Strictness and Evolutionary Stability

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August 1994

Abstract

The notion of a strict equilibrium set is introduced as a natural extension of the notion of a strict equilibrium point. The evolutionarily stable sets of a truly asymmetric contest are shown to be behaviorally equivalent to the strict equilibrium sets of an "agent representation" of the contest. Using variants of the replicator dynamic we provide dynamic characterizations of strict equilibrium sets. We do this both for truly asymmetric contests and for arbitrary normal form games modelling conflicts between several distinct species.

For stimulating discussions on the subject I would like to thank Dirk Bergemann, Oliver Kirchkamp, George Mailath, Georg Nöldeke, Daniel Probst, Klaus Ritzberger, Reinhard Selten, Avner Shaked and Karl Schlag. The paper was written while I visited the University of Pennsylvania and the Center for Rationality and Interactive Decision Theory at the Hebrew University of Jerusalem. I am grateful for the hospitality received by these institutes.

1. Introduction

We introduce the notion of a strict equilibrium set and study its relation to the notion of an evolutionarily stable set due to Thomas [32]. One relation appears in the case of so-called truly asymmetric contests. These are games modelling a conflict between the animals of a single species. With every truly asymmetric contest two normal form games are associated: There is the standard normal form of the contest, which describes the conflict ex ante, before the animals are born. This normal form game is symmetric. In addition we introduce the "agent representation" of the contest. The agent representation describes the conflict from an interim perspective, after the animals have been assigned different roles in the conflict and before the conflict actually occurs. This normal form game is typically not symmetric. We show that the evolutionarily stable sets of the normal form of a truly asymmetric contest are "behaviorally equivalent" to the strict equilibrium sets of its agent representation. This result extends a result of Selten [25] on evolutionarily stable *strategies* and strict equilibrium *points* in truly asymmetric contests.

A second relation between strict equilibrium sets and evolutionarily stable strategies appears in a dynamic context. A symmetric normal form game can model a conflict between the animals of a single species. One can define for such a game a replicator dynamic in continuous time where mixed strategies can be inherited. We characterize evolutionarily stable sets in terms of stability properties with respect to this dynamic.

Every symmetric or non-symmetric normal form game can also be interpreted as a model of a conflict between the animals of several distinct species where each player represents a different species. One can define a corresponding second replicator dynamic in continuous time where mixed strategies can be inherited. We characterize strict equilibrium sets in terms of stability properties with respect to this second dynamic. The stability properties used are the same as for evolutionarily stable sets, but they refer to a different dynamic with a different biological interpretation.

In the following we try to motivate the relevant concepts and the results in more detail: The notion of an *evolutionarily stable strategy* by Maynard Smith and Price [18] has been one of the most important conceptual innovations in non-cooperative game theory. It provided an unexpected link between normative game theoretic analysis and evolutionary biology where conscious rational behavior cannot be assumed. More recently evolutionary stability has been used by economists to discuss bounded rational behavior of economic agents.

Sometimes, e.g., for certain games in extensive form, it is desirable not to require only single strategies to be evolutionarily stable. Rather one would like to attribute evolutionary stability to a class of strategies inducing the same observed outcome. Thomas' [32] notion of an *evolutionarily stable set* of strategies allows for this possibility. An evolutionarily stable set is a set of strategies characterized by two conditions. These conditions refer to a symmetric conflict between a population of animals belonging to the same species. Roughly speaking, the first condition requires that a population using a strategy in the set cannot be invaded by a small group of mutants using a different strategy (i.e., all strategies in the set are neutral evolutionarily stable). The second condition states that the population can only drift from the use of one strategy in the set to the use of another strategy in the set.

A *strict equilibrium point* is a Nash equilibrium where each player has a unique best reply. This concept has been used for a long time in non-cooperative game theory. By definition strict equilibrium points do not suffer – in contrast to Nash equilibria in mixed strategies – from what Harsanyi and Selten [13] call the instability problem: In an equilibrium where one player has several best replies this player has no intrinsic self-interest to stick to his equilibrium strategy. If he chooses an alternative best reply he might upset the equilibrium. The instability problem may be less severe in those cases where a deviation with an alternative best reply leads again to equilibrium play. Thus we introduce *strict equilibrium sets* as sets of Nash equilibria with the following property: If a single player deviates from a Nash equilibrium in the set by using an alternative best reply then this leads to the play of a different Nash equilibrium in the set.

Both strict equilibrium points and strict equilibrium sets have an obvious evolutionary stability property: Consider a normal form game as a model of a conflict between several distinct species. Suppose that all animals of the same species use the same strategy. Suppose that this yields a strategy combination which is either a strict equilibrium point or an element of a strict equilibrium set. Consider now the situation where a few animals which belong to the same species mutate and use a different strategy. Since the animals of the other species do not mutate they will not change their behavior. Therefore the mutant strategy will die out if the mutants do not play a best reply against the strategy choices of the opponents. In case of a strict equilibrium point the mutant strategy must therefore die out. In case of a strict equilibrium set the mutant strategy can survive only if it is an alternative best reply. In the latter case drift might lead to a different strategy combination which is however again an element of the strict equilibrium set.

A strict equilibrium set is defined for an arbitrary normal form game. It is a set of *strategy combinations*. An evolutionarily stable set is defined only for a *symmetric* normal form game. It is a set of *strategies*. A link between the two concepts is provided by models of animal conflicts with role asymmetry (Selten [25], see also Selten and Hammerstein [9]) or, as they are called in van Damme [34], by *truly asymmetric contests*. As indicated, such games can be viewed from two perspectives: A view ex ante which yields a symmetric normal form game and a view from an interim perspective which yields a normal form game that is not necessarily symmetric. Formally a truly asymmetric contest is itself not a normal form game. It resembles more a Bayesian game or a game in extensive form.

As an example for a truly asymmetric contest we can take a conflict between two animals of the same species, one being the owner of a territory and the other being an intruder. The owner can either defend his territory or he can flee from his territory. The intruder can either attack or also flee. We can model the conflict as a 2×2-game between the owner and the intruder that may or may not be symmetric, depending on how we specify the payoffs. In the example this is the agent representation which describes the conflict from an interim perspective.

The standard normal form of the truly asymmetric contest describes the conflict ex ante: Since the animals are not rational they will follow the behavior encoded in their genes. This code (the ex ante "strategy") is determined before the animal is born. The "role" of being the owner or being the intruder is somehow assigned randomly to the two animals during their lifetime. The gene code must hence specify the behavior in both potential roles, for instance to defend if being an owner and to flee if being an intruder. Ex ante the conflict can hence be modelled as a 4×4 game. This game is symmetric since the animals have equal chances of being an owner or an intruder when the conflict occurs.

The agent representation can also be constructed if the number of roles exceeds the number of players: The two animals could differ in the conflict in more than two characteristics, for instance the owner could be strong or weak without the intruder being able to observe this. In that case there are three possible roles: intruder, weak and strong owner. A strategy ex ante must prescribe the behavior for all three roles. This leads us to a symmetric 8×8 game. Since there are only two players one of the three roles will remain vacant in the actual conflict. But we can write down a non-symmetric $2 \times 2 \times 2$ game where there is one player (called "agent" to distinguish) for each role. This agent representation describes, conditional on the information each animal has, the conflict at the interim stage. Its construction relies on the assumption of role asymmetry which distinguishes truly asymmetric contests from the more general class of asymmetric contests. The assumption of role asymmetry requires that in an actual conflict two animals will never find themselves in the same role.

A pure *strategy* in the symmetric normal form of a truly asymmetric contest assigns an action to each *role*. It is hence formally the same object as a pure *strategy combination* of the agent representation which assigns an action to each *agent*. This relation extends to a projection p_1 from the set of mixed strategies Φ of the normal form to the set of mixed strategy combinations Σ of the agent representation. p_1 is the first mapping in the diagram (1.1).

$$\begin{array}{ccc} \Phi & \xrightarrow{\;p_1\;} & \Sigma \\ p_2 \Big\uparrow & & \Big\uparrow p_3 \\ \Delta(\Phi) & \xrightarrow[\;p_4\;]{} & \Delta^{\times}(\Sigma) \end{array} \qquad (1.1)$$

Our central result relates to this mapping. Results concerning the other mappings in diagram (1.1) are discussed below. Our central result states that this projection induces a bijection between the evolutionarily stable sets of the normal form and the strict equilibrium sets of the agent representation. It follows by an argument as in Selten [25] that p_1 maps every evolutionarily stable set onto a strict equilibrium set. But different arguments are needed to show that the evolutionarily stable sets are exactly the preimages of strict equilibrium sets.

The truly asymmetric contest describes a *conflict within a single species.* If we assume that each member of the species can use and inherit mixed strategies, then the state of the population at a given point in time is described by a probability measure over mixed strategies. Such a probability measure tells us what fraction of the population is using what strategy. Each state determines an expected mixed strategy used by the population. This yields the projection p_2 in diagram (1.1) from the state space $\Delta(\Phi)$ to the set of mixed strategies Φ.

For the state space $\Delta(\Phi)$ we write down a *first replicator dynamic* describing how the use of mixed strategies in the population evolves over time. We consider the closed and locally asymptotically stable sets of this dynamic which consist of locally stable fixed points. For brevity we call these sets *stationary attractors* of the dynamic. We show that the projection p_2 induces a bijection between the stationary attractors of the dynamic and the evolutionarily stable sets. This result holds more generally for any symmetric normal form game. Cressman [7] obtained a very similar result.

We can formally view the agent representation of a truly asymmetric contest just isolated as some normal form game. It can then also describe a *conflict between several distinct species.* Assuming again that each animal can use and inherit mixed strategies we obtain as state space where each state describes for each species the distribution of strategies used. For each player i of the game we have hence a probability measure over the set of mixed strategies of this player. If this set of probability measures is denoted by $\Delta(\Sigma_i)$ then the state space is the Cartesian product $\Delta^{\times}(\Sigma) := \prod_{i \in N} \Delta(\Sigma_i)$. The replicator dynamic for asymmetric normal form games, as it is for instance discussed by Samuelson and Zhang [23], can be extended to this state space. This is the *second replicator dynamic* which we consider. As indicated in diagram (1.1) there is a projection p_3 that maps a state to the combination of expected mixed strategies. We show that this projection identifies stationary attractors of the second dynamic with strict equilibrium sets.

In addition, there is directly a projection p_4 from $\Delta(\Phi)$ to $\Delta^{\times}(\Sigma)$ identifying the stationary attractors of the two dynamics.

For normal form games modelling a conflict between several distinct species a related result has been obtained by Ritzberger and Weibull [19]. They consider sets that are Cartesian products of faces of the player's strategy simplices.[1] They show that such sets are closed under pure better replies if and only if they are locally asymptotically stable under the pure strategy replicator dynamic. (They actually study a more general class of dynamics.) In contrast we work here with the mixed-strategy replicator dynamic. They assume a certain geometric structure of a locally asymptotically stable set. We assume in essence that the set is a set of Nash equilibria. This assumption implies a certain geometric structure. Namely, we show that every strict equilibrium set is a *finite union* of Cartesian products of faces.

We also have a result on the geometric structure of evolutionarily stable sets: Every evolutionarily stable set is a finite union of linear subspaces intersected with the strategy simplex. We extend here a result of Cressman [7].

Furthermore we show that every strict equilibrium set of a bimatrix game contains a strategically stable set as defined by Kohlberg and Mertens [16].

As one example where the notion of strict equilibrium points does not apply while strict equilibrium sets do we will discuss Ben Porath and Dekel's [2] "burning-money" example. Using strict equilibrium sets we will obtain the same outcomes as they do using quite different arguments. Hurkens [15] obtains these outcomes by using the closely related concept of curb sets. The sequel to this paper (Balkenborg [1]) considers strict equilibrium sets in finitely repeated games and obtains a surprising relation to common interest games. We think that these examples show that our extension of Selten's [25] result to the set-valued concepts is useful.

In the paper we will have to distinguish carefully between several classes of games and study the relations between them. We will introduce them in separate sections and discuss the relevant solution concepts as we go along. The first "algebraic" part of the paper proceeds in a fashion resembling Escher's staircases: We start with arbitrary normal form games in Section 2, move down and down to more special games until we are back to arbitrary normal form games in Section 6 (see Figure 1.1). The last two sections, Section 7 and Section 8, deal with dynamic interpretations. In contrast to the biological literature we consider throughout games with an arbitrary finite number of players, not

[1] Using learning dynamics Hurkens [14] studies the stability proporties of set-valued solution concepts that are also Cartesian products of faces.

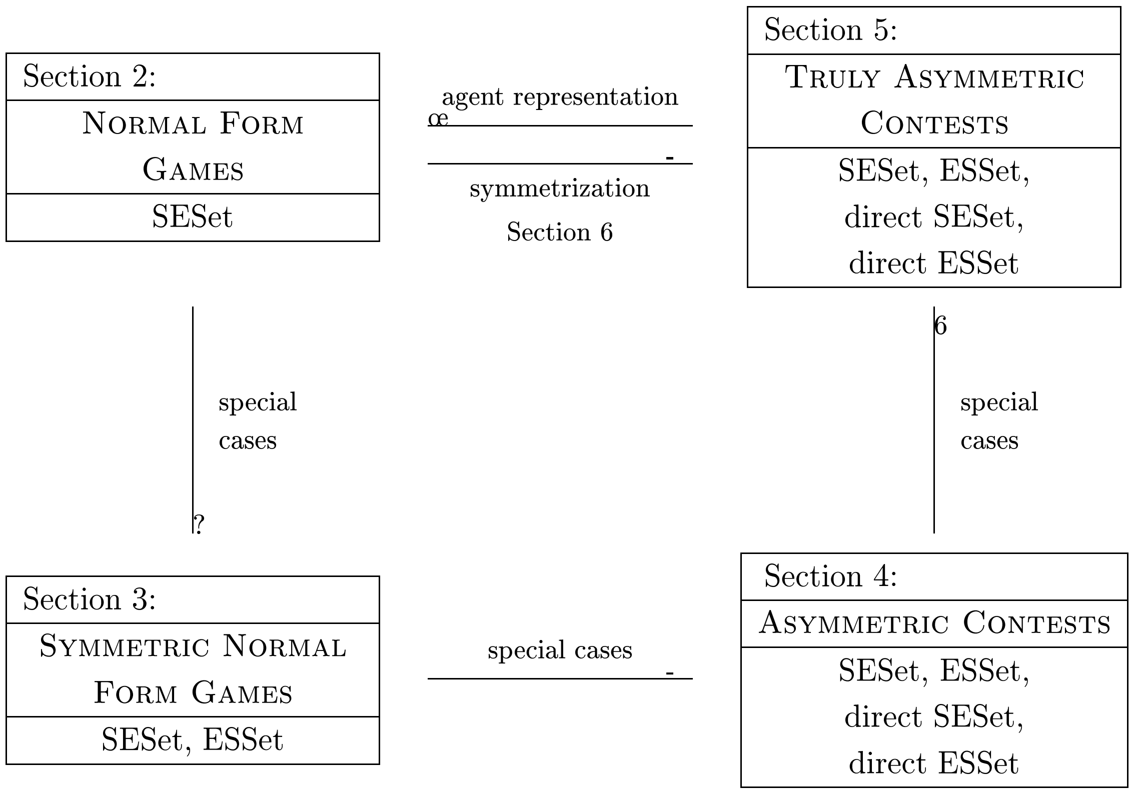

Figure 1.1: Types of games and solution concepts considered; (SESet = strict equilibrium set, ESSet = evolutionarily stable set)

just bimatrix games.

