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INFINITE SEQUENTIAL GAMES WITH PERFECT BUT INCOMPLETE INFORMATION

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Infinite Sequential Games with Perfect but Incomplete Information

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Abstract

Infinite sequential games, in which Nature chooses a Borel winning set and reveals it to one of the players, do not necessarily have a value if Nature has 3 or more choices. The value does exist if Nature has 2 choices. The value also does not necessarily exist if Nature chooses from 2 Borel payoff functions. Similarly, if Player 1 chooses the Borel winning set and does not reveal his selection to Player 2, then the game does not necessarily have a value if there are 3 or more choices; it does have a value if there are only 2 choices. If Player 1 chooses from 2 Borel payoff functions and does not reveal his choice, the game need not have a value either.

1. Introduction. Let A be an action set; we observe the two-player infinite sequential game in which Player 1 and Player 2 choose elements of A in an alternating fashion; this is a game of perfect information. If $W \subseteq A^\omega$, we denote by $\Gamma(W)$ the game in which Player 1 wins if the resulting infinite history is in W, and loses otherwise. This framework was introduced by Gale and Stewart [4], who proved that if W is open or closed, then the game is determined; that is, either Player 1 has a (pure) strategy that wins the game, regardless of Player 2's strategy, or Player 2 has a (pure) strategy that wins the game, regardless of Player 1's strategy. Blackwell [2] extended this result to the case where W is G_δ (or, by symmetry, F_σ). Eventually, Martin demonstrated [7] that if W is a Borel set, then the game is determined.

We next generalize to the following class of games: There is a finite set K, and for each $k \in K$, there is a Borel set $W_k \subseteq A^\omega$. There is also a probability distribution p on K. Nature chooses an element of K according to p and reveals the choice to Player 1 only. A pair of behavioral strategies σ, τ, along with the distribution p, determine a distribution $P_{p,\sigma,\tau}$ on $K \times A^\omega$. We denote Nature's choice by k and the infinite history of actions by $h \in A^\omega$. Player

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1 wins and receives a payoff of 1 unit from Player 2 if $h \in W_k$, and otherwise loses and receives a payoff of 0 units. We denote this game by $\Gamma(p, W_1, \dots, W_K)$.

As usual, we will denote

$$\overline{v} = \inf_{\tau} \sup_{\sigma} P_{p,\sigma,\tau}(h \in W_k), \ \ \underline{v} = \sup_{\sigma} \inf_{\tau} P_{p,\sigma,\tau}(h \in W_k)$$

where the supremum (resp. infimum) range over all the behavioral strategies of Player 1 (resp. Player 2). If $\overline{v} = \underline{v}$, then we will say that the game is *determined*, and that the game has a *value* of $v = \underline{v} = \overline{v}$.

The generalization of the sequential game with perfect information to the game with incomplete information is reminiscent of the way in which repeated games with perfect monitoring were generalized to repeated games with incomplete information in [1]. As in the case of these earlier results, it is a natural question to ask whether the games presented here are determined. The surprising answer is that they do if K consists of 2 elements (Section 2), but may not if K consists of 3 or more elements (Section 3), even when we require that each of the W_k be either open or closed. This is in stark contrast to previous studies on repeated games with incomplete information, in which the size of K makes little qualitative difference to the analysis.

The sequential games of perfect information can also be studied in the case where the payoff is given by a bounded Borel payoff function $f : A^\omega \to \mathbb{R}$. By observing the level sets of the form $\{f \leq c\}$, one easily shows (using Martin's result) that such a game has a value (and ε-optimal pure strategies). These games can then also be generalized to the case where the payoff function f_k is selected from a finite set $\{f_j\}_{j \in K}$ according to $p \in \Delta(K)$ and the selection is revealed to Player 1 only. We denote such a game by $\Gamma(p, f_1, \dots, f_K)$. As it turns out, in such a case, even if K consists of only 2 elements, the resulting game $\Gamma(p, f_\alpha, f_\beta)$ need not have a value. An example of such a game that can be described in this framework that does not have a value is given in [5, Sec. 5]. In this paper, we will construct a similar example (Section 4).

