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BARGAINING SET SOLUTION CONCEPTS IN REPEATED COOPERATIVE GAMES

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ABSTRACT. This paper is concerned with the question of extending the definition of the bargaining set, a cooperative game solution, when cooperation takes place in a repeated setting. The focus is on situations in which the players face (finite or infinite) sequences of exogenously specified TU-games and receive sequences of imputations against those static cooperative games in each time period. Two alternative definitions of what a ‘sequence of coalitions’ means in such a context are considered, in respect to which the concept of a repeated game bargaining set may be defined, and existence and non-existence results are studied. A solution concept we term subgame-perfect bargaining set sequences is also defined, and sufficient conditions are given for the non-emptiness of subgame-perfect solutions in the case of a finite number of time periods.

1. INTRODUCTION AND REVIEW OF LITERATURE

The study of repeated non-cooperative games – that is, the study of games whose structure is given by a discrete finite or infinite temporal framework in which at each time period a non-cooperative game is played and payoffs are determined accordingly – is one of the most richly studied topics in game theory. It has a history stretching back over half a century – the celebrated Folk Theorem of repeated non-cooperative game theory, to take just one example, was proved in the 1950s – and has influenced theories in several different disciplines, including political science, philosophy and evolutionary theory.

In contrast to the abundance of research in repeated non-cooperative games, the study of the analogous situation, in which the game played in each time period is a cooperative game, has been relatively sparse, and comparatively quite recent. This is perhaps surprising, because the study of repeated cooperative games can be motivated just as readily as that of repeated non-cooperative game – many, if not most, cooperative endeavours occur more than once, or repeatedly over time. Examples can be easily adduced, such as multi-year profit-sharing arrangements, cost-sharing agreements, supply relationships, labour contracts, renewable treaty negotiations, and so forth. The insights gained from further progress in this topic should be expected to have broad implications. To the best of our knowledge, the first papers devoted to the systematic study of cooperative games played iteratively appeared

in 2000, independently by [Oviedo (2000)] and [Kranich, Perea, Peters (2001)]. To those pioneering efforts have been added contributions by [Kranich, Perea, Peters (2005)], [Predtetchinski, Herings, Peters (2002)], [Predtetchinski, Herings, Peters (2004)], [Predtetchinski, Herings, Perea (2006)] and [Predtetchinski (2007)] and [Berden (2007)].

The above papers, in the main, concentrate on extensions, to the repeated setting, of the core and the Shapley value cooperative solution concepts. It is the intention of this paper to contribute to the literature by considering the bargaining set solution concept (a concept first defined in [Aumann, Maschler (1964)] and [Davis, Maschler (1967)]) in the repeated setting, largely inspired by the frameworks for studying the core in repeated situations appearing in [Oviedo (2000)] and [Kranich, Perea, Peters (2005)].

In addition to the concentration on the bargaining set, as opposed to the core, this paper also differs from these other papers in the following ways: [Oviedo (2000)] assumes throughout that paper that the underlying stage-games are super-additive. We study both superadditive and non-super-additive games. [Kranich, Perea, Peters (2005)] restrict their study to finite numbers of time periods, and to what is defined in our paper as repeated coalitions, as opposed to dynamic coalitions. On the other hand, [Kranich, Perea, Peters (2005)] work with general time-dependent utility functions. In this paper, however, we will work with time-averaged utility.

The main motivational idea in this paper is as follows: in a standard cooperative game, a set of players N , often along with a coalition structure \mathcal{R} , negotiate regarding a payoff imputation that is feasible relative to a characteristic function ν . We now add the element of time, and assume that the players will be playing a different cooperative game in each of several time periods, where the characteristic function ν^t now depends on the time period t . This leads to the concept of a *repeated game* $(N, \boldsymbol{\nu}, \mathcal{R})$, where $\boldsymbol{\nu} = (\nu^1, \dots, \nu^m)$. As in the one-shot game, the players negotiate payoffs, but now prior to the repeated game they negotiate a feasible payoff imputation in every time period. This in turn leads to the concept of an *imputation sequence*, which is a sequence of vectors $\mathbf{x} = (x^1, \dots, x^m)$ such that x^t is an imputation vector of the stage-game (N, ν^t, \mathcal{R}) for each t and $\sum_{t=1}^m x_i^t \geq \sum_{t=1}^m \nu^t(\{i\})$ for each player i .

We then consider extending the standard bargaining set concept to the repeated game setting. The standard bargaining set is the set of feasible imputations that are stable, in the sense that for any objection y a player may have to a stable imputation x , there is a counter-objection z . The extension to repeated games then essentially defines a stable imputation sequence to be an imputation sequence \mathbf{x} such that any objection \mathbf{y} a player may have (where an objection is an alternative imputation sequence) can be met by a counter-objection \mathbf{z} .

But pinning down the definition of the extension of the concept of bargaining set requires first answering the question of what is the multi-period extension of the coalition of a standard cooperative game. One straightforward extension reasons as follows: if $S \subseteq N$ is a single-period coalition, a *repeated coalition* in an m -period repeated game is $\{S, S, \dots, S\}$, (S repeated m times), where the same coalition S cooperates in every time period t , granting itself an imputation in each period that

is feasible with respect to $\nu^t(S)$. The repeated coalition bargaining set for (N, ν, \mathcal{R}) can then be defined, and denoted $\mathcal{RB}(N, \nu, \mathcal{R})$. One can, however, also consider a broader definition of a coalition in the repeated setting, a sort of ‘virtual coalition’, in which a set of players $S \subseteq N$ sign an agreement amongst themselves to form different sub-coalitions in each time period in order to create imputation vectors – if some of the stage-games are not superadditive, this could be an advantage over requiring the same coalition S to divide $\nu^t(S)$ in every time period. Calling this concept a *dynamic coalition*, the corresponding dynamic coalition bargaining set is denoted $\mathcal{DB}(N, \nu, \mathcal{R})$.

This leads to the questions: are $\mathcal{RB}(N, \nu, \mathcal{R})$ and $\mathcal{DB}(N, \nu, \mathcal{R})$ different sets? Are they guaranteed to be non-empty? That $\mathcal{DB}(N, \nu, \mathcal{R}) \subseteq \mathcal{RB}(N, \nu, \mathcal{R})$ follows from the definitions. We show by example that $\mathcal{DB}(N, \nu, \mathcal{R})$ may be empty. Proposition 1 shows that, in contrast, $\mathcal{RB}(N, \nu, \mathcal{R})$ is non-empty, from which it further follows that $\mathcal{DB}(N, \nu, \mathcal{R})$ is distinct from $\mathcal{RB}(N, \nu, \mathcal{R})$. If the stage-games are superadditive, however, $\mathcal{DB}(N, \nu, \mathcal{R}) = \mathcal{RB}(N, \nu, \mathcal{R})$ (Proposition 2).

In Section 5, we look more carefully at the set of imputation sequences in $\mathcal{RB}(N, \nu, \mathcal{R})$. Can we characterise the possible imputation sequences by establishing bounds on the payoffs that can be granted to each player in each time period? The characterisation presented in the section is based on an interpretation of imputation sequences as ‘credit sequences’, in the sense that every imputation sequence can be interpreted as some players ‘borrowing’ from other players in earlier time periods, and returning the ‘loans’ in later periods (Proposition 3). Limits on the extent of such ‘credit’ that can be granted against future payoff earnings is what establishes bounds on payoffs in imputation sequences (Proposition 4).

Finally, in Section 6, we build on the interpretation of an imputation sequence as encoding ‘inter-temporal’ borrowing, and ask what happens if a player who has borrowed heavily in earlier rounds ‘defaults’ on his debt in later periods by defecting to another coalition, breaking the imputation sequence contract? This motivates the idea of a *subgame perfect* multi-period bargaining set imputation, which meets the constraint that in each time period, no player can form a justified objection to the multi-period imputation taking into account only the remaining time periods. Denoting the set of subgame perfect imputation sequences by $\mathcal{SP}(N, \nu, \mathcal{R})$, if the number of time periods is finite, and the sequence of characteristic functions ν satisfies the mild technical assumption of being sequentially essential, then $\mathcal{SP}(N, \nu, \mathcal{R})$ is non-empty (Proposition 5).

2. PRELIMINARIES

A (*static*) *cooperative transferable utility (TU) game* consists of a pair (N, ν) such that N is a set of n elements, termed players, where n is a positive integer, and $\nu : 2^N \rightarrow \mathbb{R}$, $\nu(\emptyset) = 0$ is termed the *characteristic function* of the game. A coalition is a subset of N . For any coalition S , \mathbb{R}^S denotes the $|S|$ -dimensional Euclidean space in which the dimensions are indexed by the members of S . Given any n -tuple x and coalition $S \subset N$, $x(S) := \sum_{i \in S} x_i$.

If the characteristic function satisfies, for all coalitions $S, T \subseteq N$,

$$\nu(S \cup T) \geq \nu(S) + \nu(T) \text{ if } S \cap T = \emptyset$$

then the game is *superadditive*. Superadditivity will be assumed here only when explicitly noted. On the other hand, it will be assumed without loss of generality that $\nu(S) > 0$ for all characteristic functions and all coalitions.