In Section 2 we introduce strict equilibrium sets for arbitrary normal form games. We describe the geometric structure of these sets. We show for bimatrix games that every strict equilibrium set contains a strategically stable set as defined by Kohlberg and Mertens [16]. This result holds even when the strict equilibrium set is a cycle. It is hence very different from the results obtained by Swinkels [29] and [30]. It does, however, not extend to games with more than two players.

In Section 3 we turn to the subclass of *symmetric* normal form games and discuss Thomas' [32] notion of evolutionarily stable sets. For n-player games we discuss a lexicographic condition for evolutionary stability which coincides with the familiar equilibrium- and stability conditions in bimatrix games. We generalize a result on the geometric structure of evolutionarily stable sets that Cressman [7] obtained for symmetric bimatrix games. Moreover we establish a first link between strict equilibrium sets and evolutionarily stable sets (Proposition 3.6): In a symmetric game the set of strategies that yield symmetric Nash equilibria in a strict equilibrium set is either void or is an evolutionarily stable set. This generalizes the well-known observation that in every symmetric game every symmetric strict equilibrium point defines an evolutionarily stable strategy.

In Section 4 we consider asymmetric contests, but we do not yet require role asymmetry. For such games one has to distinguish between mixed and behavioral strategies and one has to make appropriate distinctions for the solution concepts. The direct evolutionarily stable strategies introduced by Selten [25] are defined just like evolutionarily stable strategies except that mixed strategies are replaced by behavioral strategies in the definition. Similarly direct evolutionarily stable sets can be defined. As is well known (see for instance van Damme [34]) there is a "spurious duplication problem" for evolutionarily stable strategies in asymmetric contests because many mixed strategies may correspond to the same behavioral strategy: An asymmetric contest may have a direct evolutionarily stable strategy but not an evolutionarily stable strategy. There are at least three ways out of this problem: Either one restricts attention to behavioral strategies as in Selten [25]. Or one can take the view that evolutionary stability should refer to *sets* of strategies. One can then define evolutionary stability for classes of payoff-equivalent strategies as in Schlag [24]. Or one can use a set valued concept based on internal and external stability requirements that imply invariance. This is what we do when we use Thomas' concept.

In Section 5 we turn to truly asymmetric contests and construct the agent representation. Then we can state our central result (Proposition 5.1) about the equivalence of evolutionarily stable sets and strict equilibrium sets. We prove one direction of the equivalence.

In Section 6 we construct for every normal form game a [illegible] This is a truly asymmetric contest that has the given normal form as agent representation. The symmetrization has as many players as roles. We will show that every strict equilibrium set of the given normal form game can be "extended" to a strict equilibrium set of the symmetrization. As indicated in Figure 1.1 we can take any truly asymmetric contest, construct its agent representation and for the latter the [illegible] This yields again a truly asymmetric contest, but possibly a different one. We will see that both contests have the same evolutionarily stable sets. We can then complete the proof of our central result.

In Section 7 we consider symmetric normal form games and the corresponding first replicator dynamic over mixed strategies for a conflict within a single species. We show that the stationary attractors of this dynamic correspond to the evolutionarily stable sets.

In Section 8 we consider arbitrary normal form games and the corresponding second replicator dynamic over mixed strategies for a conflict between several distinct species. If the normal form game appears as the agent representation of a truly asymmetric contest then we also have the first dynamic for a conflict within one species. We show how the second dynamic for several distinct species is embedded in the first dynamic for one species. Thus stability requirements on the latter dynamic are a priori more stringent. But as indicated above, the stationary attractors of both dynamics are essentially the same.

An appendix contains the proofs.

2. Strictness in Normal Form Games

We will introduce the notion of a strict equilibrium set and discuss some of its properties. To do so, some notations are needed.

A *finite normal form* game Γ for the set of players $N = \{1, \cdots, n\}$ consists of finite sets S_i of *pure strategies* and *payoff functions* $u_i : S \to \mathbb{R}$ for each player $i \in N$, where $S := \prod_{j \in N} S_j$ is the set of pure strategy *combinations*. A *mixed strategy* for player i is a probability measure σ_i on his set of pure strategies S_i. We write $\Sigma_i := \Delta(S_i)$ for his set of mixed strategies and $\Sigma := \prod_{i=1}^{n} \Sigma_i$ for the set of all *mixed strategy combinations* of the players. ($\Delta(\cdot)$ will be used to denote the set of probability measures over a set.) By calculating the *expected payoffs* the payoff functions extend to multilinear functions $u_i : \Sigma \to \mathbb{R}$.

For $i \in N$ we will write $\Sigma_{-i} := \prod_{j \neq i} \Sigma_j$ with typical element σ_{-i} for the set of i-incomplete strategy combinations. For $\sigma \in \Sigma$ and $\tau_i \in \Sigma_i$ $\sigma \backslash \tau_i$ denotes the strategy combination which is in all components equal to σ except that player i's strategy is replaced by τ_i.

Definition 2.1. *A non-empty set of mixed strategy combinations $R \subseteq \Sigma$ is a* **strict equilibrium set (SESet)** *if for every strategy combination $\sigma \in R$ in the set and an arbitrary strategy $\tau_i \in \Sigma_i$ of some player $i \in N$*

$$u_i(\sigma \backslash \tau_i) \leq u_i(\sigma) \tag{2.1}$$

whereby equality in (2.1) implies $\sigma \backslash \tau_i \in R$.

Consider a strategy combination σ in a strict equilibrium set R. Then for each player i and each of his strategies τ_i we have $u_i(\sigma \backslash \tau_i) \leq u_i(\sigma \backslash \sigma_i)$, i.e., σ_i is a *best reply* against σ.[2] Since this must hold for every player, σ is a *Nash equilibrium.* Against a Nash equilibrium a player might have several best replies. The self-interest of a player alone

[2]The term "best reply" is used here in three ways: A *strategy* of a player i can be a best reply against an i-incomplete strategy combination of the opponents or against a strategy combination of *all* players including a strategy for player i since the latter information is redundant for the definition. A *strategy combination* is a best reply against another strategy combination, if each of its components is a best reply against the other strategy combination.

does then not force him to use his equilibrium strategy if he believes that his opponents use their equilibrium strategies. If he uses an alternative best reply, then no equilibrium play might result. This cannot happen for a Nash equilibrium in a strict equilibrium set: If a single player deviates by choosing an alternative best reply, then we obtain again a Nash equilibrium which like the old one is stable against deviations by a player using alternative best replies. The notion of a strict equilibrium set expresses this requirement in a circular manner.

Suppose a strict equilibrium set consists of a single strategy combination σ. Then it is a Nash equilibrium and furthermore the best reply of every player against this equilibrium must be unique. For every strategy $\tau_i \neq \sigma_i$ of a player the *strict inequality*

$$u_i\left(\sigma \backslash \tau_i\right) < u_i\left(\sigma\right) \tag{2.2}$$

is therefore satisfied, i.e., σ is a **strict equilibrium point** (Harsanyi [12], Harsanyi and Selten [13][3], van Damme [35]). In particular σ must be a Nash equilibrium in pure strategies. This follows because in a best reply a player will mix only between different pure strategies that yield him the same expected payoff and are hence also best replies against the strategy combination of the opponents.

A set of strict equilibrium *points* is a strict equilibrium *set* while not every strict equilibrium set is a set of strict equilibrium points, as the examples below show.

The argument that showed that every strict equilibrium point is in pure strategies also implies that every strict equilibrium set must contain a Nash equilibrium in pure strategies: Take any σ in a SESet. Take any pure strategy s_1 that player 1 chooses with positive probability when he plays σ_1. Then s_1 is a best reply against σ and hence $\sigma \backslash s_1$ is in the SESet. Now take a pure strategy s_2 that player 2 chooses with positive probability when he plays σ_2. σ_2 and hence s_2 are best replies against $\sigma \backslash s_1$. Therefore $\left(\sigma \backslash s_1\right) \backslash s_2$ is also in the SESet. Proceeding inductively we find a pure strategy combination in the SESet.

To describe the structure of SESets more sharply some more notations and terminology are needed.

For a mixed strategy $\sigma_i \in \Sigma_i$ we define the *support*

$$\operatorname{supp}\left(\sigma_i\right) := \{s_i \in S_i | \sigma_i(s_i) > 0\}$$

as the set of pure strategies that are chosen with positive probability by σ_i. A *face* $\Delta\left(T_i\right)$ of the mixed strategy simplex Σ_i is a set of all mixed strategies whose supports are contained in a given set $T_i \subseteq S_i$ of pure strategies. A *commutative set Nash equilibria* is a set of strategy combinations such that each strategy combination in the set is a best

[3] Harsanyi and Selten use the term "strong equilibrium point".

reply (i.e., is in each component a best reply) to every other strategy combination in the set.

Proposition 2.2. *A strict equilibrium set is a union of Cartesian products of faces for each player. Each of these Cartesian products is a commutative set of Nash equilibria.*

Since each player has only finitely many strategies a SESet is a *finite* union of Cartesian products of faces and hence a *closed* set.

Let us call for a mixed strategy σ_i the set $\Delta(\mathrm{supp}(\sigma_i))$ the *face generated by* σ_i and for a strategy combination σ the set $\prod_{i\in N}\Delta(\mathrm{supp}(\sigma_i))$ the *Cartesian product faces* generated by σ. If an SESet contains σ then it contains the Cartesian product of faces generated by σ.

The *relative interior* of a face $\Delta(T_i)$ is the set of all strategies σ_i with support equal to T_i. The relative interior of a Cartesian product of faces is then the Cartesian product of the relative interiors of its components. Let me call a Cartesian product of faces Π an *interior Nash equilibrium set* if Π is a commutative set of Nash equilibria with the property that a strategy combination belongs to Π if and only if it is a best reply against a strategy combination in the relative interior of Π. Let us finally call a Cartesian product of faces contained in an SESet *maximal* if no strictly larger product of faces is contained in the SESet.

Proposition 2.3. *A Cartesian products of faces is* maximal *in the strict equilibrium set if and only if it is an interior Nash equilibrium set.*

Interior Nash equilibrium sets may be of interest on there own. For instance Börgers and Samuelson [4] (see also Samuelson [22]) introduce *consistent pairs* as sets of strategy combinations consistent with common knowledge of admissibility. Interior Nash equilibrium sets yield examples of consistent pairs.

	l		r	
U	1	1	1	1
D	1	1	0	0

Figure 2.1: SESet with dominated strategies

	L		R	
KU	5	1	0	0
KD	0	0	1	5

Keep

	l		r	
BU	3	1	0	0
BD	0	0	-1	5

Burn

Figure 2.2: Burning Money

We use three examples to illustrate strict equilibrium sets. In the first example, Figure 2.1, the unique strict equilibrium set is the set of strategy combinations where both players receive the payoff 1. It is the union of the two interior Nash equilibrium sets $\Delta(U,D) \times \{l\}$ and $\{U\} \times \Delta(l,r)$. We observe that strict equilibrium sets can contain strategy combinations where one player is using a weakly dominated strategy. The example is taken from Samuelson and Zhang [23], who observed that such strategy combination may be locally stable under evolutionary processes.

The second example is due to Ben-Porath and Dekel [2] (see also van Damme [34]). Two players have to play the 2×2-game on the left of Figure 2.2. But before the play player 1 has the opportunity to burn \$2 in front of his opponent. If he does so and then the 2×2-game is played, the net-payoffs will be the ones in the right 2×2-game of Figure 2.2 (in the example we identify payoffs with dollar values). In the reduced normal form of the game with the additional option to burn money both players have four strategies: Player 1's pure strategy set is $\{KU, KD, BU, BD\}$, where for instance BU is the strategy to burn the money and then to play "up" in the 2×2-game and KD is the strategy where he keeps, does not burn, the money and then plays "down" in the 2×2-game. Player 2's strategy set can be denoted by $\{Ll, Lr, Rl, Rr\}$ where for instance Lr is the strategy to choose "left" in the 2×2-game if player 1 does not burn the money and to choose "right" otherwise. The set $\{KU\} \times \Delta(Ll, Lr)$ is a strict equilibrium set: If player 2 chooses left with probability 1 when no money is burned the only optimal response for player 1 is not to burn money and to play "up" in the 2×2-game. Provided player 1 does not burn money and chooses "up" in the 2×2-game only those strategies of player 2 can be optimal where he chooses "left" with probability one after he observed that the opponent did not burn money. It is important here that the strategies of player 2 in the SESet can assign arbitrary behavior in the 2×2-game after money was burnt. A single-valued solution concept would not capture the robustness of this solution.

The game has no other strict equilibrium sets: We know that a strict equilibrium set must contain a pure Nash equilibrium. The game has four pure Nash equilibria: (KU, Ll) and (KU, Lr), which generate the strict equilibrium set we discussed, (KD, Rr) and (BU, Rl). (KD, Rr) does not belong to a strict equilibrium set: Since Rl is also a best reply to KD, (KD, Rl) would have to be in the strict equilibrium set, but BU yields a strictly higher payoff against Rl than KD. If (BU, Rl) would belong to a strict equilibrium set, then also (BU, Ll) would have to be in it but KU yields a higher payoff against Ll than BU.[4]

[4]Ben-Porath and Dekel consider any model where player 1 may burn multiples of a smallest money unit before a bimatrix game is played. They show: Under the condition that the bimatrix game has a unique strict equilibrium point that yields for player 1 the highest payoff he could achieve from any strategy combination in the bimatrix game and if the money unit is sufficiently small, then a specific procedure of iterated elimination of weakly dominated strategies leads to the outcome where no money is burned and player 1 receives the highest feasible payoff from the stage game. Under the conditions

The preceding example shows a crucial property of strict equilibrium sets when applied to the (reduced) normal form of a game in extensive form: Take a pure strategy combination s contained in an SESet of such a game. Consider the path it induces in the extensive game and an information set of a player that does not contain a node on this path. Then one can replace in the player's strategy s_i his choice at the information set by an arbitrary choice to obtain a new best reply s'_i against s. Therefore $s \backslash s'_i$ must also be in the SESet. In this sense, what a player does off the path must be irrelevant in an SESet.

One might expect that strict equilibrium sets are like strict equilibrium points consistent with most equilibrium refinements. The third example in Figure 2.3 shows however that strict equilibrium sets may not contain strategically stable sets as defined in Kohlberg and Mertens [16]:[5] In this $3 \times 3 \times 3$ normal form game the unique strict equilibrium set is a cycle. It consists of all strategy combinations where each player gets the payoff 2. It is the union of six line segments. In each of these segments one player uses his r-strategy, one uses his s-strategy and the third one mixes arbitrarily between his r- and his s-strategy. The t-strategy is a strictly dominated strategy for each player.

There is no Nash equilibrium near to the cycle in the trembling-hand perturbation where each player makes with small probability an error and then mixes with equal probabilities between his three strategies.

r_3:

	r_2	s_2	t_2
r_1	1 1 1	2 2 2	1 0 2
s_1	2 2 2	2 2 2	2 0 1
t_1	0 2 1	0 1 2	0 0 0

s_3:

	r_2	s_2	t_2
r_1	2 2 2	2 2 2	2 0 1
s_1	2 2 2	1 1 1	1 0 2
t_1	0 1 2	0 2 1	0 0 0

t_3:

	r_2	s_2	t_2
r_1	2 1 0	1 2 0	0 0 0
s_1	1 2 0	2 1 0	0 0 0
t_1	0 0 0	0 0 0	0 0 0

Figure 2.3: An SESet forming a cycle

Let us consider for example the segment of the cycle where player 1 mixes arbitrarily between his r- and his s-strategy while player 2 uses his s-strategy and player 3 his r-strategy. Against any strategy combination in this segment the unique best reply of player 1 is to use his s-strategy: If he uses his r- or his s-strategy he gets the payoff 2 with a

stated the set of strategy combinations inducing this outcome is also the unique strict equilibrium set of the model.