It is also possible to describe a win/loss game $\Gamma^*(p, W_1, \dots, W_k)$, in which Nature chooses $k \in K$ according to $p \in \Delta(K)$, and reveals the choice to neither player; Player 1 wins if and only if the infinite history is in W_k, where k is Nature's choice. In this case, we define an auxiliary game $\overline{\Gamma}(p, W_1, \dots, W_k)$ in which the players play an infinite sequential game of perfect information, choosing elements of A in an alternating fashion, and the payoff is given by a Borel payoff function f on space of infinite histories given by

$$f(h) = \sum_{j \in K} p_j \chi_{\{h \in W_j\}}$$

where χ_B is the indicator function of a set $B \subseteq A^\omega$. When A is finite, such games are known to have a value by another celebrated result of Martin [8][1].

We are grateful to Eran Shmaya; his work [9] inspired this research. In that paper, Shmaya introduced an information structure called *Eventual Perfect Monitoring* (henceforth, EPM). In the sequential games there, at each stage, the player whose turn it is does not know the history of the actions of his opponent, but rather knows which element from a partition over these histories the correct history is in. The EPM assumption requires that each player will be able to differentiate, at some stage, between any two different infinite histories of the opponent. These games were then shown to have a value for any Borel winning set in games with a finite action set. However, it is not known whether games with Borel payoffs which satisfy the EPM assumption must have a value. Hence, in the study of sequential games in which the EPM assumption does not hold, examples of games with Borel winning sets which do not have a value are sharper than examples of games with Borel payoffs which do not have a value.

Further communications with Shmaya were exceedingly helpful, and we are grateful for those as well. We are also grateful to Abraham Neyman, for many helpful suggestions and remarks on the research and formulation of this note.

2. Determinacy for Two States of Nature with Borel Winning Sets. We contend that for any $p \in \Delta(\{A, B\})$ and any pair of Borel sets $W_A, W_B \subseteq A^\omega$, the game $\Gamma(p, W_A, W_B)$ has a value, and that each player has a pure optimal strategy. We deal with all possible cases:

- If Player 2 has a (pure) strategy τ that wins simultaneously in $\Gamma(W_A)$ and $\Gamma(W_B)$ (i.e. τ wins in $\Gamma(W_A \bigcup W_B)$), then he can play it in $\Gamma(p, W_A, W_B)$ and the value is 0.
- If both $\Gamma(W_A)$ and $\Gamma(W_B)$ are both determined in Player 1's favor, the value is 1, as Player 1 knows which set is being used and will play a winning strategy for that set.
- If one of the perfect information games is determined in Player 1's favor while the other is determined in Player 2's favor, each player has an optimal strategy of playing a strategy that is winning in the game determined in his favor. If $\Gamma(W_A)$ is determined in Player 1's favor, the value is $p(A)$.
- The interesting case is when both perfect information games are determined in Player 2's favor, but he does not have a strategy that guarantees a win in both simultaneously. In this case, the game $\Gamma(W_A \bigcup W_B)$ is determined in Player 1's favor, so let σ be a winning pure strategy for

[1] For infinite - but discrete - action set A, [6] extends Martin's result and shows the value exists if the players are allowed to play mixed actions which are only required to be finitely additive.

Player 1. Player 2 can guarantee that the expected payoff will be at most $p' = \min(p, 1-p)$ by playing a winning strategy in whichever game occurs with higher probability. Player 1 can play σ and guarantee p' against any pure strategy τ of Player 2; for either τ is winning in $\Gamma(W_A)$ or τ is winning in $\Gamma(W_B)$ (which lead to wins for σ with probabilities $p(B)$ or $p(A)$, respectively) or neither. Therefore, Player 1 guarantees p' against any mixed strategy as well.

3. Indeterminacy for Three States of Nature with Open or Closed Winning Sets. We contend that there exists a collection $\mathbf{W} = \{W_T, W_M, W_B\}$ of subsets of A^ω, where $A = \{L, R\}$, and $p \in \Delta(\{T, M, B\})$, such that each W_k is either open or closed, and that the game $\Gamma(p, \mathbf{W})$ does not have a value. We define:

- W_T is the set in which Player 2 plays R before Player 1, or Player 1 plays R before Player 2 and the first time he does so, Player 2 immediately answers with L.
- W_M is the set in which Player 2 plays R before Player 1, or Player 1 plays R before Player 2 and the first time he does so, Player 2 immediately answers with R.
- W_B is the set in which Player 1 always plays L and Player 2 always plays L.

