (1-p)p_{n_C}^C]$, and so

$$(*) \quad R_5([A, p; C, 1-p]) = pR_5(A) + (1-p)R_5(C).$$

.

- PRL holds. If $A = [-a, p]$ and $B = [-b, p]$, where $a, b \geq 0$ and $p \in (0,1]$. Then $a > b \Longrightarrow R_5(A) = (-a)^2 p > (-b)^2 p = R_5(B)$.

[5] $R_4([A, p; C, 1-p]) = -\sum_{a_j} pp_i^A a_i - \sum_{c_i} (1-p)p_i^C c_i = -p \sum_{a_i} p_i^A a_i - (1-p) \sum_{c_i} p_i^C c_i = pR_4(A) + (1-p)R_4(C)$.

- *In* holds. Let A, B, C be gambles, and $p \in [0,1)$. Then by (*) we have that $R_5(A) > R_5(B) \iff R_5([A,p;C,1-p]) > R_5([B,p;C,1-p])$.
- Co holds. Let A,B,C be three gambles such that $R_5(A) > R_5(B) > R_5(C)$. Then from the continuity of $f(q) = qR_5(A)+(1-q)R_5(C) = R_5([A,q;C,1-q])$ together with $f(1) = R_5(A) > R_5(B) > R_5(C) = f(0)$ we can deduce that there exists $p \in (0,1)$ s.t. $f(p) = R_5(B)$ or $R_5([A,p;C,1-p]) = R_5(B)$.
- IPG holds. Let $A_- = B_-$ with $A = [a_1,p_1;a_2,p_2;...;a_n,p_n]$ and $B = [b_1,q_1;b_2,q_2;...;b_m,q_m]$, then by definition of R_5, we have $R_5(A) = \sum_{a_i<0} p_i a_i^2 = \sum_{b_i<0} q_i b_i^2 = R_5(B)$.
- IT doesn't hold. Let $A = [-1.1, 0.15]$, and $B = [-0.97, 0.2]$. Then $R_5(A) = 0.15 \cdot (-1.1)^2 = 0.1815 < 0.18818 = 0.2 \cdot (-0.97)^2 = R_5(B)$. However, if we let $p = 0.14$ and $c = -0.5$, we get $R_5([-1.6, 0.14; -1.1, 0.01; 0, 0.85]) = 0.14 \cdot (-1.6)^2 + 0.01 \cdot (-1.1)^2 = 0.3705$ while $R_5([-1.47, 0.14; -0.97, 0.06; 0, 0.8]) = 0.14 \cdot (-1.47)^2 + 0.06 \cdot (-0.97)^2 = 0.35898$, in contradiction to IT.

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