A coalition structure for $S \subseteq N$ is a partition of S . We will denote the set of all coalition structures over S , where $S \subseteq N$, by $\mathcal{C}(S)$. If \mathcal{R} is a coalition structure for N , we will write, by a slight abuse of notation, $\mathcal{R}(i)$ to stand for the element $Q \in \mathcal{R}$ such that $i \in Q$. Given a coalition structure \mathcal{R} for S , two players $i, j \in S$ will be said to be *partners* with respect to \mathcal{R} , denoted $i \sim_{\mathcal{R}} j$, if both $i \in P$ and $j \in P$ for the same $P \in \mathcal{R}$. Given a vector $x \in \mathbb{R}^n$ and a coalition P , x^P will denote the sub-vector of x consisting of all $x_i \in x$ such that $i \in P$.

If (N, ν) is a game and \mathcal{R} is a coalition structure for N , the triple (N, ν, \mathcal{R}) is a *game with coalition structure*. For any (N, ν, \mathcal{R}) ,

$$I(N, \nu, \mathcal{R}) = \{x \in \mathbb{R}^n \mid x(S) \leq \nu(S) \forall S \in \mathcal{R}, \text{ and } x_i \geq \nu(\{i\}) \forall i \in N\}$$

denotes the set of *imputations* of (N, ν, \mathcal{R}) .

Given $k, l \in N$ with $k \neq l$, denote $T_{kl}(N) := T_{kl} := \{S \subseteq N \setminus \{l\} \mid k \in S\}$. Then an *objection* of k against l at $x \in I(N, \nu, \mathcal{R})$ is a pair (P, y) satisfying

$$P \in T_{kl}, y \in \mathbb{R}^P, y_i \geq x_i \forall i \in P, y_k > x_k \text{ and } y(P) \leq \nu(P).$$

A *counter-objection* to an objection (P, y) of k against l at x is a pair (Q, z) satisfying

$$Q \in T_{lk}, z \in \mathbb{R}^Q, z_i \geq x_i \forall i \in Q, z_i \geq y_i \forall i \in P \cap Q \text{ and } z(Q) \leq \nu(Q).$$

An imputation $x \in I(N, \nu, \mathcal{R})$ is *stable* if for every objection at x there exists a counter-objection. The *(static) bargaining set* $M(N, \nu, \mathcal{R})$ is defined by

$$M(N, \nu, \mathcal{R}) = \{x \in I(N, \nu, \mathcal{R}) \mid x \text{ is stable}\}.$$

The concept of the bargaining set was first put forward by [Aumann, Maschler (1964)] and [Davis, Maschler (1967)]. See also [Maschler (1976)]. Variants of the concept of the bargaining set appear in [Granot, Maschler (1997)] and [Mas-Colell (1989)]. See also [Holzman (2000)].

When working with vectors, we adopt the following standard notation: If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$. Denote $\mathbb{R}_+^S = \{x \in \mathbb{R}^S \mid x \geq 0\}$. Given $S \subseteq Q$ and a vector $x \in \mathbb{R}^S$, x^S refers to a vector whose elements are enumerated by the elements of S and the value of $x_{i_j}^S$, where $i_j \in S$ is equal to x_{i_j} .

A game *without transferable utility* (*NTU game*) is a pair (N, V) where $V(S) \subseteq \mathbb{R}^S$ for each coalition S , and $V(\emptyset) = \emptyset$, along with the following additional conditions:

- (1) for all $S \neq \emptyset$, $V(S)$ is non-empty and closed
- (2) if $x \in V(S)$ and $y_i \leq x_i$ for all $i \in S$, then $y \in V(S)$

- (3) for every $i \in N$ there is an $m_i \in \mathbb{R}$ with $V(\{i\}) = \{x \in \mathbb{R} \mid x_i \leq m_i\}$
- (4) for each $S \subseteq N$, and every $x \in \mathbb{R}^S$, $V(S) \cap \{x + \mathbb{R}_+^S\}$ is bounded.

An NTU game with coalition structure \mathcal{R} is denoted (N, V, \mathcal{R}) . We will assume w.l.o.g. that (N, V, \mathcal{R}) is zero-normalised. A vector $x \in \mathbb{R}^N$ is *individually rational* if $x \geq 0$; *feasible* if $x^P \in V(P)$ for each $P \in \mathcal{R}$; *weakly Pareto optimal*, for every $P \in \mathcal{R}$, if it is feasible and if for every $y \in V(P)$ there exists an $i \in P$ such that $x_i^P \geq y_i^P$; and an *imputation* if it is individually rational and weakly Pareto optimal for every $P \in \mathcal{R}$. Denote the set of imputations of (N, V, \mathcal{R}) by $IX(N, V, \mathcal{R})$.

The definition of bargaining set for NTU games is as follows: Let $x \in IX(N, V, \mathcal{R})$ and let $k, l \in R$, $k \neq l$, for some $R \in \mathcal{R}$. An objection of k against l is a pair (P, y) such that

$$P \in T_{kl}, y \in V(P), \text{ and } y_i \geq x_i \text{ for all } i \in P, \text{ with } y_k > x_k.$$

A counter-objection to an objection (P, y) is a pair (Q, z) such that

$$Q \in T_{lk}, z \in V(Q), z^{Q \setminus P} \geq x^{Q \setminus P} \text{ and } z^{P \cap Q} \geq y^{P \cap Q}.$$

An objection (P, y) is justified if there is no counter-objection to (P, y) . A vector $x \in IX(N, V, \mathcal{R})$ is stable if there is no justified objection at x , and the bargaining set of (N, V, \mathcal{R}) is the set of stable vectors.

3. REPEATED GAMES

Turning to the intertemporal context, assume that time is divided into discrete time periods. Let m be either a non-negative integer or ω . If m is a finite integer, the relevant time periods are taken from $T = \{1, \dots, m\}$. If m is ω , T is $\{1, \dots\}$. To enable infinite and finite sequences to be dealt with in a unified manner as far as possible in this paper, a sequence of numbers written as (x^1, \dots, x^m) will be understood to stand for the infinite sequence (x^1, \dots) if m is ω .

As a general rule here, we adopt a notational convention in which upper indices denote time, and lower indices players.

Fix N , a sequence of characteristic functions $\boldsymbol{\nu} = (\nu^1, \dots, \nu^m)$ and a coalition structure \mathcal{R} . Then $(N, \boldsymbol{\nu}, \mathcal{R})$ will be termed a *repeated cooperative game*. The special case in which there is a single characteristic function ν such that $\nu^t = \nu$ all time periods t can, in analogy with what is customary in the non-cooperative case, be called a *repeated cooperative game* based on the *stage-game* (N, ν, \mathcal{R}) . In any case, for each integer $1 \leq t \leq m$, (N, ν^t, \mathcal{R}) will be called the *stage-game played at time t* . The set of stage-game imputations at time t is defined¹ by

$$I(N, \nu^t, \mathcal{R}) = \{x \in \mathbb{R}^N \mid x(Q) \leq \nu^t(Q) \text{ for every } Q \in \mathcal{R}\}$$

A sequence of vectors $\mathbf{x} = (x^1, \dots, x^m)$ such that x^t is an imputation vector of the stage-game (N, ν^t, \mathcal{R}) for each t and $\sum_{t=1}^m x_i^t \geq \sum_{t=1}^m \nu^t(\{i\})$ for each player i ,

¹ Note that we do not demand that each stage-game imputation satisfy individual rationality in its respective time period, thus enabling greater flexibility in the choice of stage-game imputations. Over-all individual rationality relative to the repeated game, however, is required.

is an *imputation sequence* of the repeated game (N, ν, \mathcal{R}) . The set of imputation sequences of the repeated game (N, ν, \mathcal{R}) will be denoted by $\mathbf{I}(N, \nu, \mathcal{R})$. Given $\mathbf{x} = (x^1, \dots, x^m)$, we will let \mathbf{x}_i refer to the sequence of real numbers (x_i^1, \dots, x_i^m) , where x_i^t is the payoff given to player $i \in N$ according to the imputation x^t at time t .

We will work with the time-average criterion, and define

$$\bar{\mathbf{x}}_i = \begin{cases} \frac{1}{m} \sum_{t=1}^m x_i^t & \text{when } m \text{ is finite} \\ \liminf_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^T x_i^t & \text{when } m \text{ is infinite} \end{cases}$$

The vector of the time-average payoffs granted to the players is thus denoted $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_n)$.

The special payoff in which every player receives in each period what he or she gains in a one-person coalition will be written

$$\mathbf{s} = (s^1, \dots, s^m)$$

where s^j for any time period j grants to each player i the payoff $\nu^j(\{i\})$.

Intuitively, a repeated cooperative game is intended to model a situation in which a group of players are to play a sequence of cooperative games m times. At each time period, a stage-game imputation determines how much each player receives from that round of play.

Analogously to the case of static cooperative games with coalition structures, it will be assumed here, at least intuitively, that within each coalition in the coalition structure \mathcal{R} the players will contend with each other regarding their shares of the imputations, and that they will do so by presenting each other with potential objections and counterobjections. In the repeated game, however, we assume that each player cares only about the *average total* of payoffs he or she receives over time, rather than particular imputations in each period – in other words, each player prefers an imputation sequence \mathbf{y} to \mathbf{x} precisely when $\bar{\mathbf{y}}_i > \bar{\mathbf{x}}_i$.

4. DYNAMIC AND REPEATED COALITIONS AND BARGAINING SETS

Again, in analogy with the static case, we assume that players communicate openly with each other and sign binding and enforceable contracts specifying coalition formation and accompanying imputations. But in repeated games, the contracts are assumed to cover all the m time periods. In a static cooperative game, objections and counter-objections are defined against all possible coalitions containing one player but not another, but in the repeated game setting one needs to consider *sequences of coalitions*, because an objection in the repeated game raised by a player to a sequence of imputations might involve different coalitions in each time period.