[5] The unique strategically stable set in the example consists of the strategy combination where each player mixes between his r- and his s-strategy with equal probabilities.

high probability. But with small probability player 2 makes an error and switches to r_2 or t_2. In both cases player 1 will get the payoff 1 with r_1 while he gets the payoff 2 with s_1 and is hence strictly better off with s_1. Also player 3 will make errors and switch with small but equal probabilities to s_2 or t_3. In these cases player 1 will get the payoffs 2 or 1 if he uses r_2 and the payoffs 1 or 2 with s_2. Both strategies are then equally good for player 1. If the error probabilities are sufficiently small we can neglect the cases where both opponents simultaneously make an error. We can interpret the perturbed game as a game with the same strategy spaces, but with perturbed payoff functions. Player 1 strictly prefers s_1 to r_1 against $(\sigma_{1,}s_2, r_3)$ in this perturbed game.

Thus the unique best reply of the players against $(r_{1,}s_2, r_3)$ is $(s_{1,}s_2, r_3)$. Symmetric calculations show that the unique best reply against $(s_{1,}s_2, r_3)$ is $(s_{1,}r_2, r_3)$, against $(s_{1,}r_2, r_3)$ it is $(s_{1,}r_2, s_3)$, against $(s_{1,}r_2, s_3)$ it is $(r_{1,}r_2, s_3)$, against $(r_{1,}r_2, s_3)$ it is $(r_{1,}s_2, s_3)$ and against $(r_{1,}s_2, s_3)$ it is $(r_{1,}s_2, r_3)$. It follows that there is no Nash equilibrium in the perturbed game on or near the cycle.

While the property of containing Nash equilibria gets lost for the cycle due to the perturbation, its strictness property remains: The cycle is the unique minimal set of strategy combinations for the perturbed game with the property that for each strategy combination σ in the set and for each strategy τ_i of a player that yields him at least weakly a higher payoff, i.e., $u_i(\sigma \backslash \tau_i) \geq u_i(\sigma)$, $\sigma \backslash \tau_i$ is also in the set.

Strict equilibrium sets are often special cases of the equilibrium evolutionarily stable sets discussed in Swinkels [28], [29] and [30]. Swinkels shows that equilibrium evolutionarily stable sets contain strategically stable sets if certain conditions are met that exclude cycles as above. It may be worthwhile to note that in the case of *bimatrix games*, i.e., normal form games with two players, strict equilibrium sets must contain strategically stable sets even if they form cycles.

Proposition 2.4. *A strict equilibrium set of a* bimatrix game *contains a strategically stable set as defined by Kohlberg and Mertens [16].*

3. Evolutionary Stability in Symmetric Games

Evolutionarily stable sets are defined for symmetric normal form games. A game is symmetric if all players have the same strategies available and if the payoffs depend only on the strategy a player uses and on those used by his opponents, but not on the identity of the players.

More precisely: A *symmetric normal form game* G for the set of players $M = \{1, \cdots, m\}$ is defined by a finite set F of pure strategies and payoff functions $E_j : F^m \to \mathbb{R}$

for each player $j \in \{1, \cdots, m\}$ which are order independent in the sense that

$$E_{\pi(j)}(f_1, \ldots, f_m) = E_j\left(f_{\pi(1)}, \ldots, f_{\pi(m)}\right)$$

holds for all permutations $\pi \in \Pi(M)$ of the names $\{1, \cdots, m\}$ of the players. Because of this order independence it suffices to know the payoff function $E := E_1$ and we can interpret $E(f_1, \ldots, f_m)$ as the payoff some player gets if he uses the first strategy f_1 in the strategy combination and his opponents use the strategies $f_2, \ldots, f_m$.

We denote by $\Phi := \Delta(F)$ the mixed strategy space. Furthermore we denote by $\varphi^k \in \Phi^k$ the k-tuple where all components are equal to φ and by $\varphi^k \times \psi^l \in \Phi^{k+l}$ the $(k+l)$-tuple where the first k components are equal to φ and the last l components are equal to ψ.[6] Instead of $E(\psi, \psi^{m-1})$ we can also briefly write $E(\psi^m)$.

Definition 3.1. *i) (***Maynard-Smith** *and* **Price** *[18]) A mixed strategy $\varphi \in \Phi$ of a symmetric normal form game is an* **evolutionarily stable strategy (ESS)** *if there exists a neighborhood $U(\varphi) \subseteq \Phi$ such that for all strategies $\psi \in U(\varphi) \setminus \{\varphi\}$ the strict inequality*

$$E\left(\psi, \psi^{m-1}\right) < E\left(\varphi, \psi^{m-1}\right). \tag{3.1}$$

holds.

*ii) (***Maynard-Smith** *[17]) A mixed strategy $\varphi \in \Phi$ is a* **neutral evolutionarily stable strategy** *if there exists a neighborhood $U(\varphi) \subseteq \Phi$ such that for all strategies $\psi \in U(\varphi)$ the weak inequality*

$$E\left(\psi, \psi^{m-1}\right) \leq E\left(\varphi, \psi^{m-1}\right). \tag{3.2}$$

holds.

*iii) (***Thomas** *[32]) A non-empty* closed set *of mixed strategies $R \subseteq \Phi$ is an* **evolutionarily stable set (ESSet)** *if for every strategy $\varphi \in R$ in the set there exists a neighborhood $U(\varphi) \subseteq \Phi$ such that for all strategies $\psi \in U(\varphi)$ the weak inequality (3.2) holds whereby equality implies that ψ is an element of R.*

The following Lemma describes a basis of neighborhoods that is often convenient to use.

Lemma 3.2. *The sets*

$$V_{\varepsilon_0}(\varphi) = \{\psi_\varepsilon | \psi \in \Phi, 0 \leq \varepsilon \leq \varepsilon_0\}$$

where $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$ form for varying $\varepsilon_0 > 0$ a basis of neighborhoods of a strategy φ in the simplex Φ.

[6] $E(\varphi, \psi^{m1})$ defines a particular playing-the-field model with state substitubility (see Hammerstein and Selten [9]).

Since $E\left(\psi_\varepsilon, \psi_\varepsilon^{m-1}\right) = (1-\varepsilon)E\left(\varphi, \psi_\varepsilon^{m-1}\right) + \varepsilon E\left(\psi, \psi_\varepsilon^{m-1}\right)$ a strategy φ is hence evolutionarily stable if and only if

$$E\left(\psi, \psi_\varepsilon^{m-1}\right) < E\left(\varphi, \psi_\varepsilon^{m-1}\right) \tag{3.3}$$

holds for all $\psi \in \Phi$ and all sufficiently small ε. "Sufficiently small" means here that we can take any ε with $0 \leq \varepsilon \leq \varepsilon_0$ for some fixed $\varepsilon_0 > 0$ that does not depend on ψ. ε_0 is a so-called uniform invasion barrier.[7]

The definition of an evolutionarily stable strategies is motivated by the following *monomorphic* scenario: Consider a large population playing the symmetric game in an anonymous random matching environment where each member of the population is matched with equal probabilities with $m-1$ of the other members of the population to play the symmetric game. Suppose that initially all members of the population play the same strategy φ but that a small fraction ε of the population switches to using a "mutant strategy" ψ. Then each player will face with a high probability only opponents who use the "regular" strategy, but with a small probability he will meet some opponents using the mutant strategy. In expectation an opponent will use the strategy $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$. The condition for an evolutionarily stable strategy (3.3) expresses that in this situation the original strategy fares better (has a higher fitness) against the expected strategy combination of the opponents than the mutant strategy and this has to hold for all possible mutant strategies. If in a dynamic framework the difference in the fitness of a mutant strategy and the average fitness measures how fast the use of a mutant strategy will grow or decline in the population, then the ESS condition expresses the idea that the use of any mutant strategy will die out over time. In the above definitions we assume that there exists for each φ an ε_0 such that the stability conditions hold for each

For a neutral evolutionarily stable strategy the strict inequality (3.1) is replaced by the weak inequality (3.2). This can be equivalently expressed as: A strategy φ is neutral evolutionarily stable if the weak inequality

$$E\left(\psi, \psi_\varepsilon^{m-1}\right) \leq E\left(\varphi, \psi_\varepsilon^{m-1}\right) \tag{3.4}$$

is satisfied for all $\psi \in \Phi$ and sufficiently small ε. If a small part of the population switches in the scenario sketched above to a mutant strategy ψ satisfying (3.4) with equality then the use of this strategy would not spread, but would also not decrease. If every once in a while a part of the population would "mutate" to use this mutant strategy then there might be a drift towards a distribution of strategies where a large majority of players use

[7] Evolutionarily stable sets are often defined without assuming a unifom invasion barrier. In the cases we study the two definitions are equivalent.

the mutant strategy. This could lead to a distribution of strategies that would no longer be stable.

An evolutionarily stable set is by definition a set of neutral evolutionarily stable strategies. However, the population could "drift" only from one strategy in the evolutionarily stable set to another strategy that is again neutral evolutionarily stable. The definition of an evolutionarily stable set expresses this property in a circular manner.

The neighborhoods appearing in the definition of an evolutionarily stable set may vary in size with the strategies in the set. If the set would not be required to be closed it could happen that the neighborhoods could shrink in size along the set and vanish to a point at a limit point which would no longer have to be neutral evolutionarily stable. The closedness requirement excludes the possibility that there are strategies arbitrarily nearby to the set from which the evolutionary process would lead away from the set.

For symmetric bimatrix games the notion of an ESS φ is often introduced using an equilibrium and a stability condition (see, e.g., van Damme [35], Definition 9.1.1). For games with arbitrary numbers of players one is led to a chain of conditions that will now be described. We will however see that these lexicographic conditions are only necessary, not sufficient for evolutionarily stable sets in games with more than three players because they do not yield uniform invasion barriers.

For $\varphi, \psi, \chi \in \Phi$ let $\eta(\chi, \varphi, \psi)$ denote the vector of payoffs

$$\eta(\chi, \varphi, \psi) = \left(E\left(\chi, \varphi^{m-1}\right), E\left(\chi, \varphi^{m-2} \times \psi\right), \ldots, E\left(\chi, \psi^{m-1}\right)\right) \in \mathbb{R}^m. \tag{3.5}$$

These are the payoffs from using the strategy χ when all opponents play φ, when one of them uses the "mutant strategy" ψ, when two opponents switch to the "mutant strategy" etc. In the above monomorphic scenario the case where all opponents use the original strategy is most likely and the resulting payoff the most important one, the case where one opponent uses the mutant strategy is less likely and the resulting payoff is less important and so on. To express this we can use the lexicographic ordering "$>_L$" on $\mathbb{R}^m$ defined by $(x_1, \ldots, x_m) >_L (y_1, \ldots, y_m)$ if $x_1 = y_1, \ldots, x_{k-1} = y_{k-1}$ and $x_k > y_k$ for some $1 \leq k \leq m$. The following lexicographic condition is useful when determining evolutionarily stable strategies. As is well known, the lexicographic condtions imply evolutionary stability in symmetric bimatrix games.

Proposition 3.3. *If strategy φ of a symmetric game is evolutionarily stable then*

$$\eta(\psi, \varphi, \psi) <_L \eta(\varphi, \varphi, \psi) \tag{3.6}$$

for all strategies $\varphi \neq \psi \in \Phi$.

For neutral evolutionarily stable strategies and for ESSets we obtain similar necessary conditions. For bimatrix games the following result is in essence due to Thomas [32]:

Proposition 3.4. *i) If a strategy $\varphi \in \Phi$ is neutral evolutionarily stable then*

$$\eta\left(\psi,\varphi,\psi\right) \leq_L \eta\left(\varphi,\varphi,\psi\right) \tag{3.7}$$

holds for every $\psi \in \Phi$. For a bimatrix game also the converse is true.

ii) If a set $R \subseteq \Phi$ is evolutionarily stable then (3.7) holds for all $\varphi \in R$ and $\psi \in \Phi$ whereby equality implies that $\psi \in R$. In a bimatrix game also the converse statement is true.

The results implies in particular that every neutral evolutionarily stable strategy φ is a *symmetric Nash equilibrium strategy*, i.e., φ^m is a Nash equilibrium.

The converse in Proposition 3.4 does not have to hold for games with more than two players, as the following symmetric $3 \times 3 \times 3$-game shows: The payoff is defined by the tri-linear form

$$E\left(\psi_1,\psi_2,\psi_3\right) := p_1p_2q_3 + p_1q_2p_3 - q_1(q_2+q_3)(p_1+q_1+r_1)$$

for $\psi_j := (p_j, q_j, r_j)$ with $p_j, q_j, r_j \geq 0$ and $p_j+q_j+r_j = 1$. ($p_1+q_1+r_1 = 1$ appears in the payoff function only to make it tri-linear.) $\varphi = (1,0,0)$ satisfies $\eta(\psi,\varphi,\psi) \leq_L \eta(\varphi,\varphi,\psi)$ for all ψ: We have $E\left(\varphi,\psi_2,\psi_3\right) = 0$ for all ψ_2,ψ_3. For all $\psi = (p,q,r)$ we have $E\left(\psi,\varphi^2\right) = 0$, $E\left(\psi,\varphi,\psi\right) = -q^2$, which is strictly negative for $q > 0$, and $E\left(\psi,\psi^2\right) = 2(p^2-q)q$, which is 0 for $q = 0$. But φ is not neutral evolutionarily stable. We can find strategies $\psi = (p,q,r)$ arbitrary close to φ with $p^2 = 2q$ and hence $E\left(\varphi,\psi^2\right) = 0 < 4q^2 = E\left(\psi,\psi^2\right)$. The "trick" is here that the quadric curve $q = p^2$ is tangential in φ to the line $q = 0$. One can similarly construct a game with five players where the line $q = 0$ satisfies the conditions in part ii) of Proposition 3.4 but is not an evolutionarily stable set because $E\left(\psi,\psi^4\right) = 2(p^2-q)(p^2-2q)q$.

To describe the geometric structure of ESSets we call a subset $L \subseteq \Phi$ a *linear subset* if it is the intersection of Φ with the set of solutions to a linear system of equations. We obtain the following generalization of a result by Cressman [6] (see also Cressman [8]):

Proposition 3.5. *An evolutionarily stable set is a finite union of linear subsets.*

Cressman shows for the bimatrix case that an ESSet can intersect the interior of a face in at most one linear subset. This is no longer true for games with more than two players.

We will finally describe a first relation between strictness and evolutionary stability. In the example most often used to illustrate evolutionary stability, the hawk- and dove game (Maynard-Smith and Price [18]), the unique evolutionarily stable strategy φ is completely mixed and hence φ is *not* a symmetric *strict* equilibrium strategy, i.e., φ^m is not a strict equilibrium point. But conversely, if φ is a symmetric strict equilibrium strategy then φ is an evolutionarily stable strategy. More generally we have:

Proposition 3.6. *The set of symmetric equilibrium strategies in a strict equilibrium set of a symmetric game is either empty or an evolutionarily stable set.*

For symmetric bimatrix games the result is easy to prove since we can use the lexicographic conditions: Suppose φ is a strategy for which (φ, φ) is in a strict equilibrium set. The equilibrium condition, $E(\psi, \varphi) \leq E(\varphi, \varphi)$ for all $\psi \in \Phi$, is then automatically satisfied. If equality holds here then (ψ, φ) is in the given strict equilibrium set and therefore φ is a best reply to ψ. Hence the stability condition $E(\psi, \psi) \leq E(\varphi, \psi)$ is satisfied. If we have equality also here, then (ψ, ψ) is in the given SESet, i.e., ψ is a symmetric equilibrium strategy.

