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GENERAL MATCHING: LATTICE STRUCTURE OF THE SET OF AGREEMENTS

by

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General Matching: Lattice Structure of the Set of Agreements *

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Abstract

The subset agreement problem generalizes all forms of two-sided matching. Two agents need to agree on some subset of a given finite set of contracts. A solution concept - agreement - generalizes the notion of a stable subset. Its definition does not require the consideration of a preference ordering on sets of contracts, but only that of the choice function that reveals the agents' preferences by choosing the best subset of any given set of contracts. Under a suitable condition, called coherence, that requires that contracts are substitutes to one another, at least one agreement always exists. A constructive proof is given that the structure of the set of agreements is a lattice.

1 Introduction and Background

1.1 What is Matching?

When we talk about matching we refer to a set of models and related concepts that try to model certain resource allocation markets. What special about those markets is that the

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decision-making agents that compete for resources also serve as resources for other agents: every agent is a resource for some other agents and every resource could be thought of as an agent. Consider, for example, a labor market consisting of hiring firms and workers. A worker is a resource that provides labor for a firm that hires him. On the other hand, the firm is a resource for the worker: the former establishes the working conditions and pays the latter.

Matching models attempt to study both statics and dynamics of matching markets. The first problem is to formulate a reasonable equilibrium solution concept, i.e., is there such a resource allocation that the markets clear. Other questions usually follow naturally from the formulation of the solution concept: Is there at least one such equilibrium? Are there more than one equilibrium? What are the dynamics (if any) that lead to such an equilibrium? How do different equilibrium solutions compare? And so on.

1.2 A Review of the Literature

The best approach to understanding of matching is to describe and compare various matching models that have been studied in the literature. Such a review also establishes the point of departure for this work. A substantial review of all the aspects of matching is out of the scope of this work. Moreover, such a review (which cites slightly fewer than 200 references) already exists: it is a monograph by Alvin Roth and Marilda Sotomayor ([27]). In our review we focus on the evolution of matching models and on the results dealing with the structure of the set of stable matchings.

1.2.1 Pure Matching Models

Serious theoretical analysis of matching and attempts to model certain market activities on its basis date back to the pioneering work of Gale and Shapley [12] that analyzes the one-to-one marriage problem[1] and the many-to-one college admissions problem and their relationship. In their model colleges have preferences over individual students, not groups. They begin with defining the solution concept — a stable matching. It is a quintessential static game-theoretic equilibrium concept in the following sense: when players find themselves in an equilibrium it is impossible for any single player, or two players from the opposite sides, to improve on their outcome. It is closely related to the solution concept in the coalitional games — the core — where in an equilibrium no subset of players can improve on the outcome of some of its members by forming a new coalition without hurting at least one member of the subset. It differs from the core in that it is

[1]See Appendix A.1

local in nature — only two player from the opposite sides are considered, not any coalition of the players [2].

Further, Gale and Shapley provide a constructive proof (an algorithm) of the existence of a stable matching for every marriage market [3], duality of the stable matchings with respect to men's and women's preferences[4], and the existence of optimal outcomes for men and for women when the preferences are strict[5].

Finally, they extend the results from one-to-one marriage market to the one-to-many college admissions market and demonstrate the existence of stable matchings via an algorithm very similar to that employed in the marriage problem, as well as existence of college-optimal and student-optimal outcomes.

The first lattice structure result appears in the book of Knuth [16] [6]. He attributes it to John Conway. He proves that the set of stable matchings is a distributive lattice[7]. This last result is especially important to us as the main result of this work is to demonstrate the lattice structure of the set of stable matchings in a more generalized setting.

In 1984 Charles Blair [7] showed that for every finite distributive lattice there exists some marriage market whose set of stable matchings equals the lattice.

The Gale-Shapley model was further generalized by Roth in [23] who considered one-to-many college admissions matching models where the colleges had preferences over the groups of individuals (unlike [12] who considered college preferences only over individuals). They do it by introducing so called *responsive preferences* where college preferences over groups of students are derived from their preferences over individuals[8].

Building in part on the work of Crawford and Knoer [10] and Kelso and Crawford [15], Alvin Roth in [22] and [25] reformulated discrete matching models in terms of an increasingly complex hierarchy: one-to-one, many-to-one, and many-to-many matching. In the many-to-many model not only firms may employ multiple workers, but also a worker may work for more than one firm. He used an important generalization introduced in [15]: preferences over groups of individuals are no longer derived from the preferences over individuals, as in responsive preferences. Rather, they can be arbitrary as long as they satisfy the *gross substitutes* property (which is equivalent to the *substitutability*

[2]It should be noted that in some (more restrictive) matching models the set of stable matchings coincides with the core.

[3]Theorem 4 in Appendix A.1.1

[4]Theorem 5 in Appendix A.1.2

[5]Theorem 6 in Appendix A.1.2

[6]In 1972 Shapley and Shubik [29] demonstrated a similar result. That, however, referred to the class of so called assignment games that are somewhat similar to matching models, but are different in important ways.

[7]Theorem 7 in Appendix A.1.2

[8]See Appendix A.2

condition of [6][9]). In their model (which is not by itself a pure matching model because they have a money element affecting the preferences) [15] demonstrate that a deferred acceptance algorithm (which, in essence, is very similar to the original algorithm of Gale and Shapley) where firms propose first can be used to achieve a stable matching that is optimal for the firms. [25] demonstrated how to remove the money element from the model making it completely symmetric, therefore automatically proving the dual theorem: a deferred acceptance algorithm where the workers propose first leads to a stable matching optimal for the workers.

Charles Blair in [6] further studied the symmetric many-to-many matching model. He demonstrated that the set of locally (i.e. pairwise) stable matchings no longer coincides with the core as in other models. The lattice structure of the set of stable matchings is preserved (and with it the duality and best/worst stable outcomes for the two sides), however the lattice is no longer distributive.

Relatively recently, Hatfield and Milgrom [14] presented a model that abstracts from the individual properties of matching algorithms and so to speak catches the essence of two-sided matching[10]. All previous models treated *separable preferences*: every *individual* has her own preferences not affected by the preferences of others. [14] abstracts from that approach: now every side as a whole has preferences over groups on the other side. When preferences satisfy the gross substitutability property separable preferences generalize to Hatfield and Milgrom's preferences. Theoretically this is useful when preferences of individuals on one side affect each other, but conserve the gross substitutability property[11].