This requires new definitions. We consider here two different possibilities for what a ‘sequence of coalitions’ may mean, and show that they have different implications for solutions of repeated games.

Let $S \subseteq N$. For each time t , denote a coalition structure over $S \subseteq N$ at time t by $\mathcal{R}^t \in \mathcal{C}(S)$. A *dynamic coalition* over S is then $\mathcal{R}(S) = (\mathcal{R}^1, \dots, \mathcal{R}^m)$, over the m time periods. In the special case that $\mathcal{R}^t = \{S\}$ for all t , $\mathcal{R}(S)$ will be termed a *repeated coalition*.

The repeated coalition concept is the simpler one — in the repeated coalition the single-period coalition S is formed again and again in each and every time period. This corresponds most closely to the naïve view of what a multi-period coalition means — a group of players who agree in each time period to cooperate together in the same coalition.

A dynamic coalition $\mathcal{R}(S)$ can perhaps be thought of as a ‘virtual coalition’, or a ‘coalition of coalitions’, with different sub-coalitions forming in each time period. It is intuitively conceived of as a group of players S who sign a multi-period contract that determines a coalition structure in each period — i.e. it determines for each period who partners with whom in a standard coalition in that period. Clearly, a repeated coalition is a special case of a dynamic coalition. Mathematically, the distinction is important because by definition a coalition S at time t divides amongst its members the output of $\nu^t(S)$. In a repeated coalition, the members at each time period produce $\nu^t(S)$, which they then share via an imputation. But in a dynamic coalition $\mathcal{R}(S)$, if ν^t is not super-additive, the members of S might find themselves better off not producing $\nu^t(S)$, but instead splitting into the sub-coalitions given by \mathcal{R}^t . The sequence $(\mathcal{R}^1, \dots, \mathcal{R}^m)$ is precisely the determination of what sub-coalitions are formed out of S at each time period. Examples 1 and 2 below show that allowing dynamic coalitions to form over S , as opposed to restricting players to repeated coalitions over S , can significantly affect bargaining powers and solutions in repeated games.

Given $k, l \in R \in \mathcal{R}$ and $\mathbf{x} \in \mathbf{I}(N, \nu, \mathcal{R})$, a *dynamic* [respectively *repeated*] *coalition objection* of k against l at \mathbf{x} is a triple $(P, \mathcal{R}(P) = (\mathcal{D}^1, \dots, \mathcal{D}^m), \mathbf{y} = (y^1, \dots, y^m))$ such that $\mathcal{R}(P)$ is a dynamic [repeated] coalition, satisfying

$$\begin{aligned} P &\in T_{kl} \\ y^t &\in \mathbb{R}^P \text{ for all } t \in \{1, \dots, m\} \\ \bar{\mathbf{y}}_i &\geq \bar{\mathbf{x}}_i \text{ for all } i \in P \text{ and } \bar{\mathbf{y}}_k > \bar{\mathbf{x}}_k \\ \bar{\mathbf{y}}_i &\geq \bar{\mathbf{s}}_i \text{ for all } i \in P \\ &\text{for each } t \in \{1, \dots, m\}, \text{ for each } D \in \mathcal{D}^t, y^t(D) \leq \nu^t(D) \end{aligned}$$

A *dynamic* [respectively *repeated*] *coalition counter-objection* to an objection $(P, \mathcal{R}(P), \mathbf{y})$ of k against l at $\mathbf{x} \in \mathbf{I}(N, \nu, \mathcal{R})$ is a triple $(Q, \mathcal{R}(Q) = (\mathcal{B}^1, \dots, \mathcal{B}^m), \mathbf{z} = (z^1, \dots, z^m))$ such that $\mathcal{R}(Q)$ is a dynamic [repeated] coalition, satisfying

$$\begin{aligned} Q &\in T_{lk} \\ z^t &\in \mathbb{R}^Q \text{ for all } t \in \{1, \dots, m\} \\ \bar{\mathbf{z}}_i &\geq \bar{\mathbf{x}}_i \text{ for all } i \in Q \\ \bar{\mathbf{z}}_i &\geq \bar{\mathbf{y}}_i \text{ for all } i \in P \cap Q \\ \bar{\mathbf{z}}_i &\geq \bar{\mathbf{s}}_i \text{ for all } i \in P \\ &\text{for each } t \in \{1, \dots, m\}, \text{ for each } B \in \mathcal{B}^t, z^t(B) \leq \nu^t(B) \end{aligned}$$

A dynamic [repeated] coalition objection of player i for which player j has no dynamic [repeated] coalition counter-objection is a *justified* dynamic [repeated] coalition objection. A sequence $\mathbf{x} \in \mathbf{I}(N, \nu, \mathcal{R})$ is *dynamic [repeated] coalition stable* if for each dynamic [repeated] coalition objection at \mathbf{x} there is a dynamic [repeated] coalition counter-objection. The dynamic [repeated] coalition bargaining set, is the set of all dynamic [repeated] coalition stable members of $\mathbf{I}(N, \nu, \mathcal{R})$.

Denote the dynamic coalition bargaining set of a repeated game (N, ν, \mathcal{R}) by $\mathcal{DB}(N, \nu, \mathcal{R})$, and the repeated coalition bargaining set by $\mathcal{RB}(N, \nu, \mathcal{R})$. Since a repeated coalition is a special case of a dynamic coalition, it is immediate that $\mathcal{DB}(N, \nu, \mathcal{R}) \subseteq \mathcal{RB}(N, \nu, \mathcal{R})$.

We now proceed to show by a series of examples that contemplation of cooperative repeated games adds new and interesting considerations beyond those encountered in static cooperative games, and that the distinction made between $\mathcal{DB}(N, \nu, \mathcal{R})$ and $\mathcal{RB}(N, \nu, \mathcal{R})$ is justified.²

Example 1. In a village in a far-away land, a machine is made available to the villagers for the annual harvest. No villager can afford to rent a machine by him or herself, and it requires two individuals to operate. But if two villagers partner in leasing and operating the machine, it can yield great rewards, with the exact amount dependent on the joint skills of the operators. If three or more people work together on the machine, however, it breaks down and yields nothing.

Two brothers, Tom and Yuval, have always partnered in renting a two-person harvester. Working together, the brothers annually harvest 48 units, which they divide equally amongst themselves. One day, two new residents arrive, Ivan and Chang, who refrain from partnering with each other, but are willing to work with others.

Tom wishes to take advantage of this development to gain leverage against Yuval. He can partner with either of the new arrivals for 74 units. The most Yuval can attain working with Ivan or Chang is 48 units.

Formally, denoting the players by T, Y, I, C , the stage-game is defined by $R = TY, I, C$ and $\nu(TY) = \nu(YI) = \nu(YC) = 48$, $\nu(TI) = \nu(TC) = 74$. The value of every other possible coalition, including single-player coalitions and the grand-coalition, is equal to zero. The stage-game imputation $x = (24, 24; 0; 0)$ is in the single-period stage-game bargaining set: any objection by T must necessarily involve either the coalition T, I or T, C , but in either case, Y can form a counter-objection by way of a coalition with whichever player was excluded in T 's objection; T is in an even weaker position than Y with respect to justifiable objections.

Up to here, we have a classic bargaining-set story. Although it might seem superficially that Tom has greater leverage than Yuval, the existence of a counter-objection in the hands of the latter means that, despite the apparent asymmetry between them, the weaker party can in this case demand a fully equal division of the spoils as the price for continuing the traditional partnership.

² Note that the repeated game in each of these three examples is actually a repeated game, because the same characteristic function is used in each time period.

But now consider adding the element of time. Tom can tell Yuval: “I will sign a two-year contract with Ivan *and* Chang. Under the terms of the contract, both of them must remain loyal to me and not work with you. In the first year, Ivan and I will operate a harvester, while Chang rests at home, and I will pay Ivan 49 units and keep 25 for myself. In the second year, the roles will reverse – Chang will work with me, Ivan will be idle, and Chang will be paid 49 units for his efforts while I again take 25. Against this, you have no counter-objection: The most you can offer either for two years of work is 48.” Yuval indeed is out-leveraged.

Formally, the objection is the triple $\{S, \mathbf{R}(S), \mathbf{y}\}$: where $S = (TIC)$, $\mathbf{R}(S) = (\{TI, C\}, \{TC, I\})$ and $\mathbf{y} = \{y^1, y^2\} \in \mathbb{R}^{S \times 2}$ is given by $y^1 = (25, 49, 0)$ with $y^2 = (25, 0, 49)$. Any proposed counter-objection can give either player I or C at most the sum 48.

Note that what has happened here is that adding a time factor enables Tom, Chang, and Ivan to form a profitable three-person ‘time-staggered coalition’ which they could not have formed in a single period, giving Chang and Ivan an average of 24.5 units a year and Tom an average of 25 – and this changes the balance of threats in the game. An interesting side-effect is that, in each year, Tom’s objection calls for a player to be idle.

If the set $S = \{T, C, I\}$ had been restricted to forming a repeated coalition, instead of a dynamic coalition, then since $\nu^1(TCI) = \nu^2(TCI) = 0$, the result would obviously be quite different. In fact, if only repeated coalitions are allowed, the repeated game here has no solution other than simply repeating the single-stage solution in every period. ♦

Example 2. This is an example with an empty multi-period bargaining set. Continuing the previous example, in addition to Yuval, Tom, Chang and Ivan, we now have two new players, Ranjit and Sanelma, and the introduction of a three-person harvester. The characteristic function in this example enables Tom and Yuval to get two units as a partnership. Tom can work with Ivan and Ranjit to yield 5 units, but Tom working with Chang and Sanelma get only 3 units. Yuval is in the opposite but symmetric situation: Yuval working with Chang and Sanelma yields 5, but the team of Yuval, Ivan and Ranjit get only 3.