Since the lexicographic condition is not sufficient for games with more than two players the general proof is quite tedious and based on the Taylor expansion of the payoff difference $E(\psi, \psi_\varepsilon^{m-1}) - E(\varphi, \psi_\varepsilon^{m-1})$.

4. Asymmetric Contests

In many conflicts between two or more animals of the same species the animals may have different information available on which they can condition their behavior. For instance one animal might be the owner of a territory and one may be an intruder. The animals might behave differently in the two roles. Still we can think of the conflict as a symmetric game with additional structure insofar as ex ante both animals might have the same probability of being assigned one of the two roles. The framework discussed here to describe asymmetric contests is – apart from some minor modifications – the one introduced in Selten [25]. It is not the most general model because the information on which the players can condition their behavior may not concern past strategic choices of an opponent. To allow for this possibility one would have to consider – as in Selten [26], [27] – symmetric games in extensive form.

An *asymmetric contest* A consists of a list

$$\left(M, N, (S_i)_{i \in N}, H, p, \left(r_h^j\right)_{\substack{j \in M \\ h \in H}}\right)$$

satisfying certain symmetry conditions to be introduced below. Hereby

— $M := \{1, \ldots, m\}$ is a finite set of *players*.
— $N := \{1, \cdots, n\}$ is a finite set of *roles* or *information situations* for the players.
— S_i describes the finite set of *actions* available to a player if he is assigned role $i \in N$.[8]
— $H \subseteq N^M$ is a set of possible *role assignments*. A role assignment is a map $h : M \to N$ that assigns to each player $j \in M$ his role $i \in N$ in the contest. When convenient we will denote a role assignment also as a m-tuple $(i_1, \cdots, i_m)$. We assume that each role i is assigned by at least one role assignment, i.e., there exist $j \in M$ and $h \in H$ with $h(j) = i$.
— p is a probability measure on the set of possible role assignments H describing the probability by which an initial chance move will select an assignment. We assume $p(h) > 0$ for each $h \in H$.
— r_h^j is a *conditional payoff function*

$$r_h^j : \prod_{j' \in M} S_{h(j')} \longrightarrow \mathbb{R}$$

that specifies for player $j \in M$ his payoff if the chance move selects the role assignment $h \in H$ and each player j' chooses then, conditional on his role, an action $s_{h(j')} \in S_{h(j')}$.

The following symmetry conditions must hold for each permutation $\pi \in \Pi(M)$ of the players:
— If $h \in H$ then also $h \circ \pi \in H$ and $p(h) = p(h \circ \pi)$. In particular, if the chance move assigns with a certain probability player j_1 to the role i_1 and player j_2 to the role i_2 then the chance move assigns with equal probability player j_1 to the role i_2 and player j_2 to the role i_1.
— For $j \in N$, $h \in H$ and $\left(s_{h(1)}, \cdots, s_{h(m)}\right) \in \prod_{j \in M} S_{h(j)}$ we require

$$r_h^{\pi(j)}\left(s_{h(1)}, \cdots, s_{h(m)}\right) = r_{h\circ\pi}^j\left(s_{h(\pi(1))}, \cdots, s_{h(\pi(m))}\right).$$

This condition ensures that the payoff of a player depends only on his and his opponents conditional choices of actions and not on the identity of the players.

A *pure strategy* of a player in the asymmetric contest selects one action for each possible role. Thus the set of pure strategies for each player $j \in M$ is

$$S := \prod_{i \in N} S_i.$$

Let $\vec{s} = \left(\left(s_i^j\right)_{i \in N}\right)_{j \in M}$ be a pure strategy combination. We define the *total (expected) payoff* for player $j \in M$ as

$$E_j(\vec{s}) := \sum_{h \in H} p(h) \cdot r_h^j\left(s_{h(1)}^1, \cdots, s_{h(m)}^m\right).$$

[8] I use these notation *because* they will not lead to confusion with the one introduced for normal form games.

By construction the asymmetric contest A defines a symmetric normal form game $G(A)$ with strategy set S and payoff functions $E_1, \dots, E_m$. As usual we will denote the set of mixed strategies by $\Phi := \Delta(S)$ and set $E := E_1$. When we talk about an evolutionarily stable set of the asymmetric contest we mean an evolutionarily stable set of the corresponding normal form game $G(A)$ etc.

In addition *behavioral strategies* can be introduced. A behavioral strategy assigns to each possible role a probability mixture over the set of actions available in that role, called a *local strategy*. If we denote by $\Sigma_i := \Delta(S_i)$ the set of local strategies for role $i \in N$, then a behavioral strategy is an element of $\Sigma := \prod_{i \in N} \Sigma_i$. By calculating expected payoffs the payoff functions extend to $E_j : \Sigma^m \to \mathbb{R}$.

With each mixed strategy $\varphi \in \Phi$ we can associate a *local strategy* for each role $i \in N$ by calculating the marginal probabilities

$$\mathrm{proj}_i(\varphi)(s_i) := \sum_{s_{-i} \in S_{-i}} \varphi(s_{-i}, s_i) \text{ for } s_i \in S_i.$$

Hence we obtain a linear, surjective mapping from the set of mixed strategies to the set of behavioral strategies defined by $\mathrm{proj}(\varphi) := (\mathrm{proj}_i(\varphi))_{i \in N}$. We denote the projection from mixed strategy combinations to behavioral strategy combinations by $\mathrm{proj}^m : \Phi^m \to \Sigma^m$.

Instead of using mixed strategies one can define evolutionarily stable strategies etc. directly in terms of behavioral strategies. To distinguish, the resulting concepts are called *direct* evolutionarily stable strategies and so on. For instance:

Definition 4.1. *A* **direct evolutionarily stable set** P *is a closed non-empty set of behavioral strategies* $\sigma \in \Sigma$ *for which there is a neighborhood* $U(\sigma)$ *such that*

$$E\left(\tau, \sigma^{m-1}\right) \leq E\left(\sigma, \sigma^{m-1}\right) \tag{4.1}$$

holds for all $\tau \in U(\sigma)$ *whereby equality in (4.1) implies* $\tau \in P$.

A **direct strict equilibrium set** R *is a non-empty set of behavioral strategy combinations* $\vec{\sigma} \in \Sigma^m$ *such that for all* $\tau \in \Sigma$

$$E(\vec{\sigma} \backslash \tau) \leq E(\vec{\sigma}) \tag{4.2}$$

whereby equality in (2.1) implies $\vec{\sigma} \backslash \tau \in R$.

If there are at least two actions available for each role and there are at least two players then the set of mixed strategies has a higher dimension than the set of behavioral strategies. van Damme [35] discusses an example where the additional information available to the players is redundant for the strategic conflict (all players know whether the weather is good or bad) and where a unique direct evolutionarily stable strategy σ exists while no evolutionarily stable strategy exists (see Schlag [24] for further discussions on this issue). No such "spurious duplication problem" occurs for evolutionarily stable sets, neutral evolutionarily stable strategies or strict equilibrium sets. In van Damme's example $\text{proj}^{-1}(\sigma)$ is a continuum of payoff-equivalent strategies forming an evolutionarily stable set.

To compare sets of solutions in one set with those of another set we will use the following terminology: Let $f : X \to Y$ be a function between two sets. Let A be a set of subsets in X and let B be a set of subsets of Y. We say that *the sets in A correspond via f to the sets in B* if f induces a bijection between the sets, i.e., if the image of each set in A is a set in B and if conversely every set in A is the preimage of a set in B.

We obtain the following invariance results:

Proposition 4.2. *For an asymmetric conflict the following holds: A mixed strategy is neutral evolutionarily stable if and only if it is mapped by the projection* proj *onto a direct neutral evolutionarily stable strategy. Evolutionarily stable sets correspond via the projection* proj *to the direct evolutionarily stable sets. The strict equilibrium sets correspond via* proj^m *to the direct strict equilibrium sets.*

5. Truly Asymmetric Contests

A truly asymmetric contest is a contest satisfying the condition of *role asymmetry*, i.e., a player never meets in the contest an opponent in the same role. Under this condition we can associate a new normal form game with the asymmetric contest describing the conflict from an interim perspective. It turns out that the direct evolutionarily stable sets of the truly asymmetric contest are identical with the strict equilibrium sets of this new game.

Thus an asymmetric contest A as defined in the previous section is *truly asymmetric* if it holds for any two different players $j_1, j_2 \in M$ and every possible role assignment $h \in H$ that $h(j_1) \neq h(j_2)$. The number of roles must then be at least as large as the number of players.

With a truly asymmetric contest A we associate now a new normal form game $\Gamma(A)$, called the *agent representation* of the truly asymmetric contest, where the set of players – called agents to distinguish – is identified with the set of roles $N = \{1, \cdots, n\}$.[9] The set of strategies for each agent i is the set of choices S_i. His payoff function $u_i : S \to \mathbb{R}$ is

[9] This game is not the agent normal form because we do not have an agent for each information situation *and* each player.

defined by

$$u_i(s) := \sum_{\substack{h\in H \\ h(1)=i}} p(h)\cdot r_h^1\left(s_{h(1)},\ldots,s_{h(m)}\right) \tag{5.1}$$

for a strategy combination $s \in S$. Thus the payoff of agent i is the payoff player 1 expects from being in role i. Since the contest is truly asymmetric the identity (5.1) also yields the expected payoffs for an agent when mixed strategies in the agent representation (= local strategies in the asymmetric contest) are used, i.e., for all $\sigma \in \Sigma$

$$u_i(\sigma) := \sum_{\substack{h\in H \\ h(1)=i}} p(h)\cdot r_h^1\left(\sigma_{h(1)},\ldots,\sigma_{h(m)}\right) .$$

For $\sigma, \tau \in \Sigma$ we obtain the following relation between the payoff functions for the truly asymmetric contest and the payoff functions for its agent representation:

$$\begin{aligned} & E(\tau,\sigma^{m-1}) \\ & = \sum_{i\in N}\sum_{\substack{h\in H \\ h(1)=i}} p(h)\cdot r_h^1\left(\tau_{h(1)},\sigma_{h(2)},\ldots,\sigma_{h(m)}\right) \\ & = \sum_{i\in N} u_i(\sigma\backslash\tau_i) . \end{aligned} \tag{5.2}$$

The central result in this paper is:

Proposition 5.1. *A set of behavioral strategies in a truly asymmetric contest is a direct evolutionarily stable set if and only if it is a strict equilibrium set in the corresponding agent representation.*

The relation (5.2) is easily seen to imply that a pure strategy is a symmetric equilibrium strategy if and only if it is a strict equilibrium point in the agent representation. Proposition 5.1 implies therefore:

Proposition 5.2. *(Selten [25]) A behavioral strategy in a truly asymmetric contest is a direct evolutionarily stable strategy if and only if it is a strict symmetric equilibrium strategy.*

Let me show why a direct evolutionarily stable set $P \subseteq \Sigma$ of the truly asymmetric contest is a strict equilibrium set of the agent representation. Let $\sigma \in P$ and let τ_{i_0} be a strategy of an agent i_0 in the agent representation. Suppose in a population originally playing σ a small fraction of players switches to playing the mutant strategy $\sigma\backslash\tau_{i_0}$. We know that $\sigma\backslash\tau_{i_0}$ can make at most the same payoff against the average strategy combination of the population as σ and if it makes the same payoff then $\sigma\backslash\tau_{i_0}$ belongs to the evolutionarily stable set. To compare the payoffs a player makes with these two strategies we only have to compare the payoffs if the player is given role i_0 since the two strategies σ and $\sigma\backslash\tau_{i_0}$ agree for all other roles. But if the player is in role i_0, then all opponents will be in other roles where the two strategies agree. Therefore both strategies will make the same payoff against the average strategy combination if and only if they make the same payoff against σ^{m-1} which can happen only if τ_{i_0} is a best reply to σ in the agent representation. Formally: We have for all sufficiently small $\varepsilon > 0$:

$$
\begin{aligned}
& E(\sigma\backslash\tau_{i_0}, ((1-\varepsilon)\sigma + \varepsilon(\sigma\backslash\tau_{i_0}))^{m-1}) \leq E(\sigma, ((1-\varepsilon)\sigma + \varepsilon(\sigma\backslash\tau_{i_0}))^{m-1}) \\
& \Leftrightarrow u_{i_0}(\sigma\backslash\tau_{i_0}) + \sum_{i \in N\backslash i_0} u_i(\sigma\backslash((1-\varepsilon)\sigma_{i_0} + \varepsilon(\tau_{i_0})) \\
& \qquad\qquad \leq u_{i_0}(\sigma) + \sum_{i \in N\backslash i_0} u_i(\sigma\backslash((1-\varepsilon)\sigma_{i_0} + \varepsilon(\tau_{i_0})) \\
& \Leftrightarrow u_{i_0}(\sigma\backslash\tau_{i_0}) \leq u_{i_0}(\sigma).
\end{aligned}
$$

Thus $u_{i_0}(\sigma\backslash\tau_{i_0}) \leq u_{i_0}(\sigma)$ whereby equality implies $\sigma\backslash\tau_{i_0} \in P$.

For the converse we need two more results. The proof must therefore be deferred to the end of Section 6.

6. Normal Form Games as Truly Asymmetric Contests

We started out by considering arbitrary normal form games. Then we considered as special cases symmetric normal form games, asymmetric and truly asymmetric contests. In this section we return to arbitrary normal form games. Via a symmetrization we construct for each normal form a truly asymmetric contest who's agent representation is the given normal form game.

Consider for example the game of chess as a normal form game. Chess is an asymmetric game because the player with the white figures moves first. But if we add a random move in the beginning deciding which player has to move which figures we obtain a symmetric game where a strategy of a player has to describe what he will do if he has to move the white figures and what he will do if he has to move the black figures.

So we consider a normal form game Γ with the set of players $N := \{1, \cdots, n\}$, sets of pure strategies S_i and payoff functions $u_i : S \to \mathbb{R}$ for each player $i \in N$. The *symmetrization* of Γ is an asymmetric contest A with the same set of players. Each player can be in one of the roles $i = 1, \cdots, n$ where he has the set S_i of actions available. The set of possible role assignments is the set of all permutations $\Pi(N)$ where each permutation has equal probabilities. The conditional payoffs are

$$r_\pi^j(s_{\pi(1)}, \ldots, s_{\pi(n)}) := n \cdot u_{\pi(j)}(s_1, \ldots, s_n).$$

One checks immediately that the agent representation of the symmetrization is indeed the given normal form game Γ.

We had seen in Section 3 that in a symmetric normal form game the set of strategies that yield symmetric Nash equilibria in a strict equilibrium set is either empty or an evolutionarily stable set. The results in Sections 4 and 5 imply that in a truly asymmetric contest the evolutionarily stable sets are behaviorally equivalent to the strict equilibrium sets of the agent representation. But this is not the converse to the previous statement and indeed for arbitrary truly asymmetric contests the converse is not true. If, however, a symmetric normal form game is obtained via the symmetrization of an arbitrary normal form game, then the converse holds.

A counter-example is provided in Figure 6.1: The game has two players, three roles and three actions for each role. If a player is chosen to be in a certain role he only knows that his opponent is in a different role but not which of the two remaining roles are assigned to his opponent. Thus the role assignment is $H = \{(i_1, j_2) \,|\, i, j \in \{1,2,3\}, i \neq j\}$ where each assignment has equal probabilities. The payoffs for the different roles are defined symmetrically. Figure 6.1 yields the payoffs for role i when the opponent is in role j or k ($\{i,j,k\} = \{1,2,3\}$).