W_B is closed as a singleton. W_T is open, as every $h \in W_T$ has an initial segment h_0 such that every infinite history that begins with h_0 is also in W_T; similarly, W_M is open. Let $\frac{1}{4} < q < \frac{1}{2}$ (e.g., $q = \frac{1}{3}$), and take $p = (q, q, 1-2q)$; we contend that $\Gamma(p, \mathbf{W})$ does not have a value. There are several ideas that are used to establish this result:

Firstly, we realize that Player 2 essentially chooses one of two options: Whether to ever play R or not. If Player 1 is not aware of τ, he will not be able to determine, from any initial segment of L's, the choice of Player 2. However, if he is aware of τ, he will be able to wait long enough and know, with as much precision as he likes, which of the two was chosen, even when τ consists of a mixture, and then either play R or continue to play L forever, accordingly.

Secondly, if Player 1 plays R with low enough probability against a string of L's from Player 2, Player 2 will be better off playing only L, as the event $T \bigcup M$ is at least as likely as the event B. If, however, R is played with high enough probability against this string given the event of $T \bigcup M$, then Player 2 should begin playing L, and if after a long enough time R is not played by Player 1, Player 2 will be able to deduce with high probability that the event B has occurred, and play R.

Let us describe a reply σ_ε to a fixed τ. Denote by σ_L the strategy for Player 1 which always plays L. Denote the event of Player 2 ever playing R, and doing so before Player 1, by Φ, and the event of this occurring within Player 2's first n actions by Φ_n. For every $\varepsilon > 0$, there is m such that $P_{p,\sigma_L,\tau}(\Phi_m) > P_{p,\sigma_L,\tau}(\Phi) - \varepsilon$. Choose such an m. σ_ε is then described in the following manner, where k denotes the choice of nature: If $k = B$, always play L. If $k = T$ or $k = M$, play L the first m times; if Φ_m has not occured, then play R. Note that

$$P_{p,\sigma,\tau}(\Phi_m) = P_{p,\sigma_L,\tau}(\Phi_m) > P_{p,\sigma_L,\tau}(\Phi) - \varepsilon \geq P_{p,\sigma,\tau}(\Phi) - \varepsilon$$

Denote $\xi_m \in \Delta(\{L, R\})$ to be the mixed action τ plays in response to Player 1 playing R for the first time (for either player) if he does so on his m-th turn. The resulting expected payoff $p_{\sigma_\varepsilon,\tau}$ to Player 1 then satisfies

$$\begin{aligned} p_{\sigma_\varepsilon,\tau} &\geq (1-2q)P_{p,\sigma,\tau}(\Phi^c) + 2q(P_{p,\sigma,\tau}(\Phi) - \varepsilon) \\ &\quad + (q\xi_{m+1}(L) + q\xi_{m+1}(R))(1 - P_{p,\sigma,\tau}(\Phi)) \end{aligned}$$

where the first term is the probability of B being chosen and a win for Player 1, the second bounds from below the probability of $T \bigcup M$ being played and Φ_m occurring, and the third bounds from below the probability of $T \bigcup M$ being played, Φ_m not occurring, and Player 1 winning after playing R. Therefore, since τ was arbitrary,

$$\begin{aligned} \overline{v} &\geq \liminf_{\varepsilon \to 0}(p_{\sigma_\varepsilon,\tau}) \geq (1-2q)(1 - P_{p,\sigma,\tau}(\Phi)) + 2qP_{p,\sigma,\tau}(\Phi) + q(1 - P_{p,\sigma,\tau}(\Phi)) \\ &= 1 - q + (3q-1)P_{p,\sigma,\tau}(\Phi) \geq \begin{cases} 1-q & \text{if } \frac{1}{3} \leq q \\ 2q & \text{if } q \leq \frac{1}{3} \end{cases} \end{aligned} \qquad (1)$$