1.2.2 Practical Relevance of Pure Matching Models

The Gale-Shapley model applies to markets without wages or transfers (or 'without money', for short). However, even such a 'simplistic' approach is proving to be extremely useful in practice. To mention just a few relatively recent examples: [24] demonstrates how the procedure used to match tens of thousands of physicians per year to medical residency programs is in fact a slightly modified Gale-Shapley Algorithm. [26] analyzes the practical aspects of the program and some of its shortcomings and offers solutions that since then have been successfully implemented. [1] analyzes some of the existing school choice plans, suggesting that a Gale-Shapley-based mechanism could remedy their serious flaws. [19] presents a theory of matching in vertical networks, generalizing Gale-Shapley matching theory.

[9] See Definition 25

[10] See Chapter 2.

[11] Practical usefulness of this approach is yet to be seen.

1.2.3 Tatonnement Algorithms

Closely related to Gale-Shapley are the tatonnement algorithms of Kelso and Crawford [15], [11], and [13]. The Kelso-Crawford model, in fact, may be treated as a many-to-many matching algorithm without money (see [27]).

1.2.4 Auction Models

Auction models of [8], [18], [5] and [4] are also closely related to matching.

1.2.5 Common Features of the Matching Models

There are certain features that are common to all the matching models without money:

- Problem setting is two-sided: the agents can be naturally divided into two groups, viz., men and women, students and colleges, doctors and hospitals, workers and companies.
- An outcome is defined as a set of offers, every offer involving two agents - one from each side. Viz., man M marries woman W; hospital H employs doctor D; company F employs worker W at wage level L.
- A set of valid outcomes (or matchings) is defined.
- The solution concept employed is that of stable matching with respect to agents' preferences on outcomes. This means, roughly speaking, that no single agent could improve by withdrawing (individual rationality) and that no two agents - one from each side - could improve by agreeing on a separate offer.
- The solution procedure is an iterative process in which agents on one side make a series of offers and, whenever an offer is rejected, the rejected agent is given an opportunity to make another offer.

The differences among the models lie in the particular limitations that are imposed on the set of valid outcomes. For example, the stable marriage problem is one-to-one, i.e. in a valid outcome every man is matched to no more than one woman, every woman is matched to no more than one man, and man M is matched to woman W iff W is matched to M. On the other hand, the hospital residency model and the Kelso-Crawford model, are one-to-many: a doctor is employed by no more than one hospital, but a hospital may employ more than one doctor. The difference between those two models is that a wage component is introduced in Kelso-Crawford.

As noted above, [14] presented a model that abstracts from the individual properties of matching algorithms and so to speak catches the essence of two-sided matching.

1.3 Our Contribution

One of the properties of matching models is that the set of solutions has a certain mathematical structure: it is a lattice. As noted above, we consider a generalized matching model of [14]. He shows that even in that general case the set of solutions form a lattice. However, his proof is non-constructive. Drawing on the work of Blair ([6]) we prove it constructively. [6] demonstrates that the set of solutions for certain many-to-many matching markets is a lattice. But due to the less restrictive nature of our model his proof breaks down, therefore we fix it.

Some other (lesser) contributions of this work are: a rigorous treatment of how classic matching models translate into the generalized matching model[12], a discussion of the relationship between cumulative partial orders and choice functions[13], and a necessary and sufficient condition for existence of a non-empty agreement[14].

2 The Model

We start with an informal description and then proceed with the formal setting.

2.1 The Informal Description

Following [14] we study a model of matching theory that unifies the class of two-sided matching models by abstracting from the content of an offer. Our basic unit of analysis is a two-party contract the content of which is opaque. Each such contract represents a possible assignment of a student to a college, or a doctor to a hospital, or a worker to a firm at a certain wage level. A set of such contracts may be considered a candidate matching, i.e. a possible assignments of students to colleges, etc. Hence any matching may be represented as a set of contracts.

Furthermore, we aggregate preferences of the players on each side arriving at a model that consists of only two aggregate preferences. Those preferences express the collective preferences of their members over the set of all possible matchings, i.e. the power set of contracts.

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[12]See Appendix A

[13]See Section 2.2

[14]See the end of Chapter 4

Most previous work describes preferences by a total order or a total partial order on some unstructured set of alternatives. It is an essential feature of our framework that any player is interested only in those contracts that he is a part of (a no-externalities assumption). In the aggregate preference relation a subset of contracts A is preferred to B only if all players weakly prefer A to B. However, it is possible that some players prefer A to B and some B to A. Therefore we have no choice but to consider relations that allow the subsets of contracts to be incomparable.

In order to make the model more general, we would have liked to allow the players to be indifferent between alternatives, so the anti-symmetry does not always hold. However, it is impossible to obtain many of the results if such a relaxation is allowed. Therefore we require anti-symmetry. Hence the preference relations we consider are reflexive, transitive, anti-symmetric, but not necessarily total, i.e. partial orders.

From the algorithmic perspective, those preferences over sets require vast amounts of information to be fully specified. Alkan and Gale [3] proposed that such preferences be revealed, only partially and when needed, by a choice function.[15] In order to employ the choice function approach we impose the following restriction on the aggregate preferences we consider: given any set A of contracts, there is a subset B of A that is preferred to any subset of A. Due to anti-symmetry this subset B is unique.

Finally, we need to define what constitutes a stable matching. Consider two players that are trying to agree on a set of contracts. Each player has preference over subsets of contracts. Those preferences correspond to the aggregate preferences of the sides that we have discussed above. Given a set of contracts any one player may remove contracts from it. Contracts may be added only one by one and only if both players agree. Adding a contract may be beneficial to a player because the resulting set contains a subset that the player prefers to the one that is on the table now. A subset of contracts is called an *agreement* if it cannot be modified. In our model agreements correspond to stable matchings, as we demonstrate in Section 3.

Our model may seem more restrictive than the classical matching models: Typically the set of alternatives is an arbitrary set that does not have any special structure. We, on the contrary, assume that the set of alternatives is the powerset of some set (the set of contracts), and has a certain structure that we use in order to restrict the preferences. However, as we shall see in Section 3, those restriction do not prevent our model from generalizing the existing matching models.

[15] Their framework is slightly different: the set of firms and the set of workers have preferences over matching matrices, which is a special case of sets of contracts.

2.2 The Formal Setting

We start with the definition of preference relations we consider:

Definition 1 *Let Ω be some finite set. A* ***cumulative partial order*** *$\succeq$ on the powerset of Ω is a reflexive, transitive, and anti-symmetric binary relation on 2^Ω such that*

$$\forall B{\in}2^\Omega\ \exists C{\subseteq}B : \forall C'{\subseteq}B\ C{\succeq}C'$$

In other words, although some subsets of Ω may be incomparable, every subset B contains a subset C that is preferred to all other subsets of B.