Ivan and Rajit in a two-person harvester are a strong team, gaining 20 units together, and Chang and Sanelma are also a great two-person team that can have 20 units. We assume Ivan and Chang cannot stand each other, Ranjit and Sanelma have a long-running feud, and the players do not know of each others existence.

Formally, with the set of players now denoted by $\{1, 2, 3, 4, 5, 6\}$, we have a repeated game with the stage-game defined by $\mathcal{R} = \{12, 3, 4, 5, 6\}$ and $\nu(12) = 2$, $\nu(1, 3, 4) = 5$, $\nu(1, 5, 6) = 3$, $\nu(2, 5, 6) = 5$, $\nu(2, 3, 4) = 3$, $\nu(3, 4) = \nu(5, 6) = 20$. The value of every other possible coalition, including single-player coalitions and the grand-coalition, is equal to zero.

Tom and Yuval bargain over how to divide the yields of two years of joint harvesting. Working as a team over two years, they divide at most 4 units between themselves, with Tom getting \mathbf{x}_1 , Yuval getting \mathbf{x}_2 , and $\mathbf{x}_1 + \mathbf{x}_2 \leq 4$. But whenever the suggested payoff grants Tom $\mathbf{x}_1 \leq 2$, Tom has an objection, in which he will

sign two-year contract with Ranjit, Ivan, Chang and Sanelma. In Year 1, he will work a three-person machine with Ivan and Ranjit, dividing the resulting 5 units by giving himself 1.25 and Ivan and Ranjit 1.875 each. At the same time, Chang and Sanelma work a two-person harvester, dividing the resulting 20 units equally between them. In Year 2, Tom, Chang and Sanelma are in the three-seater, each getting one unit for their efforts that year, and Ivan and Ranjit drive a two-person harvester, getting 10 a piece. Tom's sum total under this objection is 2.25 and Yuval has no counter-objection: if he works with exclusively with Chang and Sanelma in consecutive years, he can't come close to offering them what they receive under Tom's objection. So in a counter-objection he must work with Ivan and Ranjit at least one year, but even if he gives *both* of them *all* 3 units for working with him, taking nothing for himself, he cannot match Tom's offer.

Formally, if $\mathbf{x}_1 \leq 2$, player 1 has the objection $\{P, \mathbf{R}(P), \mathbf{y}\}$, where: $P = (1, 3, 4, 5, 6)$, $\mathbf{R}(P) = (\{134, 56\}, \{156, 34\})$, and $\mathbf{y} = \{y^1, y^2\} \in \mathbb{R}^{P \times 2}$ is given by $y^1 = (1.25, 1.875, 1.875, 10, 10)$ with $y^2 = (1, 10, 10, 1, 1)$. Player 2 has no feasible counter-objection.

What if the suggested payoff grants Yuval $\mathbf{x}_2 \leq 2$? By the symmetry of the situation, Yuval can then present Tom with the same objection as the one appearing in the previous paragraph, with the roles and payoffs of Chang and Sanelma swapped with those of Ivan and Ranjit (the three-person teams in the objection this time, of course, working with Yuval), and Tom is left without a counter-objection.

When $\mathbf{x}_1 = \mathbf{x}_2 = 2$, both Tom or Yuval can play the part of the sibling raising an unanswerable objection, depending on which one of them speaks first. The conclusion is that there is no possible division of two years of joint harvests by Yuval and Tom that is in the bargaining set. Unable to conclude their bargaining successfully, their partnership dissolves. Note in contrast that if the siblings are 'myopic', look ahead only one year at a time and care only about dividing the yield of that one year, they can easily find solutions in the bargaining set each and every year. ♦

Example 3. The Apex Corporation and the Zenith Company jointly run a network of shops with annual profits of \$100 million. Each year, the CEOs of Apex and Zenith split the profits, with Apex receiving \$75 million and Zenith \$25m. Neither company can run the operation without a partner. The only possible alternative partner is Midi Ltd. An Apex-Midi coalition can attain profits of \$100m, whilst a Zenith-Midi partnership will only receive \$50m. Any proposed deviation from the traditional 75 – 25 split will lead one or the other party to the negotiations to issue a threat to work with Midi.

Formally, with the set of players denoted by $\{1, 2, 3\}$, and the the stage-game is given by the coalition structure $\mathcal{R} = \{12, 3\}$, with $\nu(1) = \nu(2) = \nu(3) = \nu(123) = 0$, $\nu(12) = 100$, $\nu(13) = 100$, $\nu(23) = 50$. The bargaining set of the one-period stage-game consists of a single imputation, $(75, 25; 0)$.

Apex, however, decides one year that it needs an extra infusion of money to finance infrastructural investments. Knowing that any proposed deviation from the single bargaining set point will be useless, Apex proposes a three-year contract. Over three years, they will take in \$300m in profits, translating into a 75 – 25 split of

\$225m for Apex and \$75m for Zenith against which there can be no objection. But the disposition of those sums over the time periods need not follow a strict annual division \$75m to \$25m. Instead, in Year 1 Apex receive \$80m, sufficient to enable overseas investment. In Year 2, Apex will start ‘repaying’ Zenith by receiving only \$74m of that year’s \$100m and in Year 3, only \$71m.

The imputation sequence is thus $\mathbf{x} = (x^1, x^2, x^3)$, where $x^1 = (80, 20; 0)$, $x^2 = (74, 26; 0)$, and $x^3 = (71, 29; 0)$. Then it is the case that $x^t \notin I(N, \nu^t, \mathcal{R})$ for all $t = 1, 2, 3$, but never the less, $\mathbf{x} \in I(N, \nu, \mathcal{R})$. ♦

These three examples show that, in general, the dynamic coalition bargaining set $DB(N, \nu, \mathcal{R})$ may be empty, and that even when it is non-empty it is possible for the every element in an imputation sequence to be in the stage-game bargaining set for its respective time period without the sequence itself being in the repeated coalition bargaining set, while conversely, even if every element in an imputation sequence fails to be in the stage-game bargaining set, the sequence itself might still be in the repeated coalition bargaining set.

The reason that the dynamic coalition bargaining set may be empty stems from the following fact (a similar observation appears in [Kranich, Perea, Peters (2005)]): every cooperative repeated game (N, ν, \mathcal{R}) can be associated with a static nontransferable utility coalitional game:

Given $S \subseteq N$, m , and ν as above, and a coalition structure sequence $\mathcal{R}(S) = (\mathcal{R}^1, \dots, \mathcal{R}^m)$, with $\mathcal{R}^t \in \mathcal{C}(S)$ for each time period t , define $pmbI(\mathcal{R}(S))$ to be the set of all $\{(x^1, \dots, x^m)\} \in \mathbb{R}^{S \times m}$ such that, for all t , for all $P \in \mathcal{R}^t$, $x^t(P) \leq \nu^t(P)$, and for all $i \in N$, $\sum_{t=1}^m x_i^t \geq \sum_{t=1}^m \nu^t(\{i\})$.

Definition 1. The *static NTU-game associated with a cooperative repeated game* (N, ν, \mathcal{R}) is given by (N, V, \mathcal{R}) with

$$V(S) = \{(\bar{\mathbf{x}}_{i_1}, \bar{\mathbf{x}}_{i_2}, \dots, \bar{\mathbf{x}}_{i_{|S|}}) \in \mathbb{R}^S \mid \text{there is a } \mathcal{R}(S) \text{ with } \mathbf{x} \in I(\mathcal{R}(S))\}$$

where $\bar{\mathbf{x}}_i = \sum_t x_i^t$ for each $i \in S$, and $i_1, \dots, i_{|S|}$ is an enumeration of the set of players in S .

The following observation is nearly immediate from the definitions:

Lemma 1. *The dynamic coalition bargaining set of the repeated cooperative game (N, ν, \mathcal{R}) is non-empty if and only if the bargaining set of its associated static NTU game is non-empty.*

Proof: Let $\mathbf{x} \in I(N, \nu, \mathcal{R})$ be in the dynamic coalition bargaining set of (N, ν, \mathcal{R}) . Unravelling definitions, the vector $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \dots, \bar{\mathbf{x}}_{|N|}) \in \mathbb{R}^N$ is contained in $IX(N, V, \mathcal{R})$ of the associated NTU game (N, V, \mathcal{R}) . Suppose there is a justified NTU-objection (P, \hat{y}) of player k against player l at $\bar{\mathbf{x}}$ in (N, V, \mathcal{R}) . Then there is a coalition structure sequence over P , denote it $\mathcal{R}(P)$, such that $P \in T_{kl}$, and there is a $\mathbf{y} \in I(\mathcal{R}(P))$ corresponding to \hat{y} (i.e. $\mathbf{y} = \hat{y}$), such that $\bar{\mathbf{y}}_i \geq \bar{\mathbf{x}}_i$ for all $i \in P$, against which there is no counter-objection. But then $(P, \mathcal{R}(P), \mathbf{y})$ is a justified objection in the sense of the dynamic coalition bargaining set of the TU-game (N, ν, \mathcal{R}) , a contradiction. The proof in the other direction is similar. ■

Given that NTU games in general do not have non-empty bargaining sets (see [Peleg (1963)]), it is not surprising that the dynamic bargaining set of repeated TU-games may also be empty.