One checks immediately that the set of behavioral strategies where a player chooses the r-action in two roles and mixes arbitrarily between the r- and the s-choice in the remaining role is a strict equilibrium set in the agent representation and hence — as we are going to prove — a direct evolutionarily stable set in the asymmetric contest. But $((r_1, r_2, r_3), (r_1, r_2, r_3))$ cannot be in a direct strict equilibrium set for the asymmetric contest: (s_1, s_2, s_3) is a best reply to (r_1, r_2, r_3) because the s-action is optimal if the opponent chooses in both possible opposing roles the r-action. Hence $((r_1, r_2, r_3), (s_1, s_2, s_3))$ would have to be in the set. But the latter would not be a Nash equilibrium: The t-action yields in each given role a higher payoff than the r- or s-action if the opponent uses in *both* possible opposing roles the s-action.

In the example there were more roles than players. If the number of roles equals the number of players we have:

Proposition 6.1. *Every strict equilibrium set of a normal form game is the set of symmetric equilibrium strategies of some direct strict equilibrium set of the symmetrization.*

To prove the result we will take a strict equilibrium set $P \subseteq \Sigma$ in a normal form game and show that those strategies

$$\left(\left(\sigma_i^{(j)}\right)_{i\in N}\right)_{j\in N} \in \Sigma^n$$

		Role j			Role k		
		r_j	s_j	t_j	r_k	s_k	t_k
Role i:	r_i	2	2	0	2	2	0
	s_i	2	0	0	2	0	0
	t_i	0	3	0	0	3	0

Figure 6.1: A truly asymmetric contest where the strategies in an evolutionarily stable set are not the symmetric Nash equilibrium strategies of a strict equilibrium set

in the symmetrization which satisfy

$$\left(\sigma^{(1)}_{\pi(1)}, \ldots, \sigma^{(n)}_{\pi(n)}\right) \in P$$

for all permutations $\pi \in \Pi(N)$ form a direct strict equilibrium set (see Appendix).

Suppose A is a truly asymmetric contest with the set of players $M = \{1, \cdots, m\}$ and roles $N = \{1, \cdots, n\}$ and let $\Gamma(A)$ be its agent representation as defined in Section 5. The symmetrization A' of the agent representation $\Gamma(A)$ yields a new truly asymmetric contest with payoff function denoted by E'. Both A and A' have the same sets of behavioral strategies Σ, which is also the set of strategy combinations of $\Gamma(A)$. For all behavioral strategies $\sigma, \tau \in \Sigma$ we have

$$E'\left(\tau, \sigma^{n-1}\right) = \sum_{i \in N} u_i(\sigma \backslash \tau_i) = E\left(\tau, \sigma^{m-1}\right)$$

which immediately implies:

Lemma 6.2. *The direct evolutionarily stable sets of A and A' are the same.*

Proof of Proposition 5.1, second part Now let P be a strict equilibrium set of the agent representation $\Gamma(A)$. By Proposition 6.1 P is the set of symmetric equilibrium strategies in some direct strict equilibrium set R of the symmetrization A'. Propositions 4.2 and 3.6 imply that the set of symmetric equilibrium strategies in R – which is P – is a direct evolutionarily stable set of the symmetrization A'. By the previous lemma P is also a direct evolutionarily stable set of the original truly asymmetric contest A, which had to be shown.

7. The Replicator Dynamic with Inheritance of Mixed Strategies in Continuous Time

Taylor and Jonker [31] introduced for a symmetric normal form game G the (pure strategy) replicator dynamic in continuous time. This is the vectorfield $\dot{\varphi}$ defined on the simplex of mixed strategies Φ by

$$\dot{\varphi}(f) := \varphi(f) \cdot \left[E\left(f, \varphi^{m-1}\right) - E\left(\varphi, \varphi^{m-1}\right)\right]. \tag{7.1}$$

$\varphi(f)$ is interpreted here as the fraction of players in a very large population playing the pure strategy f. It is assumed that the use of a strategy grows in proportion to the fitness of this strategy (i.e., in proportion to the difference between the expected payoff of a player using this strategy when playing against randomly selected members of the population and the average expected payoff $E(\varphi, \varphi^{m-1})$ for the population). Taylor and Jonker [31] and Zeeman [37] showed that an evolutionarily stable strategy is an asymptotically stable fixed point of this dynamic. Thomas [32] gave an extension of this result for evolutionarily stable sets. In both cases evolutionary stability is only sufficient, not necessary for asymptotic stability. The situation changes if one allows for mixed strategies to be inherited. Then the asymptotic stability implies evolutionary stability. This will be made precise here for evolutionarily stable sets (for evolutionarily stable strategies see, e.g., Weissing [36], Robson [20] or Cressman [8]). Our result for evolutionarily stable sets differs from the one in Cressman [7] insofar as we discuss the asymptotic stability directly on the space of probability measures over mixed strategies and not via the induced trajectories for the mean strategies. Another difference is of course that we allow for more than two players.

Let G be a symmetric game.

We denote by δ_φ the Dirac distribution, i.e, the probability measure on Φ with $\delta_\varphi(B) = 1 \Leftrightarrow \varphi \in B$ for every (measurable) set $B \subseteq \Phi$. A probability measure on Φ with *finite support* can then be written as

$$\mu = \sum_{\varphi \in \Phi} \mu(\varphi)\, \delta_\varphi$$

with only finitely many coefficients $\mu(\varphi)$ being different from 0. $\mu(\varphi)$ is interpreted as the proportion of players using the mixed strategy in an infinite population.

Let $\Delta(\Phi)$ denote the set of all probability measures on Φ with finite support. We have a surjective projection

$$\begin{array}{cccc} \mathrm{pr}: & \Delta(\Phi) & \to & \Phi \\ & \sum_{\varphi \in \Phi} \mu(\varphi)\, \delta_\varphi & \mapsto & \sum_{\varphi \in \Phi} \mu(\varphi)\, \varphi. \end{array}$$

When μ describes the distribution of strategies in the population, then $\mathrm{proj}(\mu)$ describes the "average strategy" used by the population. The payoff function extends for $\vec{\mu} =$

$(\mu^{(1)}, \ldots, \mu^{(m)}) \in \Delta(\Phi)$ to

$$E(\vec{\mu}) := \sum_{\vec{\varphi}=(\varphi^{(1)},\ldots,\varphi^{(m)})\in\Phi^m} \left(\prod_{k=1}^{m} \mu^{(k)}\left(\varphi^{(k)}\right)\right) E(\vec{\varphi}) = E\left(\mathrm{pr}^{(m)}(\vec{\mu})\right)$$

where $\mathrm{pr}^{(m)}(\vec{\mu}) := \left(\mathrm{pr}\left(\mu^{(j)}\right)\right)_{j\in M}$.

The **replicator dynamic with inheritance of mixed strategies in continuos time** can now be defined as the vectorfield

$$\dot{\mu}(\varphi) := \mu(\varphi) \cdot \left[E\left(\delta_\varphi, \mu^{m-1}\right) - E(\mu^m)\right]. \tag{7.2}$$

For given μ the vectorfield can be thought of as being defined on the finite dimensional simplex $\Delta(T)$ where T is the support of μ. This holds because the vectorfield induced on $\Delta(T)$ coincides with the replicator dynamic where only pure strategies can be inherited for the symmetric game with pure strategy set T and the payoff function $E_{|T^m} : T^m \to \mathbb{R}$. In particular there is a unique solution μ_t $(-\infty \leq t \leq \infty)$ to the differential equation (7.2) with $\mu_0 = \mu$ describing how the use of strategies evolves over time. Furthermore the support of each μ_t of the trajectory through μ_0 is T. The replicator dynamic in pure strategies coincides with the dynamic induced by (7.2) on the set $\Delta(T) = \Delta\left(\left\{\delta_{\delta_f} | f \in F\right\}\right)$. It is important that also the "monomorphic" situation underlying the motivation in Section 3, where most players use a strategy φ and a small proportion switches to using a mutant strategy ψ, is covered. This case emerges when we consider the dynamic on the set $\Delta(T) = \Delta(\{\delta_\varphi, \delta_\psi\})$.

$\Delta(\Phi)$ can be thought of as the "collection" of all mixed strategy models that can be constructed from G. In contrast to , e.g., Thomas [33] we will discuss stability not for any particular mixed strategy model generated by finitely many mixed strategies but allow for mutations from "outside" any such set. This avoids the problems due to non-generic choices of mixed strategy models discussed in Thomas [33].

In order to discuss stability properties we have to fix a topology on $\Delta(\Phi)$. We will work here with the topology defined by the norm

$$||\mu - \mu'|| := \max_{\varphi\in\Phi} |\mu(\varphi) - \mu'(\varphi)|.$$

$\mu \in \Delta(\Phi)$ is a **locally stable fixed point** of the replicator dynamic if every neighborhood of μ contains some neighborhood $V(\mu)$ such that every trajectory $\{\mu_t\}$ starting in $\mu' \in V(\mu)$ (i.e., $\mu_0 = \mu'$) remains in $V(\mu)$ (i.e., $\mu_t \in V(\mu)$ for $t \geq 0$).

A subset $P \subseteq \Delta(\Phi)$ is **locally asymptotically stable** if there exists a neighborhood $U(P)$ of the set such that for every neighborhood $V(P)$ of P all trajectories starting in $U(P)$ arrive in $V(P)$ in finite time and then remain in $V(P)$.

Let me call for brevity a subset $P' \subseteq \Delta(\Phi)$ a **stationary attractor** if P' is a non-empty closed and locally asymptotically stable set and each $\mu \in P'$ is a locally stable fixed point.

We obtain the following dynamic interpretation of evolutionarily stable sets:

Proposition 7.1. *The stationary attractors of the replicator dynamic in continuous time with inheritance of mixed strategies correspond via the projection* pr *to the evolutionarily stable sets of the symmetric game.*

That the preimage of an ESSet is a stationary attractor is proved using the arguments by which Thomas [32] showed that an ESSet is a stationary attractor of the replicator dynamic with pure strategies. It is crucial that Zeeman's Lyapunov function used in the proof defines open neighborhoods in the norm topology of $\Delta(\Phi)$. For the converse we show first that with any probability distribution in the stationary attractor also the Dirac distribution of its average strategy is in the stationary attractor. We can then use the monomorphic scenarios to show the evolutionary stability of the image of the attractor.

8. The Replicator Dynamic for Conflicts Between Several Distinct Species.

In the previous section we gave a dynamic interpretation of evolutionarily stable sets in symmetric games. This implies indirectly, via truly asymmetric contests, a dynamic interpretation of strict equilibrium sets for arbitrary normal form games.

For any normal form game Γ (we continue use the notations of the previous sections) we can also study directly a replicator dynamic for conflicts between different species represented by the different players of the normal form game as, e.g., in Samuelson and Zhang [23]. For the case of pure strategies this is the vectorfield defined for $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ by $\dot{\sigma} = (\dot{\sigma}_1, \ldots, \dot{\sigma}_n)$ where

$$\dot{\sigma}_i(s_i) := \sigma_i(s_i) \cdot [u_i(\sigma \backslash s_i) - u_i(\sigma)] \tag{8.1}$$

for $s_i \in S_i$ and $i \in N$.

Suppose now that $\Gamma = \Gamma(A)$ is the agent representation of some truly asymmetric contest A. Then we have the replicator dynamic for a single species of the symmetric normal form $G(A)$. It is defined for $\varphi \in \Phi = \Delta(S)$ by

$$\dot{\varphi}(s) = \varphi(s) \cdot \left[E\left(s, \varphi^{m-1}\right) - E(\varphi^m)\right] \tag{8.2}$$

for $s \in S$.

To obtain a link between these two dynamics we consider strategy combinations in Σ as uncorrelated probability distributions on S, i.e., we look at the embedding

$$\iota : \Sigma = \prod_{i \in N} \Delta(S_i) \longrightarrow \Phi = \Delta\left(\prod_{i \in N} S_i\right)$$

where $\varphi = \iota(\sigma_1, \ldots, \sigma_n)$ is defined by

$$\varphi(s_1, \ldots s_n) := \prod_{i \in N} \sigma_i(s_i)$$

for $(s_1, \ldots, s_n) \in S$. Consider as an example a 2×2-normal form game. Here the set of mixed strategies of each player correponds to the unit interval and hence the set of strategy combinations is a square. If we symmetrize the game we obtain a 4×4-game where the set of mixed strategies of a player is described by a tetrahedron. ι maps the square onto a non-linear surface in the tetrahedron. We claim now that a trajectory of (8.1) in the square is mapped onto a trajectory of (8.2) in the tetrahedron. In particular we claim that a trajectory of (8.2) which starts on the surface remains on this surface.

Proposition 8.1. *The image of a trajectory of the dynamic (8.1) under the embedding ι is a trajectory of the dynamic (8.2).*

Since the image of ι is a proper submanifold of Φ there are many trajectories of the dynamic (8.2) we do not consider when looking at the dynamic (8.1). We will see, however, that as far as the stability properties of strict equilibrium sets are concerned this difference does not matter.

To compare the corresponding dynamics for distributions with finite support over mixed strategies we distinguish:

1. The set of distributions over *mixed* strategies $\Delta(\Phi)$ in the symmetric normal form $G(A)$ with the dynamic (7.2) specified in the preceding section.

2. The set of distributions over *behavioral* strategies $\Delta(\Sigma)$ of the truly asymmetric contest A with the dynamic defined for each $\mu \in \Delta(\Sigma)$ and $\sigma \in \Sigma$ by

$$\dot{\mu}(\sigma) := \mu(\sigma) \cdot \left[E\left(\delta_\sigma, \mu^{m-1}\right) - E\left(\mu^{m-1}\right)\right]. \tag{8.3}$$

3. The set of combinations of distributions over mixed strategies

$$\Delta^\times(\Sigma) := \prod_{i \in N} \Delta(\Sigma_i)$$

for the normal form game $\Gamma(A)$ with the dynamic defined for $\mu = (\mu_i)_{i \in N} \in \prod_{i \in N} \Delta(\Sigma_i)$ by

$$\dot{\mu}_i(\sigma_i) := \mu_i(\sigma_i) \cdot \left[u_i\left(\mu \backslash \delta_{\sigma_i}\right) - u_i(\mu)\right]. \tag{8.4}$$

Here the payoff functions are canonically extended.