Apart from being defined on a powerset, this definition differs from the usual definition of a preference relation in that we do not require it to be total, i.e. we allow elements of 2^Ω to be *incomparable*. Also, we require anti-symmetry.

In what follows whenever we write *preference relation* we mean *cumulative partial order*, unless indicated otherwise. As usual, $a{\succ}b$ indicates that $a{\succeq}b$ and $a{\neq}b$.

In analyzing our model we take the approach of choice functions and revealed preferences very similar to that of [3] and implicit in [6]. Given a set of contracts A we are only interested in the most preferred subset of A. That property can be described by a function from sets of contracts to sets of contracts that for every set A picks its most preferred subset. The following definition captures that property:

Definition 2 *Let $\succeq$ be a cumulative partial order on 2^Ω and $f : 2^\Omega \rightarrow 2^\Omega$ be a function from subsets of Ω to subsets of Ω. We call f the* ***choice function*** *for $\succeq$ if*

$$\forall A{\in}2^\Omega, \forall B \subseteq A : f(A) \subseteq A \text{ and } f(A) \succeq B$$

Note that due to cumulativity of $\succeq$ such a function exists, and due to anti-symmetry of $\succeq$ such a function is unique.

The following definition encapsulates the properties we want to assume about the function f. We claim that in all previously studied matching problems the preferences of the players may be described by *coherent* choice functions:

Definition 3 *Let $f : 2^\Omega \rightarrow 2^\Omega$ be a function from subsets of Ω to subsets of Ω. We define three properties:*

- *(Contraction)* $\forall X \subseteq \Omega\ f(X) \subseteq X$
- *(Cumulativity)* $\forall X, Y \subseteq \Omega$ *if* $f(X) \subseteq Y \subseteq X$, *then* $f(Y) = f(X)$

- *(Sα) $\forall X, Y \subseteq \Omega$ if $X \subseteq Y$, then $X \cap f(Y) \subseteq f(X)$.*

We call f a ***choice function*** *if it satisfies Contraction; we call it a* ***cumulative choice function*** *if it satisfies both Contraction and Cumulativity; and we call it a* ***coherent function*** *if it satisfies all three of the properties.*

For the justification of those requirements consider a choice function f for $\succeq$. Remember that $f(X)$ is the most preferred subset if only elements of X are allowed. Contraction just requires that only elements of X be present.

Cumulativity is obviously satisfied if $f(X)$ is the most preferred subset of X (see Claim 1).

The property Sα is Sen's [28] property α. The property appears in Chernoff's [9]. It is essentially the *substitutability condition* of [6] (see Definition 25) and [22]. The *persistence* property of [3] is very similar. This property says that if a contract is chosen out a larger set Y, it will still be chosen out of any subset of Y which contains it. In Blair's words ([6, p. 620]) it means that "members of Y are wanted for their own sake, not because of potential benefits of their interaction with other members".

Those three properties have been extensively used for non monotonic logics in [17]. In Appendix B we demonstrate some properties of coherent functions.

Claim 1 *For any cumulative partial order its choice function is cumulative.*

Proof: Let f be the choice function for $\succeq$. Clearly f satisfies Contraction.

Suppose $X, Y \subseteq \Omega$ and $f(X) \subseteq Y \subseteq X$. Now, $f(Y)$ is a subset of Y, so it is a subset of X. Therefore $f(X) \succeq f(Y)$. On the other hand, since $f(X)$ is a subset of Y we have $f(Y) \succeq f(X)$. Therefore, by anti-symmetry, $f(X) = f(Y)$. Hence f is a cumulative choice function. ▮

Having demonstrated that there is a mapping from cumulative partial orders to cumulative choice functions, we would like to establish next a mapping in the reverse direction: from cumulative choice functions to cumulative partial orders. The following claim shows that it is impossible in the unrestricted case:

Claim 2 *There exists a cumulative choice function that is not a choice function for any cumulative partial order.*

Proof: Let $\Omega = \{a, b, c\}$. Define $f(\Omega) = \Omega$, $f(\{a, b\}) = f(\{a\}) = \{a\}$, $f(\{b, c\}) = f(\{b\}) = \{b\}$, $f(\{a, c\}) = f(\{c\}) = \{c\}$, $f(\emptyset) = \emptyset$. It is easy to verify that f is a cumulative choice function.

Suppose that f is a choice function for some cumulative partial order $\succeq$. Since $f(\{a,b\}) = \{a\}$ and $\{b\} \subset \{a,b\}$ it must be the case that $\{a\} \succeq \{b\}$ (see Definition 2). Similarly, from $f(\{b,c\}) = \{b\}$ we obtain $\{b\} \succeq \{c\}$, and from $f(\{a,c\}) = \{c\}$ we obtain $\{c\} \succeq \{a\}$. By transitivity and anti-symmetry it must be the case that $\{a\} = \{b\} = \{c\}$. Contradiction. ▮

In order to establish a mapping from choice functions to cumulative partial orders we restrict our attention to *coherent partial orders*:

Definition 4 *A cumulative partial order $\succeq$ is called* **coherent** *if its choice function is coherent.*

The following claim establishes a mapping from coherent choice functions to coherent partial orders.

Claim 3 *Let $f : 2^\Omega \rightarrow 2^\Omega$ be a coherent choice function. Let binary relation $\succeq_{\mathrm{f}}$ on 2^Ω be defined as:*

- $X \succeq_{\mathrm{f}} X$
- *if $X \neq Y$ then $X \succeq_{\mathrm{f}} Y$ iff there exist $Z \subseteq \Omega$ such that $X = f(Z)$ and $Y \subseteq Z$*

Then $\succeq_{\mathrm{f}}$ is a coherent partial order, f being its choice function.

Proof: $\succeq_{\mathrm{f}}$ is reflexive by definition.

Suppose $A \succeq_{\mathrm{f}} B$ and $B \succeq_{\mathrm{f}} C$, where $A \neq B$ and $B \neq C$. It means that there exist D' and D'' such that $A = f(D')$, $B \subseteq D'$, $B = f(D'')$, and $C \subseteq D''$. Let $D = D' \cup D''$. Due to coherence of f we have:

$$D'' \cap f(D) \subseteq f(D'') = B \subseteq D'$$

Therefore, due to Contraction, $f(D) \subseteq D'$, and so $f(D) \subseteq D' \subseteq D$. By Cumulativity $f(D) = f(D') = A$. Since $C \subseteq D'' \subseteq D$, by definition of $\succeq_{\mathrm{f}}$ we obtain $A \succeq_{\mathrm{f}} C$. So $\succeq_{\mathrm{f}}$ is transitive.