In contrast to the dynamic coalition bargaining set, the repeated coalition bargaining set $\mathcal{RB}(N, \nu, \mathcal{R})$ is guaranteed to be non-empty.

Definition 2. Given (N, ν, \mathcal{R}) , $S \subseteq N$, and $t \leq m$, define

$$q^t(S) := \begin{cases} \frac{1}{(m-t)+1} \sum_{l=t}^m \nu^l(S) & \text{when } m \text{ is finite} \\ \liminf_{T \rightarrow \infty} \frac{1}{T+1} \sum_{l=1}^T \nu^l(S) & \text{when } m \text{ is infinite} \end{cases}$$

Definition 3. The *static TU-game associated with a repeated game* (N, ν, \mathcal{R}) is given by (N, q, \mathcal{R}) , where $q(S) := q^0(S)$ for $S \subseteq N$.

Definition 4. Given (N, q, \mathcal{R}) associated with (N, ν, \mathcal{R}) and an element $x \in M(N, q, \mathcal{R})$, define each player i 's relative share with respect to x as $\alpha_i(x) = \frac{x_i}{q(\mathcal{R}(i))}$. The *monotonic imputation sequence with respect to x* , written $\mathbf{a}(x) = (a^1, \dots, a^m)$, is defined by setting $a_i^t = \alpha_i(x) \nu^t(\mathcal{R}(i))$.

When the context of $x \in M(N, q, \mathcal{R})$ is clear, $\alpha_i(x)$ will sometimes be written here simply as α_i .

Proposition 1. *If $I(N, q, \mathcal{R})$ is not empty, the repeated coalition bargaining set $\mathcal{RB}(N, \nu, \mathcal{R})$ is not empty – for each element in the associated static TU-game, $x \in M(N, q, \mathcal{R})$, every feasible imputation sequence $\mathbf{c} = (c^1, \dots, c^m)$ such that $\bar{\mathbf{c}}_i = \mathbf{a}(x)_i$ for each player i , including $\mathbf{a}(x)$ itself, is in $\mathcal{RB}(N, \nu, \mathcal{R})$.*

Proof: By well-known results (see, for example, Peleg, Sudhölter (2007)), the associated static TU-game (N, q, \mathcal{R}) has a non-empty bargaining set — i.e. there exists at least one vector $x = (x_1, \dots, x_n) \in M(N, q, \mathcal{R})$, such that no player k has a justified objection against another player l at x relative to the characteristic function q . Let x be an arbitrary such vector.

Writing $\alpha_i := \alpha_i(x)$, trivially, for each $S \in \mathcal{R}$, $\sum_{i \in S} \alpha_i \leq 1$, because $\sum_{i \in S} x_i \leq q(S)$. Let $\mathbf{a} = (a^1, \dots, a^m)$ be the monotonic imputation sequence with respect to x . The sequence is feasible for each $S \in \mathcal{R}$, because $a^t(S) = (\sum_{i \in S} \alpha_i) \nu^t(S) \leq q^t(S)$. We have in addition that for each $i \in N$, $\bar{\mathbf{a}}_i = \alpha_i q(\mathcal{R}(i)) = x_i$, so that $\mathbf{a} = (a^1, \dots, a^m)$ represents a way of granting each player an amount in each time period in such a way that the sum total over all time periods is exactly equal to the vector (x_1, \dots, x_n) . By definition, the same applies to any feasible imputation sequence $\mathbf{c} = (c^1, \dots, c^m)$ such that $\bar{\mathbf{c}}_i = \bar{\mathbf{a}}_i$ for each player i .

Suppose that (P, \mathbf{y}) is a repeated coalition objection of player k against player l at \mathbf{c} . Then $\bar{\mathbf{y}}_i \geq \bar{\mathbf{c}}_i$ for all $i \in P$ and $\bar{\mathbf{y}}_k \geq \bar{\mathbf{c}}_k$. By the definition of repeated coalitions, it must be the case that for each time period t , $y^t(P) \leq \nu^t(P)$, hence $\bar{\mathbf{y}}(P) \leq q(P)$. But, because $\bar{\mathbf{c}}_i = x_i$, this means that the pair $(P, \bar{\mathbf{y}})$ is an objection of player k against player l at x in the static game (N, q, \mathcal{R}) . As x is in the bargaining set of (N, q, \mathcal{R}) , there is by definition a justified counter-objection (Q, z) to $(P, \bar{\mathbf{y}})$ of l against k .

Defining the fractions $\beta_i = \frac{z_i}{q(\mathcal{R}(i))}$ and setting $\mathbf{b} = (b^1, \dots, b^m)$, $b_i^t = \beta_i \nu^t(\mathcal{R}(i))$, it follows that $\bar{\mathbf{b}}_i = z_i$, so $\bar{\mathbf{b}}_i \geq \bar{\mathbf{c}}_i$ for all $i \in Q$, $\bar{\mathbf{b}}_i \geq \bar{\mathbf{y}}_i$ for all $i \in P \cap Q$, and hence (Q, \mathbf{b}) is a justified repeated coalition counter-objection to (P, \mathbf{y}) at \mathbf{a} in (N, ν, \mathcal{R}) .

The conclusion is that \mathbf{c} is in the repeated coalition bargaining set of (N, ν, \mathcal{R}) . ■

Corollary 1. *In general, $\mathcal{DB}(N, \nu, \mathcal{R})$ may not equal $\mathcal{RB}(N, \nu, \mathcal{R})$.*

Proof: It was shown by example that $\mathcal{DB}(N, \nu, \mathcal{R})$ may be empty, even if $I(N, q, \mathcal{R})$ is not empty. But by the above proposition, $\mathcal{RB}(N, \nu, \mathcal{R})$ is guaranteed to be non-empty. ■

If the stage-games are superadditive, however, there is no need for distinguishing between dynamic coalitions and repeated coalitions:

Proposition 2. *If the stage-games of (N, ν, \mathcal{R}) are superadditive, then $\mathcal{DB}(N, \nu, \mathcal{R}) = \mathcal{RB}(N, \nu, \mathcal{R})$.*

Proof: Again, form the associated single-stage TU-game (N, q, \mathcal{R}) by setting, for every $S \subseteq N$, $q(S) = \sum_t \nu^t(S)$. Assuming $I(N, \nu, \mathcal{R})$ is not empty, select arbitrarily an imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ in $I(N, \nu, \mathcal{R})$.

Suppose that $(P, \mathcal{R}(P) = (\mathcal{D}^1, \dots, \mathcal{D}^m), \mathbf{y} = (y^1, \dots, y^m))$ is a dynamic coalition objection of k against l at \mathbf{x} . Denote by $(d_1^t, d_2^t, \dots, d_{E(t)}^t)$ the partition of P given by each \mathcal{D}^t . For each t , super-additivity implies $\nu^t(\bigcup_{j=1}^{E(t)} d_j^t) \geq \sum_{j=1}^{E(t)} \nu^t(d_j^t)$. Since the objection imputation at each time t must be feasible, for each $d_j^t \in \mathcal{D}^t$, $y^t(d_j^t) \leq \nu^t(d_j^t)$. By definition, $P = \bigcup_{j=1}^{E(t)} d_j^t$, so it follows that $\nu^t(P) \geq y^t(P)$ and therefore $\bar{\mathbf{y}}(P) \leq q(P)$.

Forming the sequence $\mathbf{a} = (a^1, \dots, a^m)$ by setting $\alpha_i = \frac{y_i}{q(P)}$ and $a_i^t = \alpha_i \nu^t(P)$ for each $i \in P$, it follows that $\bar{\mathbf{a}}_i = \bar{\mathbf{y}}_i$, and hence the net effect of the repeated coalition objection (P, \mathbf{a}) is equivalent to the net effect of the dynamic coalition objection $(P, \mathcal{R}(P), \mathbf{y})$.

Similarly, any dynamic coalition counter-objection $(Q, \mathcal{R}(Q), \mathbf{z})$ can be achieved equally well by the repeated coalition counter-objection (Q, \mathbf{b}) where $\mathbf{b} = (b^1, \dots, b^m)$ is derived by setting $\beta_i = \frac{y_i}{q(Q)}$ and $b_i^t = \beta_i \nu^t(Q)$ for each $i \in Q$. The conclusion is that under the assumptions of the proposition, any $\mathbf{x} \in I(N, \nu, \mathcal{R})$ is dynamic coalition stable if and only if it is repeated coalition stable. ■

These results give a general principle: if the possible coalitions are ‘fixed’ once and for all over all the time periods (or alternatively, if the characteristic functions are superadditive) then the players can always identify bargaining set points by following a two-stage process: first play a ‘grand TU-game’ in which the characteristic function for each coalition is the time average of the payoffs in the separate time periods, and agree on a bargaining set solution. In the second stage, determine payoffs in particular for each time period, subject to the constraint that the time-average each player receives over all time equals the bargaining-set payoff agreed upon in the first stage. This adds great flexibility, as shown in the last example: for each grand game bargaining set point, there are an infinite number of time-stream

payoffs that are possible, which would not be conceivable if the players were to play a separate and isolated cooperative game in each separate period.

The reason dynamic coalitions in non-supperadditive situations can lead to repeated game bargaining set solutions that diverge from solutions derived directly from stage game solutions – as in Example 1 presented above – is because in that situation, the associated TU-game does not reflect accurately what can be attained in the repeated game. A dynamic coalition over $S \subseteq N$ can, in certain cases, attain for its members a total pay-off greater than $q(S)$ by cleverly arranging different coalition structures over S in different time periods – but that total pay-off may not necessarily be freely transferable between the members of S .