If we denote by $\iota' : \Delta^{\times}(\Sigma) \to \Delta(\Sigma)$ the embedding that maps $(\mu_1, \ldots, \mu_n)$ onto $\nu = \sum_{\sigma\in\Sigma} \nu(\sigma) \cdot \delta_\sigma$ with

$$\nu(\sigma_1, \ldots, \sigma_n) := \prod_{i\in N} \mu_i(\sigma_i)$$

the above proposition extends to:

Proposition 8.2. *The image of a trajectory of the dynamic (8.4) under the embedding ι' is a trajectory of the dynamic (8.3).*

To compare the various stationary attractors (defined using norm topologies as in the previous section) we use projections as indicated in the commutative diagram

$$\begin{array}{ccccc} \Phi & \xrightarrow{\text{proj}} & \Sigma & \xrightarrow{\text{id}} & \Sigma \\ \uparrow{\scriptstyle \text{pr}} & & \uparrow{\scriptstyle \text{pr}'} & & \uparrow{\scriptstyle \text{pr}''} \\ \Delta(\Phi) & \xrightarrow[\text{proj}']{} & \Delta(\Sigma) & \xrightarrow[\text{proj}'']{} & \Delta^{\times}(\Sigma) \end{array} \tag{8.5}$$

Here $\text{proj} : \Phi \to \Sigma$ maps onto the marginal distributions (see Section 4). It induces the projection

$$\begin{array}{rccc} \text{proj}' : & \Delta(\Phi) & \longrightarrow & \Delta(\Sigma) \\ & \sum_{\varphi\in\Phi} \mu(\varphi)\, \delta_\varphi & \longmapsto & \sum_{\varphi\in\Phi} \mu(\varphi) \delta_{\text{proj}(\varphi)} \end{array}$$

for which we have:

Proposition 8.3. *The image of a trajectory of the dynamic (7.2) under the projection* proj' *is a trajectory of the dynamic (8.3).*

proj'' is also obtained by taking the marginal distribution. The mappings pr' and pr'' onto (combinations of) average strategies are constructed in analogy to pr in the previous section. We summarize our findings in:

Proposition 8.4. *The stationary attractors of the dynamic (7.2) correspond via* pr *to the evolutionarily stable sets in Φ.*

The stationary attractors of the dynamic (8.3) and (8.4) correspond via pr' *respectively* pr'' *to the strict equilibrium sets in Σ.*

The stationary attractors of the dynamic (7.2) correspond via proj' *to those of the dynamic (8.3) and the latter via* proj'' *to those of the dynamic (8.4).*

The first statement in the proposition restates our result from the previous section. Then the invariance (Proposition 4.2) and Proposition 8.3 imply that the stationary attractors of the dynamic (8.3) correspond to the direct evolutionarily stable sets. Our main result, Proposition 5.1, together with Proposition 8.1 yields then that the preimage of an SESet under pr$''$ is a stationary attractor of the dynamic (8.4). The converse follows as in the previous section: It is first shown that with every combination of probability distributions in a stationary attractor of (8.4) also the combination of the Dirac distributions of the mean strategies belong to the attractor. Then one can use monomorphic scenarios where only one population mutates to deduce that the preimage of the attractor is an SESet (see Appendix).

A. Proofs

Proofs for Section 2

Proof of Proposition 2.2 Let σ be a strategy combination in the SESet and hence a Nash equilibrium. Then every strategy combination τ with $\text{supp}(\tau_i) \subseteq \text{supp}(\sigma_i)$ for $i \in N$ is first of all a best reply against σ. It is also contained in the SESet: Suppose $(\tau_1, \ldots, \tau_{i-1}, \sigma_i, \ldots \sigma_n)$ with $1 \leq i < n$ is in the SESet (which is true for $i = 1$). Because $\text{supp}(\tau_i) \subseteq \text{supp}(\sigma_i)$ the strategy τ_i is a best reply to this strategy combination and hence $(\tau_1, \ldots, \tau_{i-1}, \tau_i, \sigma_{i+1}, \ldots \sigma_n)$ is also in the SESet. It follows by induction that τ is in the SESet. To prove that σ is a best reply against τ we take a sequence $\left(\tau^k\right)$ with $\text{supp}\left(\tau_i^k\right) = \text{supp}(\sigma_i)$ converging to τ. Because σ is a best reply to each τ^k (which is an element of the SESet) and the best reply correspondence is upper-hemi continuous it is also a best reply against τ. Thus with every strategy combination σ in the SESet the Cartesian product $\prod_{i\in N} \Delta(\text{supp}(\sigma_i))$ is a commuting set of Nash equilibria contained in the SESet. ■

Proof of Proposition 2.3 Let $\Pi = \prod_{i\in N} \Delta(P_i)$ be a maximal Cartesian product of faces in the SESet. We have just seen that for any $\sigma, \tau \in \Pi$ with $\text{supp}(\sigma_i) = P_i$ for $i \in N$ τ is a best reply to σ. Suppose that ρ is an arbitrary best reply to σ. Let τ_i $(i \in N)$ be a mixed strategy whose support is the union of the support of σ_i and ρ_i. Since every pure strategy contained in the support of σ_i and ρ_i is a best reply to σ, τ_i is a also a best reply to σ. Therefore $\sigma \backslash \tau_i$ is in the SESet and the Cartesian product of faces generated by $\sigma \backslash \tau_i$ is contained in the SESet. Since we have chosen Π maximal we obtain $\text{supp}(\rho_i) \subseteq \text{supp}(\tau_i) = \text{supp}(\sigma_i) = P_i$ for each i and hence $\rho \in \Pi$.

Suppose Π and Π' are Cartesian products of faces contained in the SESet and that Π is a proper subset of Π'. Let σ be a strategy combination generating Π and let σ' be a strategy combination generating Π'. Then σ' is a best reply to σ but not contained in Π. ■

Proof of Proposition 2.4 A trembling-hand perturbed game can be modelled either as a game with restricted mixed strategy sets or as a game with the original strategy sets and special perturbed payoff functions. We work here with the latter interpretation.

A trembling-hand perturbation of the bimatrix game is specified by a vector

$$((\varepsilon_1, \tau_1), (\varepsilon_2, \tau_2))$$

where $0 < \varepsilon_i < 1$ is the probability of a player $i \in \{1,2\}$ to make an error and $\tau_i \in \Sigma_i$ the strategy chosen if he makes an error. We can assume that the error probabilities are so small that all strategies of a player that are optimal against a *pure* strategy of his opponent in the perturbed game are also best replies in the unperturbed game. Consider a pure strategy combination (s_1, s_2) in a strict equilibrium set R for which the payoff vector $(u_1(s_1, \tau_2), u_2(\tau_1, s_2))$ is Pareto efficient within the set

$$\{(u_1(t_1, \tau_2), u_2(\tau_1, t_2)) \mid (t_1, t_2) \text{ is a pure strategy combination in } R\}.$$

Then (s_1, s_2) is a Nash equilibrium in the perturbed game:

Otherwise we could find a pure strategy, say, t_1 of player 1 against s_2 with

$$u_1(t_1, (1-\varepsilon_2)s_2 + \varepsilon_2\tau_2) > u_1(s_1, (1-\varepsilon_2)s_2 + \varepsilon_2\tau_2).$$

By the choice of the error probabilities t_1 is a best reply against s_2 in the unperturbed game. Therefore the pure strategy combination (t_1, s_2) is in the SESet and $u_1(t_1, \tau_2) > u_1(s_1, \tau_2)$. Consequently $(u_1(s_1, \tau_2), u_2(\tau_1, s_2))$ is not Pareto-efficient in the set mentioned above in contradiction to the choice of (s_1, s_2). The claim follows because strict equilibrium sets depend only on the reduced form of a normal form game. ■

Proofs for Section 3

Proof of Lemma 3.2 For the norm $||\psi - \varphi|| := \max_{f\in F} |\psi(f) - \varphi(f)|$ the sets

$$U_\delta = \{\psi \in \Phi \mid ||\psi - \varphi|| \leq \delta\}$$

form for vaying $\delta > 0$ a basis of neighborhoods of φ. We have $||\psi_\varepsilon - \varphi|| = \varepsilon||\psi - \varphi|| \leq \varepsilon$ for all ψ and hence $V_\delta \subseteq U_\delta$. To prove that every U_δ is for sufficiently small δ contained in a given V_{ε_0} we note that every $\chi \in \Phi$ can be written in a unique way as a convex combination $\chi = \psi_\varepsilon = (1-\varepsilon)\varphi + \varepsilon\psi$ where ψ is on a face of Φ that does not contain φ. The union of these faces is a compact set and the continuous function $||\psi - \varphi||$ is strictly positive on this set. Hence there exists a constant $\alpha > 0$ with $||\psi - \varphi|| \geq \alpha$ for all ψ in this union. For $\delta := \varepsilon_0 \cdot \alpha$, $\psi_\varepsilon = (1-\varepsilon)\varphi + \varepsilon\psi$ with ψ in the union and $||\psi_\varepsilon - \varphi|| \leq \delta$ we obtain $||\psi_\varepsilon - \varphi|| = \varepsilon||\psi - \varphi|| \leq \varepsilon_0 \cdot \alpha$, hence $\varepsilon \leq \varepsilon_0$. Therefore $U_\delta \subseteq V_{\varepsilon_0}$. ■

Proof of Proposition 3.3 Fix $\psi \in \Phi\backslash\{\varphi\}$. For fixed φ and ψ the inequality $E(\psi, \psi_\varepsilon^{m-1}) < E(\varphi, \psi_\varepsilon^{m-1})$ with $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$ is equivalent to

$$\sum_{k=1}^{m-1} \binom{m-1}{k-1} (1-\varepsilon)^{m-k} \varepsilon^{k-1} \left(E\left(\varphi, \varphi^{m-k-1} \times \psi^k\right) - E\left(\psi, \varphi^{m-k-1} \times \psi^k\right)\right) < 0. \qquad \text{(A.1)}$$

Suppose for some k we have $E(\varphi, \varphi^{m-l-1} \times \psi^l) - E(\psi, \varphi^{m-l-1} \times \psi^l) = 0$ for all $1 \leq l < k \leq m-1$. (This condition is void and hence true for $k = 1$.) If $E(\varphi, \varphi^{m-k-1} \times \psi^k) - E(\psi, \varphi^{m-k-1} \times \psi^k)$ is strictly positive, then the left hand side of (A.1) is strictly positive for all small ε. If the difference is strictly negative, the left hand side is strictly negative for all small ε. If the difference is equal to 0, then we have to look at the difference $E(\varphi, \varphi^{m-k-2} \times \psi^{k+1}) - E(\psi, \varphi^{m-k-2} \times \psi^{k+1})$ provided $k < m-1$. If $k = m-1$ then $E(\psi, ((1-\varepsilon)\varphi + \varepsilon\psi)^{m-1}) = E(\varphi, ((1-\varepsilon)\varphi + \varepsilon\psi)^{m-1})$ for all ε. Thus (3.6) is equivalent to (3.3) being true for all sufficiently small ε. Therefore an ESS has to satisfy (3.6) for all ψ. ■

Proof of Proposition 3.4 The first part of i) follows as in Proposition 3.3. The converse for bimatrix games is Lemma 1 in Thomas [32].

To prove the first part of ii) let $\psi \in \Phi$ and $\varphi \in R$ with $\eta(\psi, \varphi, \psi) = \eta(\varphi, \varphi, \psi)$. Hence $E(\psi, \psi_\varepsilon^{m-1}) = E(\varphi, \psi_\varepsilon^{m-1})$ for all $\varepsilon > 0$, where $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$. Since R is evolutionarily stable, $\psi_\varepsilon \in R$ for all sufficiently small ε. Let ε' be the supremum of all ε'' satisfying $\psi_\varepsilon \in R$ for all $\varepsilon \leq \varepsilon''$. Since R is closed $\psi_{\varepsilon'} \in R$. Hence there exists a small neighborhood of ψ_ε such that for all χ in this neighborhood $E(\chi, \chi^{m-1}) = E(\psi_{\varepsilon'}, \chi^{m-1})$ implies $\chi \in R$. If $\varepsilon' < 1$ we would have for all $\varepsilon > \varepsilon'$ sufficiently close to ε' $E(\psi_{\varepsilon'}, \psi_\varepsilon^{m-1}) = E(\psi_\varepsilon, \psi_\varepsilon^{m-1})$. This would imply $E(\psi, \psi_\varepsilon^{m-1}) = E(\psi_\varepsilon, \psi_\varepsilon^{m-1})$ and hence $\psi_\varepsilon \in R$, contradicting our choice of ε'. Therefore $\psi \in R$.

For the converse in bimatrix games we use first Lemma 1 in Thomas [32] to conclude that R is a set of neutral evolutionarily stable strategies with the property that for all $\varphi \in R$ any ψ is in R which lies in a sufficiently small neighborhood of φ and satisfies $E(\psi, \psi) = E(\varphi, \psi)$. To show that R is a closed set let $\{\varphi_k\}$ be a sequence in R converging to the strategy φ. Selecting if necessary a subsequence, we can assume that all strategies in the sequence have equal support. Therefore φ is a best reply whenever one element in the sequence is. Since each element in the sequence is a symmetric Nash equilibrium strategy we obtain $E(\varphi, \varphi_k) = E(\varphi_k, \varphi_k)$ for all φ_k. Because each φ_k is neutral evolutionarily stable $E(\varphi_k, \varphi) \geq E(\varphi, \varphi)$ by part i) above. For every $\psi \in \Phi$ and every k we have $E(\psi, \varphi_k) \leq E(\varphi_k, \varphi_k)$ and, by taking the limit, $E(\psi, \varphi) \leq E(\varphi, \varphi)$. In particular we have for every k $E(\varphi_k, \varphi) \leq E(\varphi, \varphi)$. Therefore $\eta(\varphi_k, \varphi, \varphi_k) = \eta(\varphi, \varphi, \varphi_k)$ for all k which implies $\varphi \in R$. Thus R is closed. ■

Proof of Proposition 3.5 Let $P \subseteq \Phi$ denote the ESSet.

i) By definition we can find for every $\varphi \in P$ an open neighborhood $U(\varphi)$ such that for all $\psi \in U(\varphi)$ $E(\varphi, \psi^{m-1}) \geq E(\psi^m)$ whereby equality implies $\psi \in P$. It actually follows that equality holds here *if and only if* ψ is in the ESSet: We know that each $\psi \in U(\varphi) \cap P$ is a symmetric Nash equilibrium strategy. Hence $E(\varphi, \psi^{m-1}) \geq E(\psi^m)$ for such ψ and therefore $E(\varphi, \psi^{m-1}) = E(\psi^m)$.

ii) Suppose $E(\varphi, \psi^{m-1}) = E(\psi^m)$ for some $\varphi \neq \psi \in U(\varphi)$. Then all strategies

$$\chi_\varepsilon = (1-\varepsilon) \cdot \varphi + \varepsilon \cdot \psi \tag{A.2}$$

in Φ with $\varepsilon \in \mathbb{R}$ belong to the ESSet:

Since ψ is in the ESSet we have for all $0 < \varepsilon < 1$ sufficiently close to 1 :

$$E\left(\psi, \chi_\varepsilon^{m-1}\right) - E\left(\chi_\varepsilon, \chi_\varepsilon^{m-1}\right) = -(1-\varepsilon)\left(E\left(\varphi, \chi_\varepsilon^{m-1}\right) - E\left(\psi, \chi_\varepsilon^{m-1}\right)\right) \leq 0.$$

Since $\chi_\varepsilon \in U(\varphi)$ we obtain for all such ε

$$E\left(\varphi, \chi_\varepsilon^{m-1}\right) - E\left(\chi_\varepsilon, \chi_\varepsilon^{m-1}\right) = \varepsilon\left(E\left(\varphi, \chi_\varepsilon^{m-1}\right) - E\left(\psi, \chi_\varepsilon^{m-1}\right)\right) \leq 0$$

and therefore $E(\varphi, \chi_\varepsilon^{m-1}) - E(\chi, \chi_\varepsilon^{m-1}) = 0$. Thus the polynomial $E(\varphi, \chi_\varepsilon^{m-1}) - E(\chi, \chi_\varepsilon^{m-1})$ in ε is the 0-polynomial. Consequently, all χ_ε that are in $U(\varphi)$ are in the ESSet. Let ε_0 be the infimum of all $\varepsilon > 0$ with $\chi_\varepsilon \notin \Phi$ or $\chi_\varepsilon \notin P$. Since P is closed χ_{ε_0} is in the ESSet. Our argument yields for all ε slightly larger than ε_0 that either $\chi_\varepsilon \in P$, contradicting our assumption, or that $\chi_\varepsilon \notin \Phi$. The symmetric argument for negative ε proves the claim.

iii) The set of points in the ESSet P where P is locally an analytic submanifold is a dense open subset of P consisting of finitely many path-connected components $P_1, \ldots, P_l$.