Suppose $A \succeq_{\mathrm{f}} B$ and $B \succeq_{\mathrm{f}} A$. It means that there exist E' and E'' such that $A = f(E')$, $B \subseteq E'$, $B = f(E'')$, and $A \subseteq E''$. Let $E = E' \cap E''$. By Contraction $A \subseteq E'$. Since $A \subseteq E''$ we have $f(E') = A \subseteq E$. Similarly $f(E'') \subseteq E$. Since $E \subseteq E'$, by coherence of f, $E \cap f(E') \subseteq f(E) \Rightarrow f(E') \subseteq f(E) \Rightarrow f(E') \subseteq f(E) \subseteq E'$. Hence, by Cumulativity, $f(E') = f(E)$. Similarly, $f(E'') = f(E)$. Therefore $A = B$ and $\succeq_{\mathrm{f}}$ is anti-symmetric.

Let $A \subseteq B$. Then, by definition of $\succeq_{\mathrm{f}}$, $f(B) \succeq_{\mathrm{f}} A$. Thus $\succeq_{\mathrm{f}}$ is a cumulative partial order. It is also immediate that f is the choice function for $\succeq_{\mathrm{f}}$, so $\succeq_{\mathrm{f}}$ is coherent. ▮

Note that in general the reverse of Claim 3 does not hold: there may exist many coherent partial orders having the same choice function.

Suppose that we somehow managed to translate individual preference in a two-sided matching model into aggregate preferences of the two sides (see Section 3 on how it can be done for classical matching models). Our next goal is to provide a suitable definition for the resulting model, including the definition of stable matchings via the new aggregate preferences.

Definition 5 *A* ***two-party subset agreement problem*** *(or* ***2-SAP*** *for short)* $(\Omega, \succeq_1, \succeq_2)$ *consists of the finites set of* ***contracts*** Ω*, and binary relations* $\succeq_1$ *and* $\succeq_2$ *on* 2^Ω*. Where* $\succeq_i$ *,* $i \in \{1, 2\}$ *is understood as a preference relation of player i over subsets of* Ω*.*

In order to obtain interesting results we have to restrict our attention to 2-SAPs where the player preferences are coherent.

Definition 6 *A 2-SAP* $(\Omega, \succeq_1, \succeq_2)$ *is called* **coherent** *if both* $\succeq_1$ *and* $\succeq_2$ *are coherent. In such a case we say that the coherent choice functions for* $\succeq_1$ *and* $\succeq_2$ **characterize** *the 2-SAP.*

We follow with two definitions of stable agreements. The first one defines an agreement via player preferences. The second - via the choice functions. While the first approach provides the intuition behind the definition, the choice functions definition is easier to work with technically. Of course we will have to demonstrate the equivalence of the two definitions.

Definition 7 *Given a two-party subset agreement problem* $(\Omega, \succeq_1, \succeq_2)$ *and a set* $A \subseteq 2^\Omega$*, we say that A is* ***a stable agreement*** *or simply* ***an agreement*** *if both of the following conditions hold:*

1. *(Individual Rationality)* $\forall B \subseteq A : \ A \succeq_1 B$ *and* $A \succeq_2 B$
2. *(Stability)* $\forall x \in \Omega \ \exists i \in \{1, 2\} : \ (\forall B \subseteq A \cup \{x\} : \ A \succeq_i B)$

To justify the definitions, remember that $\succeq_i$ are aggregate preferences: $A \succeq_i B$ means that all individuals on side i (weakly) prefer A to B. Each contract, on the other hand, is between two individuals on both sides. The Individual Rationality property means that individuals are not forced to enter into contracts and may drop any contract if the result is more suitable to them. The Stability property means that if a new contract is introduced *all* the individuals on at least one of the sides are not going to be willing to enter into it (possibly by dropping some other contracts). Note that Stability is a *local* condition in the following sense: we require A only to be resistant to addition to it of any single contract, not any arbitrary subset of contracts.

Definition 8 *Given two coherent choice functions* $f_1, f_2 : 2^\Omega \to 2^\Omega$ *we call* $A \subseteq \Omega$ *an* **agreement with respect to** f_1 **and** f_2 *if both of the following two conditions hold:*

1. $f_1(A) = f_2(A) = A$
2. $\forall x \in \Omega : f_1(A \cup \{x\}) \cap f_2(A \cup \{x\}) \subseteq A.$

When the identity of f_1 *and* f_2 *is clear from the context we will refer to such subsets as simply agreements.*

The following claim establishes equivalence between the two definitions.

Claim 4 *Suppose a coherent 2-SAP* $(\Omega, \succeq_1, \succeq_2)$ *is characterized by coherent choice functions* f_1 *and* f_2*. Then* A *is an agreement with respect to* f_1 *and* f_2 *if and only if* A *is an agreement in the 2-SAP.*

Proof: Suppose A is an agreement with respect to f_1 and f_2. We show that A satisfies Individual Rationality and Stability and therefore is an agreement in the 2-SAP:

By definition $f_i(A) = A$, where $i \in \{1, 2\}$. Since f_i is the choice function for $\succeq_i$, by Definition 2 we obtain Individual Rationality: $\forall B \subseteq A : A \succeq_i B$.

Fix $x \in \Omega$ and let $C \subseteq A \cup \{x\}$. Since $f_1(A \cup \{x\}) \cap f_2(A \cup \{x\}) \subseteq A$, it must be the case that either $f_1(A \cup \{x\}) \subseteq A$ or $f_2(A \cup \{x\}) \subseteq A$. Suppose without loss of generality that the former holds, then $f_1(A \cup \{x\}) \subseteq A \subseteq A \cup \{x\}$, and by Cumulativity, $f_1(A \cup \{x\}) = A$. Since f_1 is the choice function for $\succeq_1$ we have $A \succeq_1 C$. But x and C were chosen arbitrarily, therefore we obtain Stability.

Suppose now that A is an agreement in the 2-SAP. We show that it is an agreement with respect to f_1 and f_2:

From Individual Rationality and Definition 2 it immediately follows that $f_i(A) = A$ for $i \in \{1, 2\}$.