In turn, let σ be a fixed strategy of Player 1, and $\varepsilon > 0$. We will describe a reply of τ_ε of Player 2. Denote by τ_L the strategy for Player 2 which always plays L. If Player 1 plays R first, τ responds with $(\frac{1}{2}, \frac{1}{2})$. Let γ denote the probability that σ will play R given that Player 2 plays only L and given that $T \bigcup M$ occurs, and let γ_m denote the probability of it occurring within Player 1's first m actions; formally,

$$\gamma = P_{p,\sigma,\tau_L}\left(\{(h_1, L, h_2, L, h_3, L, \cdots) \mid \exists i \in \mathbb{N} \ s.t. \ h_i = R\} \mid T \bigcup M\right)$$

and

$$\gamma_m = P_{p,\sigma,\tau_L}\left(\{(h_1, L, h_2, L, h_3, L, \cdots) \mid \exists i \leq m \ s.t. \ h_i = R\} \mid T \bigcup M\right)$$

Then, there is m such that $\gamma_m > \gamma - \varepsilon$; choose such an m.

Let $\lambda \in [0, 1]$; λ will later be chosen according to q. We describe τ in two cases: If $\gamma < \lambda$, τ will always play L. Otherwise, τ will play L for m stages; if R has not been played by Player 1, Player 2 will play R. Note that in this latter case of $\gamma \geq \lambda$, $(\sigma, \tau_\varepsilon)$ will always give Player 2 a win if Nature chooses B. In the case $\gamma < \lambda$, the expected payoff $p_{\sigma,\tau_\varepsilon}$ to Player 1 satisfies

$$p_{\sigma,\tau_\varepsilon} \leq (1-2q) + 2q \cdot \gamma \cdot \frac{1}{2}$$

where the first term represents Player 1's chance of winning via B being chosen by Nature, and the second represents his chances of winning via $T \bigcup M$ being chosen by Nature, him eventually playing R, and winning. Therefore, in this case,

$$p_{\sigma,\tau_\varepsilon} \leq (1-2q) + q\lambda \tag{2}$$

In the case $\gamma \geq \lambda$,

$$p_{\sigma,\tau_\varepsilon} \leq 2q[(1-\gamma_m) + \frac{1}{2}\gamma] \leq 2q(1 - \frac{1}{2}\lambda + \varepsilon)$$

where $2q(1-\gamma_m)$ is the probability $T \bigcup M$ is chosen by Nature and that R is not played by Player 1 in the first m stages, and $2q \cdot \frac{1}{2}\gamma$ is the probability $T \bigcup M$ is chosen by Nature and Player 1 plays R and wins. Therefore,

$$\limsup_{\varepsilon \to 0}(p_{\sigma,\tau_\varepsilon}) \leq 2q - q\lambda \tag{3}$$

Therefore, since σ was arbitrary, $\underline{v}$ is at most the maximum of the right-most sides of (3) and (2). One then shows that for all $q \in (\frac{1}{4}, \frac{1}{2})$, if we select $\lambda = \lambda(q)$ satisfying

$$3 - \frac{1}{q} < \lambda < 4 - \frac{1}{q}$$

(e.g., $\lambda(q) = 2 - \frac{1}{2q}$, which satisfies $0 < \lambda(q) < 1$ for $q \in (\frac{1}{4}, \frac{1}{2})$) then the right-most sides of (3) and (2) are both strictly smaller than the right-hand side of (1).

We do, however, state the following observation:

Proposition. *If the action set A is finite, and if all the W_k are closed (resp. open), then the game does have value.*

Proof. In this case, payoff function is then an upper (resp. lower) semi-continuous function of the players pure strategies. Since the pure action spaces are compact in the Tychonoff topology, the game has a value (e.g., [3, Thm. 2]) in mixed strategies, which are equivalent to behavioral strategies by Kuhn's theorem. □

4. Indeterminacy for Borel Payoff Functions The example in Section 3 allows us to present an example of a game $\Gamma = \Gamma((p, 1-p), f_\alpha, f_\beta)$ (where $f_\alpha, f_\beta : A^\omega \to \mathbb{R}$ are bounded Borel payoff functions) which does not have a value. Choose $q \in (\frac{1}{4}, \frac{1}{2})$ and set $p = 2q$. Let W_T, W_M, W_B be as in Section 3, and let χ_W denote the indicator function of a set W. Then set $f_\beta = \chi_{W_B}$ and $f_\alpha = \frac{1}{2}\chi_{W_T} + \frac{1}{2}\chi_{W_M}$. The analysis of Section 3 then follows in the same way, with the choice of Nature of α replacing the choice of $T \bigcup M$, and the choice of β replacing the choice of B.