To prove this we check first that P is a closed semi-algebraic set: Since P is compact there exist finitely many $\varphi_1, \ldots, \varphi_k \in P$ with semi-algebraic neighborhoods $U(\varphi_\kappa)$ covering P such that by i)

$$P \cap U(\varphi_\kappa) = \left\{\psi \in U(\varphi_\kappa) \mid E\left(\varphi, \psi^{m-1}\right) = E(\psi^m)\right\}.$$

Each $P \cap U(\varphi_\kappa)$ is therefore semi-algebraic. Hence P is as a finite union of semi-algebraic sets itself semi-algebraic. It is well known (see, e.g., Hardt [10]) that every semi-algebraic set can be described as a finite union of disjoint, pathwise connected analytic manifolds with nowhere dense boundaries such that the following is satisfied: If one of these manifolds intersects the closure of another one of these manifolds, then it is contained in the boundary of the latter. We choose the P_λ as those manifolds not contained in any boundary.

iv) Let P_λ be one of the analytic submanifolds mentioned in the preceding paragraph and denote by k_λ its dimension. We are going to construct inductively sequences of affine independent points $\varphi_0, \varphi_1, \ldots, \varphi_k$, $k \leq k_\lambda$, and convex open neighborhoods $\varphi_\kappa \in U(\varphi_\kappa)$ such that

— $U(\varphi_{\kappa-1}) \supseteq U(\varphi_\kappa)$ for $0 < \kappa \leq k$,

— each $U(\varphi_\kappa)$ satisfies

$$U(\varphi_\kappa) \cap P = U(\varphi_\kappa) \cap P_\lambda = \left\{\psi \in U(\varphi_\kappa) \mid E\left(\varphi_\kappa, \psi^{m-1}\right) = E(\psi^m)\right\}. \tag{A.3}$$

— the affine span L_λ^k of these points in Φ is contained in P and its intersection with $U(\varphi_k)$ equals the intersection of P_λ with $U(\varphi_k)$.

We start the sequence with an arbitrary point $\varphi_0 \in P_\lambda$ and a convex open neighborhood $U(\varphi_0)$ satisfying (A.3). Suppose we have found a sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ as desired. Suppose $U(\varphi_k) \cap P$ contains some point φ_{k+1} not in L_λ^k. By ii) P contains the line segment through φ_k and φ_{k+1}. Since this line segment is contained in $U(\varphi_{k-1})$ P contains by ii) the triangle with the vertices φ_{k-1}, φ_k and φ_{k+1}. Proceeding inductively we find that P_λ contains the convex hull of $\varphi_0, \ldots, \varphi_k, \varphi_{k+1}$. Now we take a point in the relative interior of this convex hull and a sufficiently small neighborhood of this point. By ii) P contains the intersection of the line through this point and any sufficiently nearby point in the convex hull with Φ. Hence the affine hull of $\varphi_0, \ldots, \varphi_k, \varphi_{k+1}$ intersected with Φ is contained in P. Finally we choose a convex open neighborhood $U(\varphi_{k+1})$ of φ_{k+1} in $U(\varphi_k)$ satisfying (A.3).

In this manner we extend our sequence until k equals the dimension k_λ of P_λ. Now we cannot find a point in $U(\varphi_{k_\lambda}) \cap P_\lambda$ that is not in the affine hull $L_\lambda := L_\lambda^{k_\lambda}$ of $\varphi_0, \ldots, \varphi_{k_\lambda}$ because otherwise we could continue our construction in contradiction to P_λ having dimension k_λ. Thus $U(\varphi_{k_\lambda}) \cap P_\lambda = U(\varphi_{k_\lambda}) \cap L_\lambda$. Since P_λ is a path-connected analytic manifold we obtain, using the identity theorem for analytic functions, $U(\varphi_0) \cap P_\lambda = U(\varphi_0) \cap L_\lambda$ and then $P_\lambda \subseteq L_\lambda \cap \Phi$. Furthermore $L_\lambda \cap \Phi \subseteq P$. Since P is the closure of $\bigcup_{\lambda=1,\ldots,l} P_\lambda$ and

$$\bigcup_{\lambda=1,\ldots,l} P_\lambda \subseteq \bigcup_{\lambda=1,\ldots,l} L_\lambda \cap \Phi \subseteq P$$

we obtain $P = \bigcup_{\lambda=1,\ldots,l} L_\lambda \cap \Phi$. ■

Proof of Proposition 3.6 Let R be a strict equilibrium set of the symmetric game.

We consider first special combinations of strategies: I will call a combination with k strategies $\psi^{(k)} = (\psi_1, \ldots, \psi_k) \in \Phi^k$ $(1 \leq k \leq m)$ *admissible* (with respect to φ) if for all subcombinations $\psi^{(l,\pi)}$ of $\psi^{(k)}$ (i.e., the combinations $(\psi_{\pi(1)}, \ldots, \psi_{\pi(l)})$ with $1 \leq l \leq k$ and $\pi : \{1, \ldots, l\} \rightarrow \{1, \ldots, m\}$ injective)

$$E(\psi_{\pi(\lambda)}, \varphi^{m-l} \times \psi^{(l,\pi)}_{-\pi(\lambda)}) = E(\varphi, \varphi^{m-l} \times \psi^{(l,\pi)}_{-\pi(\lambda)}) \tag{A.4}$$

for all $1 \leq \lambda \leq l$.

If $\psi^{(k)}$ is admissible then it follows inductively that for every subcombination of $\psi^{(l,\pi)}$ of $\psi^{(k)}$ of length $1 \leq l \leq k$ the combination $\varphi^{m-l} \times \psi^{(l,\pi)}$ is in R: If $\psi^{(l,\pi)}$ contains only one component , i.e., $\psi^{(l,\pi)} = (\psi_\kappa)$ for some $1 \leq \kappa \leq k$ then (A.4) implies $\varphi^{m-1} \times \psi^{(l,\pi)} \in R$. If we know for the subcombination $\psi^{(l,\pi)}$ of length $1 \leq l \leq k$ that $\varphi^{m-l+1} \times \psi^{(l,\pi)}_{-\pi(\lambda)} \in R$ for every $1 \leq \lambda \leq l$ then (A.4) implies $\varphi^{m-l} \times \psi^{(l,\pi)} \in R$.

A combination $\psi^{(k)} = (\psi_1, \ldots, \psi_k) \in \Phi^k$ is called *critical* (with respect to φ) if every subcombination with strictly less components is admissible while for at least some $1 \leq \kappa \leq k$

$$E(\psi_\kappa, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}) \neq E(\varphi, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}).$$

If $\psi^{(k)}$ is a critical combination of length k, then we have for *all* $1 \leq \kappa \leq k$

$$E(\psi_\kappa, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}) < E(\varphi, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}). \tag{A.5}$$

This follows because for all κ $\varphi^{m-k+1} \times \psi^{(k)}_{-\kappa} \in R$ and hence

$$E(\psi_\kappa, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}) \leq E(\varphi, \varphi^{m-k} \times \psi^{(k)}_{-\kappa}). \tag{A.6}$$

If we had equality in (A.6) for *some* κ then $\varphi^{m-k} \times \psi^{(k)} \in R$ and therefore we would have equality in (A.6) for *all* κ.

Let $T^k \subseteq F^k$ denote the set of all critical combinations in pure strategies of length k. Since there are only finitely many critical k-tuples there exists a constant $A > 0$ such that

$$E(f_\kappa, \varphi^{m-k} \times f_{-\kappa}^{(k)}) < E(\varphi, \varphi^{m-k} \times f_{-\kappa}^{(k)}) - A \tag{A.7}$$

holds simultaneously for all $1 \leq k \leq m$, all $f^{(k)} = (f_1, \dots, f_k) \in T^k$ and all $1 \leq \kappa \leq k$.

It follows: Any k-tupel $f^{(k)} \in F^k$ of length at most m belongs to one of the following disjoint categories: Either $f^{(k)}$ is admissible or $f^{(k)}$ is critical or $f^{\ (k)}$ contains a critical subcombination of strictly smaller length.

We have to show that for all $\varphi \in P$ there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and for all $\psi \in \Phi$ the inequality $E(\psi, \psi_\varepsilon^{m-1}) \leq E(\varphi, \psi_\varepsilon^{m-1})$ holds with $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$ whereby equality implies $\psi \in P$.

I claim that the inequality

$$\begin{aligned} &E(\psi, \psi_\varepsilon^{m-1}) - E(\varphi, \psi_\varepsilon^{m-1}) \leq \\ &\sum_{k=1}^{m} \sum_{(f_1,\dots,f_k)\in T_k} \varepsilon^{k-1} \cdot \left(\prod_{\kappa=1}^{k} \psi(f_\kappa)\right) \Big(-(1-\varepsilon)^{m-k} kA + \varepsilon \cdot p_{(f_1,\dots,f_k)}(\varepsilon) \cdot C\Big) \end{aligned} \tag{A.8}$$

holds where T_k is the set of critical combinations in pure strategies of length k, $A > 0$ is the constant we found above, $C > 0$ is an upper bound on the difference between the payoffs resulting from any two strategy combinations in the game and the $p_{(f_1,\dots,f_k)}(\varepsilon)$ are polynomials in ε with real coefficients.

To prove the inequality we observe that

$$\begin{aligned} &E(\psi, \psi_\varepsilon^{m-1}) - E(\varphi, \psi_\varepsilon^{m-1}) \\ &= \sum_{k=0}^{m-1} (1-\varepsilon)^{m-k-1} \varepsilon^k \left(E(\psi, \varphi^{m-k-1} \times \psi^k) - E(\varphi, \varphi^{m-k-1} \times \psi^k)\right) \\ &= \sum_{k=1}^{m} \sum_{(f_1,\dots,f_k)\in F^k} \sum_{\kappa=1}^{k} (1-\varepsilon)^{m-k} \varepsilon^{k-1} \left(\prod_{\kappa=1}^{k} \psi(f_\kappa)\right) \cdot \\ &\qquad \Big(E(f_\kappa, \varphi^{m-k+1} \times (f_1, \dots, f_k)_{-\kappa}) - E(\varphi, \varphi^{m-k+1} \times (f_1, \dots, f_k)_{-\kappa})\Big) . \end{aligned} \tag{A.9}$$

For any admissible combination $(f_1, \dots, f_k) \in F^k$ the equalities (A.4) imply that all differences in the payoffs and hence the whole sum for $(f_1, \dots, f_k)$ vanishes in the last term of (A.9). If $(f_1, \dots, f_k)$ is critical, then each payoff difference appearing in the summand for $(f_1, \dots, f_k)$ is at most $-A$ and hence the whole summand is at most

$$\varepsilon^{k-1} \cdot \left(\prod_{\kappa=1}^{k} \psi(f_\kappa)\right) \Big(-(1-\varepsilon)^{m-k} kA\Big) .$$

If $(f_1, \dots, f_k)$ belongs to none of these cases, then $(f_1, \dots, f_k)$ contains some critical combination of smaller length, say $(f_1, \dots, f_l)$ with $l < k$. The payoff differences appearing in the summand for $(f_1, \dots, f_k)$ can be at most C. The summand can be at most

$$(1-\varepsilon)^{m-k} \varepsilon^{k-1} \left(\prod_{\kappa=1}^{k} \psi(f_\kappa)\right) \cdot k \cdot C \leq \varepsilon^{l-1} \cdot \varepsilon \left(\prod_{\kappa=1}^{l} \psi(f_\kappa)\right) \cdot k \cdot (1-\varepsilon)^{m-k} \cdot C. \tag{A.10}$$

$k \cdot (1-\varepsilon)^{m-k}$ is a polynomial in ε because $m \geq k \geq 2$. The expression on the right of (A.10) is hence captured in the inequality (A.8) by the expression

$$\varepsilon^{k-1} \cdot \left(\prod_{\kappa=1}^{k} \psi(f_\kappa) \right) \cdot \varepsilon \cdot p_{(f_1,\ldots,f_k)}(\varepsilon) \cdot C$$

for the critical subcombination we picked. This proves the inequality (A.8).

We can choose finally $\varepsilon_0 > 0$ such that

$$-(1-\varepsilon)^{m-k} kA + \varepsilon \cdot p_{(f_1,\ldots,f_k)}(\varepsilon) \cdot C < 0$$

holds for all $0 \leq \varepsilon \leq \varepsilon_0$ and for all critical combinations $(f_1, \ldots, f_k)$. Consequently $E(\psi, \psi_\varepsilon^{m-1}) \leq E(\varphi, \psi_\varepsilon^{m-1})$ holds and equality implies that $\prod_{\kappa=1}^{l} \psi(f_\kappa) \neq 0$ only when $(f_1, \ldots, f_k)$ is admissible. Therefore, if equality holds, we obtain for all $0 \leq k \leq m-1$

$$\begin{aligned} &E(\psi, \varphi^{m-k} \times \psi^k) - E(\varphi, \varphi^{m-k} \times \psi^k) \\ &= \sum_{(f_1,\ldots,f_k) \in F^k} \left(\prod_{\kappa=1}^{k} \psi(f_\kappa) \right) \cdot \\ &\qquad \left(E\left(f_\kappa, \varphi^{m-k+1} \times (f_1, \ldots, f_k)_{-\kappa}\right) - E\left(\varphi, \varphi^{m-k+1} \times (f_1, \ldots, f_k)_{-\kappa}\right) \right) = 0. \end{aligned}$$

Hence it follows inductively that $\psi \times \varphi^{m-1}, \psi^2 \times \varphi^{m-2}, \ldots, \psi^m$ are admissible with respect to φ. We obtain in particular that $\psi^m \in R$ and therefore $\psi \in P$. ■

Proofs for Section 4

Proof of Proposition 4.2 The proof follows immediately from the following facts: i) For each strategy combination $\vec{\varphi} \in \Phi^m$ we have

$$E(\vec{\varphi}) = E\left(\text{proj}^m(\vec{\varphi})\right).$$

ii) The projection proj : $\Phi \to \Sigma$ is linear and continuous. It is open because it maps each neighborhood $\{(1-\varepsilon)\varphi + \varepsilon\psi | \psi \in \Phi, 0 \leq \varepsilon \leq \varepsilon_0\}$ of $\varphi \in \Phi$ onto the neighborhood $\{(1-\varepsilon)\sigma + \varepsilon\tau | \sigma \in \Sigma, 0 \leq \varepsilon \leq \varepsilon_0\}$ of $\sigma = \text{proj}(\varphi) \in \Sigma$. ■

Proofs for Section 6

Proof of Proposition 6.1 Let $P \subseteq \Sigma$ be a strict equilibrium set. We define R as the set of all behavioral strategy combinations $\vec{\sigma} = ((\sigma_i^j)_{i \in N})_{j \in N}$ of the symmetrization with the property that $(\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)})$ is in P for all $\pi \in \Pi(N)$. P is clearly the set of symmetric equilibrium strategies in R. Suppose for $\tau \in \Sigma$ and $\vec{\sigma} = ((\sigma_i^{(j)})_{i \in N})_{j \in N} \in R$

$$E(\vec{\sigma} \backslash \tau) \geq E(\vec{\sigma}).$$

Then

$$\sum_{\pi \in \Pi(n)} \left(u_{\pi^{-1}(j)}((\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)}) \backslash \tau_{\pi(j)}) - u_{\pi^{-1}(j)}(\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)})\right) \geq 0.$$

But since for each permutation π each $(\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)})$ is a Nash equilibrium in the original game it follows that the above inequality can only be satisfied if $\tau_{\pi(j)}$ is a best reply to the Nash equilibrium $(\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)})$ for each permutation π and hence $(\sigma_1^{\pi(1)}, \ldots, \sigma_n^{\pi(n)}) \backslash \tau_{\pi(j)} \in P$. It follows that $\vec{\sigma} \backslash \tau \in R$. ■

Proofs for Section 7

Some technical problems can be avoided in the following proof because for most arguments we can restrict attention to finite dimensional subsets of $\Delta(\Phi)$.