For the sake of contradiction suppose that $\exists x \in \Omega$ such that $f_1(A \cup \{x\}) \cap f_2(A \cup \{x\}) \nsubseteq A$. This means that $f_1(A \cup \{x\}) \neq A$ and $f_2(A \cup \{x\}) \neq A$. By Definition 2, $f_1(A \cup \{x\}) \succeq_1 A$. Since $f_1(A \cup \{x\}) \neq A$ it cannot be the case that $A \succeq_1 f_1(A \cup \{x\})$. Similarly it cannot be the case that $A \succeq_2 f_2(A \cup \{x\})$. Thus Stability is violated. We conclude by contradiction that $\forall x \in \Omega : f_1(A \cup \{x\}) \cap f_2(A \cup \{x\}) \subseteq A$. ∎

It follows from Claim 3 and Claim 4 that if we are interested only in the set of agreements, coherent preferences approach and coherent choice function approach may be used interchangeably.

The set of all possibilities that a player faces may be huge. Consider, for example, a hospital that wants to hire 50 residents out 10,000 newly graduated medical doctors. The number of all possible assignments is around 3×10^{135}. The preference relation over such a set of possibilities cannot be calculated. On the other hand, given a subset of doctors, it is a relatively easy task to choose 50 (or less) best suited candidates among them. It is therefore *the choice function approach that really matters*. And it is the approach that we use as *the main definition of stable agreements*.

3 Model Comparisons

In this section we justify the definitions given in Section 2. We demonstrate how the classical matching models can be translated into 2-SAPs and prove that stable agreements in those models map one-to-one to stable agreements in the corresponding 2-SAPs.

We formally consider 2 classical matching models: Stable Marriage and One-to-many Job Matching (those models are formally presented in Appendix A).

3.1 The Marriage Problem

The Marriage Problem is the simplest matching model. Although due to its simplicity lacking practical application, it is very useful in establishing basic intuitions about matching. Therefore we start with it.

We first would like to translate a marriage market into a 2-SAP and show that the agreements in the two models coincide. We do it indirectly by constructing choice functions for men and women such that the agreements with respect to the functions coincide with the stable marriages. We are assured of the existence of suitable preferences by Claim 3, and therefore we obtain a 2-SAP. Claim 4 then guarantees that the agreements in the 2-SAP and the marriage market coincide.

One of the common features of classical treatments of matching problems is the special emphasis given to the *legal* (or feasible) solutions. Preferences of the parties are then considered only over such legal solutions. We, on the other hand, consider preferences over both legal and illegal solutions. Instead of ruling out by law a *marriage à trois*, of, say, a woman accepting contracts with two different men, we simply make the assumption that every player prefers a legal subset of contracts (in this case, any set consisting of no more than one contract) to any illegal set.

Let $(M, W, (\succeq_i)_{i \in M \cup W})$ be a marriage market. The set Ω of contracts will be the set of all possible man-woman pairings:

$$\Omega = \{(m, w) | \; m \in M, w \in W\}$$

Next we define two partitions of Ω: one for M and one for W. Let $p \in M \cup W$, define Ω_p as:

$$\Omega_p = \{x \in \Omega |\ x = (p, w)\ or\ x = (m, p)\}.$$

Then the two sets $\{\Omega_m\}_{m\in M}$ and $\{\Omega_w\}_{w\in W}$ are partitions of Ω.

For notation convenience denote $X_p \equiv X \cap \Omega_p$ for $X \subseteq \Omega$. Now for every $m \in M$ we define a choice function $f_m : 2^\Omega \to 2^\Omega$ as:

$$f_m(X) = (X - \Omega_m) \cup \{(m, w) \in X_m|\ w \succ_m \emptyset\ and\ \forall (m, w') \in X_m, w \succeq_m w'\}$$

Analogously, for every $w \in W$:

$$f_w(X) = (X - \Omega_w) \cup \{(m, w) \in X_w|\ m \succ_w \emptyset\ and\ \forall (m', w) \in X_w, m \succeq_w m'\}$$

In other words, given a set of contracts X, $f_p(X)$ consists of all the contracts in X to which p is not a party, plus the contract p prefers the most among the members of X_p (provided that p prefers that contract to staying single).

It is an easy exercise to verify that:

- f_p's are coherent
- By Definition 29, $\{f_m\}_{m\in M}$ form a party and $\{f_w\}_{w\in W}$ form a party

Therefore, by Lemma 14, the function $f_M(X) = \bigcap_{m\in M} f_m(X)$ and the function $f_W(X) = \bigcap_{w\in W} f_w(X)$ are coherent.

We next establish the equivalence of the set of agreements in the marriage market and the set of agreements with respect to $f_W(X)$ and $f_M(X)$.

Let $A \subseteq \Omega$ and let $p \in M \cup W$ be a player in the underlying marriage game. We define $\pi_p(A)$ as the set of all players whose name appears opposite p on contracts in A to which p is a party:

$$\pi_p(A) = \{w|\ (p, w) \in A\} \cup \{m|\ (m, p) \in A\}$$

Note that if $p \in M$ then $\pi_p(A) \subseteq W$, and if $p \in W$ then $\pi_p(A) \subseteq M$.

Let $A \subseteq \Omega$ be an agreement. We define $\mu_A : M \cup W \to 2^{M\cup W}$ as follows:

$$\mu_A(p) = \pi_p(A)$$

Since $f_M(A) = f_W(A) = A$ it follows that for any player p we have $f_p(A) = A$. Which means that $|\pi_p(A)| \leq 1$ and thus $|\mu_A(p)| \leq 1$ for any p. Other properties of a matching readily follow from the definition of π_p and the fact that $|\pi_p(A)| \leq 1$. Therefore μ_A is a matching.

Suppose μ_A is not individually rational. Without loss of generality assume that $\emptyset \succ_m \mu_A(m) = \{w\}$ for some $m{\in}M$ and $w{\in}W$. It follows that $(m, w){\in}A$. But then $(m, w) \notin f_m(A)$, and therefore $f_M(A) \neq A$. Contradiction. Therefore μ_A must be individually rational.

Suppose that μ_A is blocked by a pair (m, w). Then $(m, w) \in f_M(A \cup (m, w))$ and $(m, w) \in f_W(A \cup (m, w))$. Clearly (m, w) is not in A. Hence $f_M(A \cup (m, w)) \cap f_W(A \cup (m, w)) \nsubseteq A$. Contradiction. Therefore it must be the case that μ_A is not blocked by any pair, and thus it is a stable matching.