5. Determinacy for an Unobserved Action of Player 1. The most natural model in which the EPM assumption of [9] does not hold is a game in which one player's initial move is unobserved, after which the players play a sequential game which has perfect information except for the initial action. Equivalently, we define, for each finite collection of subsets $\mathbf{W} = (W_k)_{k \in K}$ of A^ω, a game $\Gamma(\mathbf{W})$ in the following manner: Player 1 makes a choice of k, which is not observed by Player 2, after which the players play a perfect information game. Player 1 wins and receives a payoff of 1 unit from Player 2 if the resulting history is in W_k, and otherwise loses and receives a payoff of 0 units.

It is easy to modify the proof of Section 2 to show[2] that, if $|K| = 2$, such a game either has a value of 0 (in the case that Player 2 has a strategy that wins simultaneously in both perfect information games), a value of 1 (in the case that one of the two perfect information games is determined in Player 1's favor), or a value of $\frac{1}{2}$ (in which case, Player 1 chooses the set via a fair coin toss, and then plays as he would in $\Gamma((\frac{1}{2}, \frac{1}{2}), W_1, W_2)$).

It is also possible to modify the game presented in Section 3 to the case where Player 1 chooses one of the options T, M, B before play begins, and this choice is not revealed to Player 2. This game also will not have a value; a symmetry argument shows we need only consider strategies that assign the same probability to T and M. If Player 1 knows τ, then by (1) and by choosing $q = \frac{1}{3}$, Player 1 can assure, for any $\varepsilon > 0$, a payoff of at least $\frac{2}{3} - \varepsilon$. If Player 2 knows σ, and σ chooses from K with $(q, q, 1-2q)$, then Player 2 can assure, for any $\varepsilon > 0$, a payoff (to Player 1) of at most $\frac{1}{2} + \varepsilon$ by choosing $\lambda = 2 - \frac{1}{2q}$ if $\frac{1}{4} < q$ (by (3) and (2)), or $\lambda = 0$ if $q \leq \frac{1}{4}$ (by (3); (2) is irrelevant if $\lambda = 0$).

References

[1] Aumann, R.J. and Maschler, M. (1995), *Repeated Games with Incomplete Information* (with the collaboration of R. E. Stearns), MIT Press, Cambridge, MA.

[2] Blackwell, D. (1969) Infinite G_δ Games of Imperfect Information, *Applicationes Mathematicae*, Vol. X., pp. 99–101, Hugo Steinhaus Jubilee Volume.

[3] Fan, K. (1953) Minimax theorems, *Proceedings of the National Academy of Sciences of the U.S.A.* 39, 42-47

[4] Gale, D. and Stewart, F.M. (1953) Infinite Games with Perfect Information, in *Contributions to Theory Theory of Games*, Vol. 2, *Annals of Mathematical Studies* 28, 245–266

[2]This result was communicated to us by E. Shmaya.

[5] Guriel-Gurevich, O. (2009) Pursuit-Evasion Games with Incomplete Information in Discrete Time, *International Journal of Game Theory* 38, 3, 367–376

[6] Maitra, A. and Sudderth, W. (1998), Finitely Additive Stochastic Games with Borel–Measurable Payoffs, *International Journal of Game Theory* 27, 257-267

[7] Martin, D. (1975), Borel Determinacy, *Annals of Mathematics* 102, 363–371

[8] Martin, D. (1998), The Determinacy of Blackwell Games, *The Journal of Symbolic Logic*, 63, 1565–1581

[9] Shmaya, E. The Determinacy of Infinite Games with Eventual Perfect Monitoring, Pre-Print.