5. CREDIT SEQUENCES

Assume in this section that in imputation sequences $\mathbf{x} = (x^1, \dots, x^m)$, $x^t(S) = \nu^t(S)$ for all time periods t . It will also be assumed in this section (and the next) that m is always a finite number.

By Proposition 1 of the previous section, the set of bargaining set solutions for a repeated game (N, ν, \mathcal{R}) is at least as large as the set of bargaining set solutions of its associated static TU-game (N, q, \mathcal{R}) – for each solution $x \in M(N, q, \mathcal{R})$, the monotonic imputation sequence $\mathbf{a}(x)$ is a solution of (N, ν, \mathcal{R}) . But this by no means begins to exhaust the set of solutions of the bargaining set of (N, ν, \mathcal{R}) , as the same proposition extends that set to any feasible imputation sequence $\mathbf{c} = (c^1, \dots, c^m)$ such that $\bar{c}_i = \bar{a}(x)_i$ for each player i .

Consider the repeated cooperative game based on the stage-game (N, ν, \mathcal{R}) of Example 3 above. The associated static TU-game of that example has a single bargaining set solution given by $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = 0$, in terms of player share. In sharp contrast, the bargaining set of $(N, \nu, \mathcal{R}, 3)$ has an infinite number of solutions. Example 3 exhibits one such solution, which deviates from the monotonic imputation sequence in each time period. But clearly a solution in the bargaining set of the repeated cooperative game cannot allow *every* feasible imputation in *every* period – for example, any imputation sequence $\mathbf{x} = (x^1, x^2, x^3)$ with $x^1 = (25, 75; 0)$ must lie outside the bargaining set of (N, ν, \mathcal{R}) , even if x^2 and x^3 are each feasible imputations in their respective time periods.

The intention of this section is to give a finer characterisation of the possible solutions of the bargaining set of a repeated cooperative game by establishing bounds on the payoffs that can be granted to each player in each time period within the context of a repeated coalition bargaining set solution.

Returning to Example 3, consider a solution given by $\mathbf{x} = (x^1, x^2, x^3)$, $x^1 = (80, 20; 0)$, $x^2 = (74, 26; 0)$ and $x^3 = (71, 29; 0)$. One way to regard this solution is to interpret it as if player 1 ‘justifies’ the first-period imputation of $(80, 20; 0)$, which deviates from the static bargaining set solution of $(75, 25; 0)$, by ‘borrowing’ 5 units from player 2. The debt is then re-paid over the next two time periods.

This is, of course, an anthropomorphic story that is overlaid over a particular solution to a mathematical construct – and in fact, there are many such ‘stories’

that can be told relative to each solution – but it so enhances intuitive insight that we present here a formalisation of the idea of players borrowing and repaying debts over time. Within the context of studying the core in dynamic situations, [Kranich, Perea, Peters (2001)] and [Berden (2007)] consider what they term inter-temporal transfers, in which players receive more in some time periods at the expense of less in other time periods. This takes the form of postulating a sequence $\{c_i^t\}$ for each player, such that $\sum_{t=1}^m c_i^t = 0$, which changes the player's payoff in each time period relative to an imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ from x_i^t to $x_i^t + c_i^t$. We expand here on this idea, explicitly taking into account the need for one player to borrow from another player in each time period in order to increase his personal payoff, which then also imposes a requirement on the debtor to repay the creditors in a later time period.

This motivates the following:

The players first select a ‘goal’ vector $g \in M(N, q, \mathcal{R})$ from the bargaining set of the associated TU-game. This is intended to be interpreted as an agreement between them that the sum-total of the imputation sequence of the m -period game they are to play will be the vector g . The vector g , along with the array of associated relative shares per player $\alpha_i(g)$, can be regarded as determining an over-all canonical ‘income distribution’. The monotonic imputation sequence given by these relative shares with respect to g , $\mathbf{a}(g) = (a^1, \dots, a^m)$, also serves as a ‘baseline’ against which deviations in imputations in particular time periods are interpreted as credits and debits.

Definition 5. A *credit sequence* relative to a repeated cooperative game (N, ν, \mathcal{R}) and a vector $g \in M(N, q, \mathcal{R})$ is composed of real numbers $\{d_{i,j}^t\}$, $\{p_{i,j}^t\}$ defined inductively at each time period t for each pair of players $i, j \in N$, subject to the following list of constraints:

- (1) $d_{i,j}^t \geq 0$; $d_{i,j}^t = 1$ whenever i and j are not in the same partition of \mathcal{R} ; $d_{i,i}^t = 0$ for all $i \in N$ and times $t \geq 0$. We also define $d_{i,j}^0 = 0$ to initiate the induction, for all $i, j \in N$.
- (2) $p_{i,j}^t \geq 0$; $p_{i,j}^t = 0$ whenever i and j are not in the same partition of \mathcal{R} ; $p_{i,i}^t = 0$ for all $i \in N$ and times $t \geq 1$. We also define $p_{i,j}^0 = 0$ to initiate the induction, for all $i, j \in N$.
- (3) $p_{i,j}^t \leq \sum_{l=1}^{t-1} (d_{i,j}^l - p_{i,j}^l)$ for all $t \geq 0$, for all $i, j \in N$.
- (4) For each $i \in N$ and time period t , defining $c_{i,j}^t := c_{i,j}^{t-1} + d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t$ for each j and $c_i^t := \sum_{j \in N} c_{i,j}^t$, constrain c_i^t to be $c_i^t \leq \sum_{l=t+1}^m a_i^l$ (with the understanding that $\sum_{l=m+1}^m a_i^l = 0$), where a_i^t is given at each t by the monotonic imputation sequence $\mathbf{a}(g)$.
- (5) $\sum_{j \in N} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t \leq a_i^t$ for each time period t and each player $i \in N$.

Given a credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ relative to $g \in M(N, q, \mathcal{R})$, a feasible imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ will be said to be *derivable from* $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ if for each $i \in N$ and time period t , $x_i^t = a_i^t + \sum_{j \in N} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t$.

These constraints are justified by intuitive interpretations. Each $d_{i,j}^t$ is to be interpreted as saying that ‘ i is indebted to (or borrows from) j in period t the

amount $d_{i,j}^t$ ’ or equivalently that ‘ j has an IOU written by i in period t for $d_{i,j}^t$ units’, and each $p_{i,j}^t$ is intended to represent a (possibly partial) re-payment of a debt made by player i to j in period t . Constraint 1 then says that debts are always counted in positive units, that players only borrow from their partners relative to the coalition structure, and that a player can never borrow from himself. Constraint 2 says much the same about debt repayments.

The term $\sum_{l=1}^{t-1} d_{i,j}^l - p_{i,j}^l$ represents ‘total outstanding debt’ owed by player i to player j in time period t – it sums up all IOUs given over all previous time periods by i to j , and subtracts all repayments made against them. Constraint 3 then states that in time period t player i can not give player j more, in debt re-payments, than the total outstanding debt he owes to j (this does not prevent i from giving j more than $\sum_{l=2}^{t-1} d_{i,j}^l - p_{i,j}^l$ – but any transfer from i to j greater than that sum will be counted as a loan from i to j).

In each time period t , c_i^t represents the ‘cumulative debt portfolio’ held by player i , as it takes into account all loans given to other players, all loans taken and all the respective re-payments to date. Note that although each $d_{i,j}^t$ is greater than or equal to zero, $c_{i,j}^t$ may be positive or negative – if it is positive, then player i is a net debtor with respect to j , and if it is negative, i is a net creditor with respect to j . It also follows from the definitions that $c_{i,j}^t = -c_{j,i}^t$, and hence that in any single time period and for any single $S \in \mathcal{R}$, $\sum_{i \in S} c_i^t = 0$.

Under that interpretation, constraint 4 establishes an important ‘credit limit’ for each player i , in the following sense. The vector g determines the ‘total income’ for player i as g_i , which by definition equals $\mathbf{a}_i = \sum_t a_i^t$. At each time period t , therefore, $\sum_{l=t+1}^m a_i^l$ represents player i ’s future income stream. Constraint 4 is intuitively a ‘no-default’ condition: at no time is a player permitted to have outstanding positive cumulative debt which is greater than his future income stream – total debt in this system is always staked against future income.

Note, however, that because constraint 4 applies to the total debt portfolio of a player, there is an implication under this system that a player can borrow both against future income and against previously issued IOUs he holds. In effect, ‘debt securities’ which are tradable and negotiable instruments arise naturally from the system.

Constraint 5 exists to ensure that under any imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ derivable from a credit sequence, no player transfers to others so much in loans and repayments that he receives less than zero.

Proposition 3. *In the context of a repeated cooperative game (N, ν, \mathcal{R}) , each imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ derivable from a credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ relative to a vector $g \in M(N, q, \mathcal{R})$ is located within the repeated coalition bargaining set $\mathcal{RB}(N, \nu, \mathcal{R})$. Conversely, for each $\mathbf{x} = (x^1, \dots, x^m)$ in $\mathcal{RB}(N, \nu, \mathcal{R})$, there is at least one credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ relative to $g = \bar{\mathbf{x}} = (\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_n)$ such that \mathbf{x} is derivable from $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$.*

The proof appears in the appendix.