Thus we have demonstrated how to map agreements into stable matchings. Now we show how to map stable matchings into agreements. Let μ be a stable matching. Define A_μ as follows:

$$A_\mu = \{(m, w) |\ \mu(m) = \{w\}\}$$

or, equivalently:

$$A_\mu = \{(m, w) |\ \mu(w) = \{m\}\}$$

It is easy to verify that $\pi_p(A_\mu) = \mu(p)$. In particular, there is no more than one contract in A_μ for any player p.

Suppose that $f_M(A_\mu) \neq A_\mu$. It follows that $f_m(A_\mu) \neq A_\mu$ for some $m \in M$. But that is possible only if $(m, w) \in A_\mu$ and $(m, w) \notin f_m(A_\mu)$ for some $w \in W$. Since (m, w) is the only contract in A_μ for m, it must be the case that $\emptyset \succ_m w$. But then μ is not individually rational. Contradiction. Therefore $f_M(A_\mu) = A_\mu$ and, similarly, $f_W(A_\mu) = A_\mu$.

Suppose that $f_M(A_\mu \cup (m, w)) \cap f_W(A_\mu \cup (m, w)) \nsubseteq A_\mu$ for some $(m, w) \in \Omega$. Clearly $(m, w) \notin A_\mu$. Then $(m, w) \in f_m(A_\mu \cup (m, w))$. That is possible only if $w \succ_m \mu(m)$. Similarly, $m \succ_w \mu(w)$. Therefore μ is blocked by (m, w). Contradiction. We conclude that $f_M(A_\mu \cup (m, w)) \cap f_W(A_\mu \cup (m, w)) \subseteq A_\mu$ for all $(m, w) \in \Omega$.

Therefore A_μ is an agreement with respect to f_M and f_W.

It is an easy exercise to show that the mappings $A \to \mu_A$ and $\mu \to A_\mu$ are inverses of each other, thus completing the construction.

3.2 One-to-many Job Matching

The translation of One-to-many Job Matching model into a 2-SAP follows the same pattern as the Marriage problem translation.

Let $M = (F, W, S, (L_p)_{p \in F \cup W}, (\succ_p)_{p \in F \cup W})$ be a job market. The set Ω consists of all possible triplets of firm, worker, and salary:

$$\Omega = \{(f, w, s) |\ f{\in}F, w{\in}W, s{\in}S\}$$

Let $A \subseteq \Omega$ and let $p \in P = F \cup W$ be a player in the underlying job matching model. We define $\pi_p(A)$ as the set of all player-salary pairs that appear on contracts in A to which p is a party:

$$\pi_p(A) = \{(w,s) | \ (p,w,s) \in A\} \cup \{(f,s) | \ (f,p,s) \in A\}$$

We next define two partitions of Ω: one for F and one for W. Let $p \in F \cup W$, define Ω_p as:

$$\Omega_p = \{x \in \Omega | \ x = (p,w,s) \ or \ x = (w,p,s)\}.$$

Then the sets $\{\Omega_f\}_{f \in F}$ and $\{\Omega_w\}_{w \in W}$ form partitions of Ω.

We naturally extend the concept of the choice set (see Definition 23) to the sets of contracts. Let $f \in F$, then $C_f^{2-SAP} : 2^{\Omega_f} \to 2^{\Omega_f}$ is defined as:

$$C_f^{2-SAP}(X) \ = \ \{(f,w,s) \in X | \ (w,s) \in C_f(\pi_f(X))\}$$

Analogously, for $w \in W$, $C_w^{2-SAP} : 2^{\Omega_w} \to 2^{\Omega_w}$ is defined as:

$$C_w^{2-SAP}(X) \ = \ \{(f,w,s) \in X | \ (f,s) \in C_w(\pi_w(X))\}$$

We are now ready to define a choice function $g_p : 2^{\Omega} \to 2^{\Omega}$ for every $p \in F \cup W$ as:

$$g_p(X) = (X - \Omega_p) \cup C_p^{2-SAP}(X)$$

In other words, given a set of contracts X, $g_p(X)$ is the subset of contracts p prefers the most among members of X to which p is a party, together with all the contracts in X to which p is not a party.

It is an easy exercise to verify that:

- g_p's are coherent (Sα, of course, follows from the Substitutability property of choice sets)
- By Definition 29 $\{g_f\}_{f \in F}$ form a party, and $\{g_w\}_{w \in W}$ form a party

Therefore by Lemma 14, function $g_F(X) = \bigcap_{f \in F} g_f(X)$ is coherent, and the function $g_W(X) = \bigcap_{w \in W} g_w(X)$ is also coherent.

We next show that there is a one-to-one correspondence between agreements with respect to $g_F()$ and $g_W()$ and stable matchings in the job market.

Let $A \subseteq \Omega$ be an agreement, $P = F \cup W$. We define $\nu_A : P \to 2^{P \times S}$ as follows:

$$\nu_A(p) = \pi_p(A)$$

Since A is an agreement, we have $g_F(A) = g_W(A) = A$. Therefore for any $p \in P$, $g_p(A) = A$. By definitions of $g_p()$ and $C_p^{2-SAP}()$ it must be the case that $C_p(\pi_p(A)) = \pi_p(A)$. Since $\nu_A(p) = \pi_p(A)$ and $C_p(\pi_p(A)) \in L_p$ by definition, we conclude that $\nu_A(p) \in L_p$ for any p.

Suppose, without loss of generality, that $(w, s) \in \nu_A(f)$ for some $w \in W$. Then $(w, s) \in \pi_f(A)$, which means that $f \in F$. Moreover, $(f, w, s) \in A$, and therefore $(f, s) \in \pi_w(A) \Rightarrow (f, s) \in \nu_A(w)$. We conclude that ν_A is a matching in M.

We next show that ν_A is stable. We've already demonstrated that for any $p \in P$, $C_p(\nu_A(p)) = \nu_A(p)$.

Suppose that for some $w \in W,\ f \in F,\ and\ s \in S$, $(w, s) \in C_f(\nu_A(f) \cup \{(w, s)\})$. If $(w, s) \in \nu_A(f)$ then it immediately follows from the definition of matching and the equivalence $C_p(\nu_A(p)) = \nu_A(p)$ that $C_w(\nu_A(w) \cup \{(f, s)\}) = \nu_A(w)$.

Suppose then that $(w, s) \notin \nu_A(f)$ and, for the sake of contradiction, that $C_w(\nu_A(w) \cup \{(f, s)\}) \neq \nu_A(w)$. Since $C_w(\nu_A(w) \cup \{(f, s)\}) \neq \nu_A(w)$, it must be the case that $(f, s) \notin \nu_A(w)$ and $(f, s) \in C_w(\nu_A(w) \cup \{(f, s)\})$. It follows that $(f, w, s) \notin A$, but $(f, w, s) \in g_w(A \cup \{(f, w, s)\}) \Rightarrow (f, w, s) \in g_W(A \cup \{(f, w, s)\})$ and $(f, w, s) \in g_f(A \cup \{(f, w, s)\}) \Rightarrow (f, w, s) \in g_F(A \cup \{(f, w, s)\})$, violating Stability. Contradiction.