Proof of Proposition 7.1 i) Let $P \subseteq \Phi$ be an evolutionarily stable set and $P' = \mathrm{pr}^{-1}(P)$. Since pr is continuous P' is closed. For each $\mu \in P'$ we can find a neighborhood $U(\mathrm{pr}(\mu))$ such that $E(\psi^m) \leq E(\mathrm{pr}(\mu), \psi^{m-1})$ for all $\psi \in U(\mathrm{pr}(\mu))$, whereby equality implies $\psi \in P$. $U(\mu) := \mathrm{pr}^{-1}(U(\mathrm{pr}(\mu)))$ is then a neighborhood of μ with $E(\nu^m) \leq E(\mu, \nu^{m-1})$ for all $\nu \in U(\mu)$ whereby equality implies $\nu \in P'$.

For $\mu \in P'$ denote by T the support of μ. We define a function $L_\mu : \Delta(\Phi) \to \mathbb{R}$ by

$$L_\mu(\nu) := 1 - \prod_{\varphi \in T} \left(\frac{\nu(\varphi)}{\mu(\varphi)} \right)^{\mu(\varphi)}.$$

L_μ depends only on the probabilities a probability measure assigns to the strategies in T and is hence continuous in the norm topology. Straightforward computations show that L_μ satisfies $0 \leq L_\mu(\nu) \leq 1$ for all $\nu \in \Delta(\Phi)$, that μ is the unique minimum of the function with value 0 and that $L_\mu(\nu) = 1$ if and only if ν does not contain T in its support. Furthermore we can find for each $\delta > 0$ an $\varepsilon > 0$ such that $L_\mu(\nu) < \varepsilon$ implies $||\mu - \nu|| < \delta$: Otherwise one could find a sequence $\varepsilon^k > 0$ converging to 0 and μ^k with $||\mu^k - \mu|| \geq \delta$ and $L_\mu\left(\mu^k\right) < \varepsilon^k$. Since L_μ depends only on the probabilities assigned to strategies in T, we can assume that each μ^k assigns positive probability to at most one strategy not in T and that this strategy is the same for all μ^k. Hence all μ^k lie in the same compact subset of $\Delta(\Phi)$. We can therefore find a subsequence converging to some $\hat{\mu}$ with $||\hat{\mu} - \mu|| \geq \delta$. Continuity implies $L_\mu(\hat{\mu}) = 0$ in contradiction to μ being the unique minimum of L_μ.

Thus the sets $V_\varepsilon(\mu) := \{L_\mu(\nu) < \varepsilon\}$ $(\varepsilon > 0)$ form a basis of neighborhoods of μ.

Each $\mu \in P'$ is locally stable: Let $U(\mu)$ be a neighborhood of μ with $E(\nu^m) \leq E\left(\mu, \nu^{m-1}\right)$ for all $\nu \in U(\mu)$ whereby equality implies $\nu \in P'$. For every sufficiently small $\varepsilon > 0$ we have $V_\varepsilon(\mu) \subseteq U(\mu)$ and T is contained in the support of each $\nu \in V_\varepsilon(\mu)$. For each $\nu \in V_\varepsilon(\mu)$ it suffices to consider the restriction of the vectorfield (7.2) to the finite dimensional simplex $\Delta(T')$, where T' is the support of ν. The proof of Theorem 1 ii) in Thomas [32] – where $L_\mu(\nu)$ is used as a Lyapunov function – shows that the trajectory starting in ν remains in $V_\varepsilon(\mu) \cap \Delta(T')$.

It also follows from the proofs of Theorem 2 and Corollary 3 in Thomas [32] that each trajectory starting in $V_\varepsilon(\mu) \cap \Delta(T')$ converges to a probability measure in $V_\varepsilon(\mu) \cap P'$. Selecting for each $\mu \in P'$ a neighborhood $V_\varepsilon(\mu)$ as just described and taking the union of these neighborhoods we obtain a neighborhood of P' such that every trajectory starting in this neighborhood converges to a probability measure in P'. In particular P' is locally asymptotically stable.

ii) Suppose $P' \subseteq \Delta(\Phi)$ is a local stationary attractor of the replicator dynamics (7.2). Observe that every fixed point in a sufficiently small neighborhood of P' must belong to P' since P' is locally asymptotically stable.

Fix any $\mu \in P'$. Then also $\delta_\varphi \in P'$ for $\varphi := \mathrm{pr}(\mu)$: We have for every $0 \le \varepsilon \le 1$ and $\mu_\varepsilon := (1-\varepsilon)\mu + \varepsilon\delta_\varphi$

$$E\left(\delta_\varphi, \mu_\varepsilon^{m-1}\right) = E\left(\varphi^m\right) = E\left(\mu^m\right).$$

Furthermore we obtain for each ψ in the support of μ

$$E\left(\delta_\psi, \mu_\varepsilon^{m-1}\right) = E\left(\psi, \varphi^{m-1}\right) = E\left(\varphi^m\right) = E\left(\mu_\varepsilon^m\right)$$

since μ is a fixed point of (7.2). Hence μ_ε is also a fixed point of (7.2) for all $0 \le \varepsilon \le 1$. Since P' is locally asymptotically stable $\mu_\varepsilon \in P'$ for all sufficiently small ε. Let $\bar{\varepsilon}$ be the supremum of all ε' for which $\mu_\varepsilon \in P'$ for all $\varepsilon \le \varepsilon'$. Since P' is closed $\mu_{\bar{\varepsilon}} \in P'$. We obtain $\bar{\varepsilon} = 1$ since otherwise there would be fixed points arbitrarily close to $\mu_{\bar{\varepsilon}}$ not in P', in contradiction to P' being locally asymptotically stable.

Hence $P := \mathrm{pr}(P') = \mathrm{pr}(P') \cap \{\delta_\psi | \psi \in \Phi\}$.

We show next that every $\varphi \in P$ is a neutral evolutionarily stable strategy. Let

$$V_{\varepsilon_0}(\delta_\varphi) = \{\mu \in \Delta(\Phi) |\, ||\mu - \delta_\varphi|| < \varepsilon_0\}$$

be a neighborhood of δ_φ such that every trajectory of (7.2) starting in this neighborhood remains in this neighborhood and has only ω-limit points in P'. For the neighborhood

$$U_{\varepsilon_0}(\varphi) = \{\psi_\varepsilon \in \Phi | \psi \in \Phi,\, 0 \le \varepsilon \le \varepsilon_0\}$$

with $\psi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\psi$ we have $U_{\varepsilon_0}(\varphi) \subseteq \mathrm{pr}(V_{\varepsilon_0}(\delta_\varphi))$: We obtain for any $\psi \neq \varphi$ and $\mu_\varepsilon := (1-\varepsilon)\delta_\varphi + \varepsilon\delta_\psi$ $\mathrm{pr}(\mu_\varepsilon) = \psi_\varepsilon$ and $||\mu_\varepsilon - \delta_\varphi|| = \varepsilon||\delta_\psi - \delta_\varphi|| = \varepsilon$. Therefore

$$\varepsilon \le \varepsilon_0 \Leftrightarrow \mu_\varepsilon \in V_{\varepsilon_0}(\delta_\varphi) \Leftrightarrow \psi_\varepsilon \in U_{\varepsilon_0}(\varphi).$$

Now suppose that for some $\psi_\varepsilon \in U_{\varepsilon_0}(\varphi)$ $E\left(\psi_\varepsilon^m\right) = E\left(\varphi, \psi_\varepsilon^{m-1}\right)$. Then also $E\left(\psi, \psi_\varepsilon^{m-1}\right) = E\left(\varphi, \psi_\varepsilon^{m-1}\right) = E\left(\psi_\varepsilon^m\right)$. Therefore μ_ε is a fixed point of (7.2) and hence in P'.

Suppose next that $E\left(\psi_\varepsilon^m\right) > E\left(\varphi, \psi_\varepsilon^{m-1}\right)$ for some $\psi_\varepsilon \in U_{\varepsilon_0}(\varphi)$ or equivalently that $E\left(\psi, \psi_\varepsilon^{m-1}\right) > E\left(\varphi, \psi_\varepsilon^{m-1}\right)$. We note that the payoff difference $E\left(\psi, \psi_\varepsilon^{m-1}\right) - E\left(\varphi, \psi_\varepsilon^{m-1}\right)$ is a polynomial in ε for fixed φ and ψ with a zero at $\varepsilon = 0$. The assumption implies that the polynomial is not the 0-polynomial and that there is a smallest $\bar{\varepsilon} < \varepsilon_0$ such that $E\left(\psi, \psi_\varepsilon^{m-1}\right) - E\left(\varphi, \psi_\varepsilon^{m-1}\right) \le 0$ for all $0 \le \varepsilon \le \bar{\varepsilon}$ and $E\left(\psi, \psi_\varepsilon^{m-1}\right) - E\left(\varphi, \psi_\varepsilon^{m-1}\right) > 0$ for all $\bar{\varepsilon} < \varepsilon \le \varepsilon' \le \varepsilon_0$. Again $\mu_{\bar{\varepsilon}}$ is a fixed point and hence in P'. But this leads to a contradiction because $\mu_{\bar{\varepsilon}}$ cannot be locally stable: for all $\bar{\varepsilon} < \varepsilon \le \varepsilon' \le \varepsilon_0$ the trajectory starting in μ_ε remains on the line $\{\mu_\varepsilon | 0 \le \varepsilon \le 1\}$ and moves away from $\mu_{\bar{\varepsilon}}$ in the direction of δ_ψ. Any ε-neighborhood of $\mu_{\bar{\varepsilon}}$ with $0 < \varepsilon \le \varepsilon' - \bar{\varepsilon}$ contains therefore points from which the trajectory starting in this point leaves the neighborhood.

Thus every $\varphi \in P$ is a neutral evolutionarily stable strategy and furthermore $E(\psi^m) = E(\varphi, \psi^{m-1})$ implies $\psi \in P$ for all ψ in a sufficiently small neighborhood of φ.

It follows from the proof of part i) that $\mathrm{pr}^{-1}(\varphi)$ is a set of locally stable fixed points of the replicator dynamics. For any $\mu \in \mathrm{pr}^{-1}(\varphi)$ the set

$$\{(1-\varepsilon)\delta_\varphi + \varepsilon\mu | 0 \leq \varepsilon \leq 1\}$$

is therefore a set of fixed points of the replicator dynamics and hence $\mathrm{pr}^{-1}(\varphi) \subseteq P'$. Thus $P' = \mathrm{pr}^{-1}(P)$.

pr maps the simplex $\Delta\left(\left\{\delta_{\delta_f} | f \in F\right\}\right)$ generated by the pure strategies with the topology induced from $\Delta(\Phi)$ homeomorphically onto Φ and maps the intersection of the two closed sets $P' \cap \left(\left\{\delta_{\delta_f} | f \in F\right\}\right)$ onto P. Hence P is closed, which completes the proof. ■

Proofs for Section 7

Proofs of Propositions 8.1 and 8.2 A straightforward computation yields for $\mu \in \Delta^\times(\Sigma)$ and a tangential vector $v = (v_1, \ldots, v_n)$:

$$\left(\underline{d\iota'_{|\mu}}(v)\right)(\sigma) = \sum_{i \in N}\left(v_i(\sigma_i) \cdot \prod_{j \neq i} \mu_j(\sigma_j)\right).$$

For the tangential vector $\dot{\mu} = (\dot{\mu}_1, \ldots, \dot{\mu}_n)$ of the vectorfield (8.4) it follows with $\nu = \iota'(\mu)$

$$\begin{aligned}\left(d\iota'_{|\mu}(\dot{\mu})\right)(\sigma) &= \sum_{i \in N}\left(\mu_i(\sigma_i) \cdot [u_i(\mu \backslash \delta_{\sigma_i}) - u_i(\mu)] \cdot \prod_{j \neq i} \mu_j(\sigma_j)\right) \\ &= \prod_{i \in N} \mu_i(\sigma_i) \cdot \left(\sum_{i \in N}[u_i(\mu \backslash \delta_{\sigma_i}) - u_i(\mu)]\right) \\ &= \nu(\sigma) \cdot [E(\delta_\sigma, \nu^{m-1}) - E(\nu^m)] \\ &= \dot{\nu}(\sigma)\end{aligned}$$

where $\dot{\nu}$ is defined by the vectorfield (8.3). (8.3) satisfies as ordinary differential equation the Lipschitz condition and hence our claim for ι' follows. The claim holds for ι since we can identify ι with ι' restricted to the convex hull of all $(\delta_{s_1}, \ldots, \delta_{s_n})$ with $(s_1, \ldots, s_n) \in S$. ■

Proofs of Proposition 8.3 Since proj′ is linear we have for $\mu \in \Delta(\Phi)$, $\nu = \mathrm{proj}(\mu)$ and $\dot{\mu}$ defined by (7.2)

$$\begin{aligned}&\left(d\,\mathrm{proj}'_{|\mu}(\dot{\mu})\right)(\sigma) = \mathrm{proj}'(\dot{\mu})(\sigma) \\ &= \sum_{\mathrm{proj}(\varphi)=\sigma} \mu(\varphi) \cdot [E(\delta_\varphi, \mu^{m-1}) - E(\mu^m)] \\ &= \nu(\sigma) \cdot [E(\delta_\sigma, \nu^{m-1}) - E(\nu^m)] = \dot{\nu}\end{aligned}$$

where $\dot{\nu}$ is given by the vectorfield (8.3). ■

Proofs of Proposition 8.4 Consider a stationary attractor $R' \subseteq \Delta^{\times}(\Sigma)$ of the dynamic (8.4). It follows as in the proof of Proposition 7.1 (but inductively on the number of roles) that $\delta_{\mathrm{pr}''(\mu)} \in R'$ for all $\mu \in R'$. Fix $\sigma \in R := \mathrm{pr}''(R')$ and $\tau_i \in \Sigma_i$. For all sufficiently small $\varepsilon > 0$ we must have for μ^{ε} with $\mu_i^{\varepsilon} = (1-\varepsilon) \cdot \delta_{\sigma_i} + \varepsilon \cdot \delta_{\tau_i}$ and $\mu_j^{\varepsilon} = \delta_{\sigma_j}$ for $j \neq i$:

$$[1-\varepsilon]\,\varepsilon \cdot [u_i(\sigma) - u_i(\sigma \backslash \tau_i)] = \dot{\mu}_i^{\varepsilon}(\sigma_i) \leq 0.$$

If $u_i(\sigma \backslash \tau_i) = u_i(\sigma)$ then all μ^{ε} are fixed points of the dynamics for all $\varepsilon \geq 0$ and it follows $\delta_{\sigma \backslash \tau_i} \in R'$. Thus R is a strict equilibrium set. As was discussed in Section 8 $(\mathrm{pr}'')^{-1}(R)$ is therefore a stationary attractor and it follows as in the proof of Proposition 7.1 that $R' = (\mathrm{pr}'')^{-1}(R)$. ■

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