Proposition 4. *In a repeated cooperative game (N, ν, \mathcal{R}) , if the players agree on an over-all goal vector $g \in M(N, q, \mathcal{R})$, then in any imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$, at time period t the largest value received by any player is bound by $x_i^t \leq \min(\nu^t(\mathcal{R}(i)), g_i)$ and the smallest by $x_i^t \geq \max(0, a_i^t - \sum_{j \neq i} (\sum_{l=1, l \neq t}^m a_j^l))$.*

Proof: Obviously, player i cannot receive at time t more than the total produced by the coalition to which he belongs, hence not more than $\nu^t(S)$. What he receives, however, is also limited by the fact that $\mathbf{x} = (x^1, \dots, x^m)$ must be derivable from a credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$, as per the previous proposition. Under any credit sequence, at time t he cannot borrow more than his ‘future income stream’, given by $\sum_{j \neq i} d_{i,j}^t \leq \sum_{l=t+1}^m a_i^l$, and the most he can receive in debt-repayment is limited by the most he could have lent in past periods, given by $\sum_{l=1}^{t-1} a_i^l$, so the most he can pocket in time t is $\sum_{l=t+1}^m a_i^l + \sum_{l=1}^{t-1} a_i^l + a_i^t = g_i$. The maximal value of x_i^t is then the smaller of $\nu^t(S)$ or g_i .

For calculating the least value of x_i^t , clearly player i cannot receive less than 0. Again, what he receives is also limited by the fact that $\mathbf{x} = (x^1, \dots, x^m)$ must be derivable from a credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$. As a lender, he cannot give the other players in his coalition more than their ‘credit limit’ at time t , which is represented by $\sum_{j \neq i} \sum_{l=t+1}^m a_j^l$, their ‘future income streams’. On the other hand, as a (former) borrower the most he can now repay is limited by the most they could have lent him, which is given by their *past* income streams $\sum_{j \neq i} \sum_{l=1}^{t-1} a_j^l$. This means the greatest possible downward deviation from a_i^t is limited by $\sum_{j \neq i} (\sum_{l=1}^{t-1} a_j^l + \sum_{l=t+1}^m a_j^l)$. ■

6. SUBGAME PERFECT SEQUENCES

The paradigm in which the players negotiate a target goal vector $g \in M(N, q, \mathcal{R})$ relative to a repeated cooperative game (N, ν, \mathcal{R}) , against which they then negotiate a contract establishing the detailed imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$ they will share ‘once and for all’, enfold within it implicit assumptions regarding enforcement. Cooperative game theory itself, of course, leans on an implicit enforcement postulation – even in the single-stage case, the players negotiate an imputation of the payoff they will receive for forming a coalition, with an assumed enforcement mechanism ensuring that the agreed-upon imputation will be granted to the players.

In the multi-stage case, the assumption of an enforcement mechanism is even more critical, especially given the interpretation presented in the previous section of the imputation sequence as encoding ‘inter-temporal’ borrowing from one player to another: a player who has borrowed heavily in the earlier rounds and in later rounds is expected to repay the loans by accepting imputations outside the bargaining set, has a strong incentive to defect to another coalition, thus defaulting on his debt to the detriment of the other players.

An example can further elucidate this possibility:

Example 4. Example 3 has one potential weakness: suppose the CEO of Apex is a devious person. Then, after receiving \$80m in Year 1, she can demand that the

contract be ‘renegotiated’ in Year 2, or 3, with the previous year’s deviation from the single-period bargaining set point ‘forgotten’. She can do this with the credible threat to partner with Midi if Apex does not receive \$75m in each year subsequent to Year 1. ♦

This motivates the idea of a subgame perfect multi-period bargaining set imputation, which meets the constraint that in each time period, no player can form a justified objection to the multi-period imputation taking into account only the remaining time periods. We re-iterate that in this section, as in the previous one, m will always be understood to be a finite number.

The idea of players mistrusting each other when multiple rounds of a game are being played appears in several early papers (such as [Gale (1978)] and [Becker, Chakrabarti (1995)]) and in particular has become a theme in studies of the core in dynamic cooperative games, where concepts such as the weak and strong sequential cores have been developed to analyse such situations (see [Kranich, Perea, Peters (2005)], and [Predtetchinski (2007)]). It is in that spirit that we present the following definition.

Definition 6. An imputation sequence $\mathbf{x} = (x^1, \dots, x^m)$, relative to a repeated game $(N, \boldsymbol{\nu}, \mathcal{R})$, is *subgame perfect* if for each time period t , the sub-sequence of vectors $(x^t, x^{t+1}, \dots, x^m)$ is in the bargaining set of the static TU-game (N, q^t, \mathcal{R}) defined by the characteristic function $q^t(S) = \sum_{l=t}^m \nu^l(S)$ for all $S \subseteq N$.

Denote the set of subgame perfect imputation sequences of $(N, \boldsymbol{\nu}, \mathcal{R})$ by $\mathcal{SP}(N, \boldsymbol{\nu}, \mathcal{R})$. Clearly, $\mathcal{SP}(N, \boldsymbol{\nu}, \mathcal{R}) \subseteq \mathcal{RB}(N, \boldsymbol{\nu}, \mathcal{R})$.

Subgame perfect stability guards against player defection in later rounds by replicating the stability of the bargaining set with respect to future time periods at any point in time: any suggested defection by a player with respect to future time periods by way of an objection can be met by a counter-objection. Example 4 shows that the set of subgame perfect sequences, if it exists, is generally strictly smaller than the set of repeated coalition bargaining set sequences.

Definition 7. A sequence of characteristic functions $\mathbf{v} = (v^1, \dots, v^m)$ defined relative to a set of players N and a coalition structure \mathcal{R} is *sequentially essential* if for each time period t and each $S \in \mathcal{R}$, $v^t(S) \geq \sum_{i \in S} v^t(i)$.

Proposition 5. If m is finite, and $\mathbf{v} = (v^1, \dots, v^m)$ is sequentially essential relative to N and \mathcal{R} , the set of subgame perfect sequences $\mathcal{SP}(N, \boldsymbol{\nu}, \mathcal{R})$ is not empty.

Proof: This is proved by a backwards induction argument (hence the condition of finiteness of m). Begin the induction by selecting an arbitrary element x^m in the bargaining set of (N, q^m, \mathcal{R}) .

Suppose, for $t < m$, the sequence $(x^{t+1}, x^{t+2}, \dots, x^m)$ is in the bargaining set of $(N, q^{t+1}, \mathcal{R})$. Naïvely, it might seem that in order to define x^t it would suffice to select arbitrarily an element $\hat{x} \in M(N, q^t, \mathcal{R})$ and set $x_i^t = \hat{x}_i - \sum_{l=t+1}^m x_i^l$ for each player i . The problem is that there is no guarantee this procedure will yield a non-negative value for each x_i .

This potential flaw can, however, be avoided by a tweak to the procedure. Instead of working with q^t , define the characteristic function

$$b^t(S) = \begin{cases} \sum_{l=t+1}^m x_i^l + \nu^t(\{i\}) & S = \{i\}, i \in N \\ 0 & S = \emptyset \\ q^t(S) & \text{otherwise} \end{cases}$$

It must now be shown that the set $I(N, b^t, \mathcal{R})$ is non-empty. Select arbitrarily $S \in R$, and define $r = q^t(S) - \sum_{i \in S} b^t(\{i\})$. The assumptions that $\mathbf{v} = (v^1, \dots, v^m)$ is sequentially essential and that each element of $(x^{t+1}, x^{t+2}, \dots, x^m)$ is a feasible vector at its respective time period implies that $q^t(S) = q^{t+1}(S) + \nu^t(S) \geq \sum_{i \in S} (\sum_{l=t+1}^m x_i^l + \nu^t(\{i\}))$, but the last term is equal to $\sum_{i \in S} b^t(\{i\})$, so that $r \geq 0$. Defining the $|S|$ -vector \hat{x}' by $\hat{x}'_i = b^t(\{i\}) + \frac{r}{|S|}$, we have $\hat{x}'(S) = q^t(S)$. As S was selected arbitrarily, it follows that $I(N, b^t, \mathcal{R})$ is not-empty.

We can therefore select a vector \hat{x} in the bargaining set of (N, b^t, \mathcal{R}) , and now set $x_i^t = \hat{x}_i - \sum_{l=t+1}^m x_i^l$, confident that this will not lead to negative values, and that by construction $(x^t, x^{t+1}, \dots, x^m)$ is in the bargaining set of (N, q^t, \mathcal{R}) . Continuing with this backward induction to time period 1, we are done with identifying a subgame perfect imputation sequence for $(N, \mathbf{v}, \mathcal{R})$. ■

Finally, we show by an example that the contrast between the set of repeated coalition bargaining set sequences and the set of subgame perfect sequences goes beyond the fact that the latter is generally a subset of the former. As shown in Proposition 1, in seeking a repeated-coalition bargaining set sequence, the players may first select any solution in the associated static TU-game bargaining set and then fit a sequence to that static solution. But if the players seek a subgame perfect sequence, they might not be able to rely on first considering the associated static game and then finding a sequence that fits that.

Symbolically, for any set Q of imputation sequences of a repeated game $(N, \mathbf{v}, \mathcal{R})$, denote $T(Q) := \{\bar{\mathbf{x}} \mid \mathbf{x} \in Q\}$. Then clearly $T(\mathcal{SP}(N, \mathbf{v}, \mathcal{R})) \subseteq T(\mathcal{RB}(N, \mathbf{v}, \mathcal{R}))$, but as the next example shows, equality between these sets does not hold.