This completes the proof that ν_A is a stable matching in M.

Now we show how to map stable matchings into agreements. Let ν be a stable matching. Define A_ν as follows:

$$A_\nu = \{(f, w, s) |\ (f, s) \in \nu(w)\}$$

or, equivalently:

$$A_\nu = \{(f, w, s) |\ (w, s) \in \nu(f)\}$$

Note that for any $p \in P$, $\pi_p(A_\nu) = \nu(p)$, and also, by definition of a stable matching, $\nu(p) = C_p(\nu(p))$, so $C_p(\pi_p(A_\nu)) = \nu(p)$.

Suppose $g_W(A_\nu) \neq A_\nu$. Then it must be the case that for some $w \in W$, $f \in F$, and $s \in S$: $(f, w, s) \in A_\nu$, but $(f, s) \notin C_w(\pi_w(A_\nu)) = \nu(w)$. But that contradicts the definition of A_ν. We conclude that $g_W(A_\nu) = A_\nu$ and, analogously, $g_F(A_\nu) = A_\nu$.

Suppose that for some $(f, w, s) \notin A_\nu$: $(f, w, s) \in g_W(A_\nu \cup \{(f, w, s)\})$ and $(f, w, s) \in g_F(A_\nu \cup \{(f, w, s)\}))$. Then it must be the case that $(f, s) \notin \nu(w)$ and $(f, s) \in C_w(\pi_w(A_\nu \cup \{(f, w, s)\})$. But $C_w(\pi_w(A_\nu \cup \{(f, w, s)\})) = C_w(\pi_w(A_\nu) \cup \{(f, s)\}) = C_w(\nu(w) \cup \{(f, s)\})$. Therefore $(f, s) \in C_w(\nu(w) \cup \{(f, s)\})$. Analogously, $(w, s) \notin \nu(f)$ and $(w, s) \in C_f(\nu(f) \cup \{(w, s)\})$. Since $(f, s) \in C_w(\nu(w) \cup \{(f, s)\})$ and $\nu()$ is a matching, it follows that $C_f(\nu(f) \cup \{(w, s)\}) = \nu(f)$, but that is impossible, since $(w, s) \notin \nu(f)$ and $(w, s) \in C_f(\nu(f) \cup \{(w, s)\})$. Contradiction. We conclude that A_ν is a stable agreement with respect to g_W and g_F.

One can easily verify that the mappings $A \to \nu_A$ and $\nu \to A_\nu$ are inverses of each other completing the construction.

4 Existence of an Agreement

In this section we demonstrate that there is always an agreement with respect to two coherent functions. We also produce an easily checkable necessary and sufficient condition for non-emptiness of the agreements.

Theorem 1 *Assume the choice functions f_1 and f_2 are coherent, then there exists an agreement with respect to them.*

Proof: The proof is constructive. We shall first describe the iterative process that will lead to constructing an agreement. Define function g for any $X \subseteq \Omega$ as:

$$g(X) = \ (X - f_1(X)) \cup f_2(f_1(X)).$$

Let $Y_0 = \Omega$ and for any $i \geq 0$ let $Y_{i+1} = g(Y_i)$. Since the choice functions f_i are contractions, so is g and, since Ω is finite, there is some i such that $Y_{n+1} = Y_n$. Let $Y = Y_n$ and $Z = f_1(Y)$. We shall show that Z is an agreement.

Lemma 1 *For any i: $f_2(f_1(Y_i)) \subseteq f_1(Y_{i+1})$.*

Proof: By definition of Y_{i+1}, $f_2(f_1(Y_i)) \subseteq Y_{i+1}$. Since f_2 is a contraction, $f_2(f_1(Y_i)) \subseteq f_1(Y_i)$. Therefore $f_2(f_1(Y_i)) \subseteq Y_{i+1} \cap f_1(Y_i)$. But by $S\alpha$, since $Y_{i+1} \subseteq Y_i$, we have $Y_{i+1} \cap f_1(Y_i) \subseteq f_1(Y_{i+1})$. ▮

Lemma 2 *For any $x \in \Omega$ and any i such that $x \notin f_2(f_1(Y_i) \cup \{x\})$,
we have $x \notin f_2(f_1(Y_{i+1}) \cup \{x\})$.*

Proof: By Lemma 1, $f_2(f_1(Y_i)) \cup \{x\} \subseteq f_1(Y_{i+1}) \cup \{x\}$. Therefore, by $S\alpha$:

$$\begin{aligned}(f_2(f_1(Y_i)) \cup \{x\}) \cap f_2(f_1(Y_{i+1}) \cup \{x\}) &\subseteq f_2(f_2(f_1(Y_i)) \cup \{x\}) \\ &= (\textit{by Path Independence})\ f_2(f_1(Y_i) \cup \{x\}).\end{aligned}$$

From which it immediately follows that if $x \in f_2(f_1(Y_{i+1}) \cup \{x\})$ then $x \in f_2(f_1(Y_i)) \cup \{x\}$. ▮

Lemma 3 *If $x \in Y_i - Y_{i+1}$, then $x \notin f_2(Z \cup \{x\})$.*

Proof: Since $x \in Y_i$, but $x \notin Y_{i+1}$, by definition of Y_{i+1} it must be the case that $x \in f_1(Y_i)$, but $x \notin f_2(f_1(Y_i))$. Therefore $x \notin f_2(f_1(Y_i) \cup \{x\})$. Using Lemma 2 we obtain by induction that for any $j \geq i$, $x \notin f_2(f_1(Y_j) \cup \{x\})$, and thus $x \notin f_2(Z \cup \{x\})$. ▮

Lemma 4 *For any $x \in \Omega - Z$, $x \notin f_1(Z \cup \{x\}) \cap f_2(Z \cup \{x\})$.*

Proof: If $x \in Y$, then $f_1(Y) \subseteq Z \cup \{x\} \subseteq Y$ and, by Cumulativity of f_1, $f_1(Z \cup \{x\}) = f_1(Y) = Z$, so $x \notin f_1(Z \cup \{x\})$.