Example 5. Continuing the above Example 4, let us now have, in addition to Apex, Zenith and Midi, two other companies, Generi and Anonymi. Apex and Zenith, as before share \$100m between them in Year 1. Apex can in Year 1 alternatively form a two-company partnership with each of Midi, Generi and Anonymi, and attain \$100m in profits. The only other option open to Zenith in Year 1 is to form a coalition that includes it and Midi and Generi and Anonymi, for a total of \$200m. In Year 2, market conditions are expected to change. Apex and Zenith can still obtain \$100m as partners in year 2, but if Apex wishes to have an alternative to Zenith, it must partner with Midi and Generi and Anonymi, for only \$50m, whilst a coalition of Zenith, Midi, Generi and Anonymi can attain \$100m.

Formally, let $n = 5$, with the set of players N denoted by 1, 2, 3, 4, 5. Consider a 2-period repeated game $(N, \mathbf{v}, \mathcal{R})$ with coalition structure $\mathcal{R} = \{12, 3, 4, 5\}$ and $\mathbf{v} = \{\nu^1, \nu^2\}$ defined by $\nu^1(12) = \nu^1(13) = \nu^1(14) = \nu^1(15) = 100$, $\nu^1(2345) = 200$, $\nu^2(12) = 100$, $\nu^2(1345) = 50$, $\nu^2(2345) = 100$. The value of every other possible coalition at all time periods is equal to zero.

The grand-game over two time periods has a bargaining-set solution that grants Apex \$150m and Zenith \$50m of the \$200m they can create together over two years. But there can be no subgame perfect sequence summing to (\$150m, \$50m) over two time periods, because a division of (\$25m, \$75m) between Apex and Zenith is the only possible payoff in Year 2 of any subgame perfect sequence – and then Zenith is already guaranteed a multi-period payoff greater than \$50m. ♦

In conclusion, we can state the following about subgame perfect stability: in the repeated game setting (in which the same characteristic function holds true in each time period), a subgame perfect solution always exists – even when there are an infinite number of time periods – because the canonical monotonic sequence is always subgame perfect. In the repeated game setting, the monotonic sequence might not be subgame perfect – as shown in Example 4. Example 5 shows that there might not be a subgame perfect sequence summing to each solution of the associated static game. When there are a finite number of time periods, a subgame perfect solution can be found, even in the repeated game setting, as shown in Proposition 5. It is unclear, as of this writing, whether that result can be extended to the case of an infinite number of time periods.

7. APPENDIX

Proof of Proposition 3: Suppose $\mathbf{x} = (x^1, \dots, x^m)$ is a feasible imputation sequence derivable from a credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ relative to vector g . First of all, for each t , x^t is feasible: because $d_{i,j}^t = 0$ and $p_{i,j}^t = 0$ whenever i and j are not partners in the same partition of \mathcal{R} , we can write $x_i^t = a_i^t + \sum_{j \in \mathcal{R}(i)} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t$. Given $Q \in \mathcal{R}$, $\sum_{i \in Q} \sum_{j \in Q} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t = 0$, so that we arrive at the conclusion $\sum_{i \in Q} x_i^t = \sum_{i \in Q} a_i^t = \nu^t(Q)$.

Recalling that we have assumed that m is finite, hence there is a last period m . Define $\Delta_i^t := x_i^t - a_i^t = \sum_{j \in N} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t$. By condition 4, in time period m , for each player i , $c_i^m \leq 0$. But $\sum_{i \in N} c_i^m = 0$, hence $c_i^m = 0$ must hold for each i . As by definition, $c_i^m = \sum_t \sum_{j \in N} d_{i,j}^t - p_{i,j}^t - d_{j,i}^t + p_{j,i}^t$, we conclude that $\sum_{t=1}^m \Delta_i^t = 0$, so $\sum_{t=1}^m x_i^t = \sum_{t=1}^m a_i^t = g_i$.

In the other direction, suppose that $\mathbf{x} = (x^0, x^1, \dots, x^m)$ is an imputation sequence, with the goal of exhibiting a credit sequence from which \mathbf{x} is derivable. This is done inductively, with a round of re-payments defined first in each time period, followed by a round of debt allocations. Intuitively, the construction here is rather simple: in each time period, each player strives to re-pay as much debt as possible. After that, all other deviations from x^t are ‘explained’ by way of transfers undertaken through loans.

To decrease some of the clutter of symbols, define $o_{i,j}^t := \sum_{l=1}^{t-1} d_{i,j}^l - p_{i,j}^l$ and $o_i^t := \sum_{j \in N} o_{i,j}^t$. As before, $\mathbf{a}(x) = (a^1, \dots, a^m)$ is the monotonic sequence defined against g .

Suppose that $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ has been defined for all time periods less than t . In period 0, no re-payment is effected. Otherwise, the re-payment round is defined as follows: each player ‘re-pays as much as possible’ of outstanding debt o_i^t , re-payment

capped only by a_i^t , so the sum total of re-payment by player i in time period t is given by $r = \min(a_i^t, o_i^t)$. Let D_i^{t+} be the set of players such that for each player $l_j \in D_i^{t+}$, $o_{i,l_j}^t > 0$ and order them by decreasing ‘debt’ size, i.e. l_j comes before l_k only if $o_{i,l_j}^{t-1} \geq o_{i,l_k}^{t-1}$, with arbitrary ordering when this last semi-inequality is a strict equality. Next, set $n(i)$ to be the smallest integer such that $\sum_{j=1}^{n(i)} o_{i,l_j}^{t-1} \leq a_i^t$ under this ordering. For $1 \leq j \leq n(i)$, let $p_{i,j}^t = o_{i,l_j}^{t-1}$, and for $j = n(i) + 1$, if there exists an element $l_{n(i)+1}$ in D_i^{t+} , let $p_{i,j}^t = a_i^t - \sum_{k=1}^{n(i)} o_{i,l_k}^{t-1}$.

After the round of re-payments has been completed, we have for each player i the value $f_i^t := a_i^t - \sum_{j \in N} p_{i,j}^t + \sum_{j \in N} p_{j,i}^t$, and it is against these values that the round of debt allocation is conducted. Let $\hat{\Delta}_i^t := f_i^t - a_i^t = \sum_{j \in N} p_{i,j}^t + p_{j,i}^t$, and, for an arbitrary $S \in \mathcal{R}$ order the players in S as i_1, \dots, i_k by decreasing size of $\hat{\Delta}_i^t$. Further define $\hat{\Delta}_+^t = \{i \in S | \hat{\Delta}_i^t > 0\}$ and $\hat{\Delta}_-^t = \{i \in S | \hat{\Delta}_i^t < 0\}$. List the elements $\hat{\Delta}_+^t$ as $\{j_1, \dots, j_k\}$, ordered by decreasing size of $\hat{\Delta}_j^t$, and similarly list the elements of $\hat{\Delta}_-^t$ as $\{h_1, \dots, h_l\}$, ordered by decreasing size of $|\hat{\Delta}_h^t|$.

Define for each member of $\hat{\Delta}_+^t$ a set of ‘creditors’ in $\hat{\Delta}_-^t$ as follows. Set $C_{j_1} := \{h_1, \dots, h_{m(j_1)}\}$ such that $\sum_{i=1}^{m(j_1)} |\hat{\Delta}_{h_i}^t| \geq \hat{\Delta}_{j_1}^t$, where $m(j_1)$ is the smallest integer such that this inequality holds. Set $d_{j_1, h_1}^t = |\hat{\Delta}_{h_1}^t|$ for $i < m(j_1)$, and $d_{j_1, h_{m(j_1)}} = \hat{\Delta}_{j_1}^t - \sum_{i=1}^{m(j_1)-1} |\hat{\Delta}_{h_i}^t|$.

For calculating C_j , for $s > 1$, first set

$$d_{j_s, h_{m(j_{s-1})}}^t := |\hat{\Delta}_{h_{m(j_{s-1})}}^t| - d_{j_{s-1}, h_{m(j_{s-1})}}^t$$

and then set $C_{j_s} := \{h_{m(j_{s-1})}, \dots, h_{m(j_s)}\}$ such that

$$\hat{\Delta}_{j(s)}^t \leq d_{j_s, h_{m(j_{s-1})}}^t + \sum_{i=m(j_{s-1})+1}^{m(j_s)} |\hat{\Delta}_{h_i}^t|$$

and $m(j_s)$ is the smallest integer such that this inequality holds. Set $d_{j_s, h_i}^t = |\hat{\Delta}_{h_i}^t|$ for $m(j_{s-1}) < i < m(j_s)$, and

$$d_{j_s, h_{m(j_s)}}^t = \hat{\Delta}_{j_s}^t - \sum_{i=m(j_{s-1})+1}^{m(j_s)-1} |\hat{\Delta}_{h_i}^t|.$$

It remains to be shown that following these steps leads to an admissible credit sequence $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$. Constraints 1 and 2 are trivially met by the constructed credit sequence. Constraint 3, which limits the size of re-payments, is explicitly guaranteed by the construction, as is constraint 5.

To see that constraint 5 is met, note that by the way $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$ are constructed, for any player i and time period t , $\sum_{l=1}^t x_i^l = \sum_{l=1}^t a_i^l + c_i^t$. On the other hand, by assumption $\sum_{l=1}^m x_i^l = \sum_{l=1}^m a_i^l$. Hence, if $c_i^t > \sum_{l=t+1}^m a_i^l$, $\sum_{l=1}^t x_i^l = \sum_{l=1}^m a_i^l$, which would require $\sum_{l=t+1}^m x_i^l$ to be a negative quantity in order to ensure $\sum_{l=1}^m x_i^l = g_i$. This is impossible, and we conclude constraint 5 holds. ■

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