If $x \notin Y$, there is some i such that $x \in Y_i - Y_{i+1}$. In this case by Lemma 3, $x \notin f_2(Z \cup \{x\})$. ▮

Lemma 5 $f_1(Z) = f_2(Z) = Z$

Proof: $f_1(Z) = Z$ is immediate. Now, $Y_{n+1} = Y_n = Y$. So,

$$\begin{aligned} Y = Y_{n+1} &= (Y_n - f_1(Y_n)) \cup f_2(f_1(Y_n)) \\ &= (Y - f_1(Y)) \cup f_2(f_1(Y)) \\ &= (Y - Z) \cup f_2(Z). \end{aligned}$$

Since $Z \subseteq Y$ it follows that $Z \subseteq f_2(Z)$, but $f_2(Z) \subseteq Z$, therefore $f_2(Z) = Z$. ▮

Theorem 1 then immediately obtains from Lemmas 4 and 5 and the definition of an agreement. ▮

Note that the running time of the algorithm is polynomial in $|\Omega|$ times the complexity of calls to f_i. Thus if we have an oracle that computes f_i, the whole algorithm is polynomial in $|\Omega|$.

Corollary 1 *A non-empty agreement exists if and only if $\exists x \in \Omega$ such that $f_1(\{x\}) = f_2(\{x\}) = \{x\}$.*

Proof: Suppose such an x exists. Then $f_1(\emptyset \cup \{x\}) \cap f_2(\emptyset \cup \{x\}) = \{x\} \not\subseteq \emptyset$. So $\emptyset$ is not an agreement. By Theorem 1 at least one agreement exists, so it cannot be empty.

Suppose there is no such x and there exists a non-empty agreement A. Let $y \in A$. By Sα we have: $\{y\} \cap f_1(A) \subseteq f_1(\{y\})$, since $f_1(A) = A$, it must be the case that $y \in f_1(\{y\})$, and therefore $f_1(\{y\}) = \{y\}$. Similarly, $f_2(\{y\}) = \{y\}$. Contradiction. ▮

Corollary 2 *If the empty set is an agreement, then it is the only one.*

Proof: Suppose a non-empty agreement exists. By Corollary 1 $\exists x \in \Omega$ such that $f_1(\{x\}) = f_2(\{x\}) = \{x\}$. But then, following the first part of the proof of Corollary 1, $\emptyset$ is not an agreement. Contradiction. ▮

Due to Claim 4 the result of Theorem 1, Corollary 1, and Corollary 2 immediately carry to 2-SAPs charachterized by f_1 and f_2.

5 The Set of Agreements: Its Structure

In this section we study the structure of the set of stable agreements. We demonstrate that it is a lattice with respect to the preferences revealed by the underlying choice functions.

5.1 Definitions

In Section 2 we considered certain classes of preferences and demonstrated how they can be represented by choice functions. We may take a slightly different approach: In the algorithm in Section 4 the players were only allowed to choose subsets of a given set of contracts according to their choice functions. Such a restriction affects the implicit player preferences as the relative ranking of any two given subsets of contracts now depends only on what may be "extracted" from them via the choice functions. The following definition makes precise this intuitive notion:

Definition 9 *Given a choice function $f : 2^\Omega \to 2^\Omega$, we say that a preference relation $\succeq_f$ on subsets of Ω is* **revealed** *by f if:*

$$C \succeq_f D \ \Leftrightarrow \ f(C \cup D) = C$$

Note that in general $\succeq_f$ is not anti-symmetric and therefore does not belong to the class of preference relations we considered in Section 2. In this section, however, we are going to restrict our attention to revealed preferences over fixed points of coherent functions, on which $\succeq_f$ induces a partial ordering (see Claim 5). Therefore we allow ourselves to write $\succeq_f$ instead of more technically correct $\succsim_f$.

Since the relation *revealed* by choice function f (Definition 9) and preference relations *for which* f is the choice function (Claim 3) are not the same thing, one should be careful not to confuse them. In order to avoid that confusion note that until this section we have only dealt with the latter. From now on we will deal only with the former (i.e. revealed preferences). The only exception is Claim 6 below.

In what follows fixed points of coherent functions play a prominent role. Therefore we make explicit the notion of the set of fixed points of a coherent function:

Definition 10 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. We denote by $\Phi_f \subseteq 2^\Omega$ the set of all fixed points of f: $\Phi_f = \{A \in 2^\Omega |\ f(A) = A\}$. In the context of two-sided matching we denote by Φ_1 and Φ_2 the sets of fixed points of f_1 and f_2 respectively.*

Claim 5 *It is easy to verify that $\succeq_f$ restricted to Φ_f is reflexive, antisymmetric, and transitive, and therefore is a partial ordering on Φ_f.*

The following claim provides some further justification for Definition 9:

Claim 6 *Let $\succeq_{\mathrm{f}}$ be the preference relation defined in Claim 3. Let $\succeq_f$ be the preference relation revealed by f as defined in Definition 9. Then $\succeq_{\mathrm{f}}$ and $\succeq_f$ coincide on Φ_f.*

Proof: Let $X, Y \in \Phi_f$. Suppose $X \succeq_f Y$. Then $f(X \cup Y) = f(X) = X$. Meaning that there exists a set $Z = X \cup Y$ such that $f(Z) = X$ and $Y \subseteq Z$. By definition $X \succeq_{\mathrm{f}} Y$.

Suppose now that $X \succeq_{\mathrm{f}} Y$. Then there exists a set Z such that $f(Z) = X$ and $Y \subseteq Z$. Therefore $X \cup Y \subseteq Z$. We have: $f(Z) \subseteq X \cup Y \subseteq Z$. By Cumulativity $f(X \cup Y) = f(Z) = X$, so $X \succeq_f Y$. ∎

See Appendix C for some other properties of $\succeq_f$.

Notation: *Let $f : 2^{\Omega} \to 2^{\Omega}$ be a coherent choice function. In the context of two-sided matching we denote by $\succeq_1$ and $\succeq_2$ the preference relations revealed by f_1 and f_2 respectively.*

Our goal is to prove that $\succeq_{f_1}$ (and $\succeq_{f_2}$) induce a lattice structure on the set of all agreements between f_1 and f_2 under natural meet and join. The following definitions formalize that notion:

Definition 11 *Let $f_1, f_2 : 2^{\Omega} \to 2^{\Omega}$ be coherent choice functions. We denote by $\bar{\Phi}_{f_1,f_2}$ the set of all agreements between f_1 and f_2. When the identities of f_1 and f_2 are clear from the context we will omit the subscript and just write $\bar{\Phi}$ instead of $\bar{\Phi}_{f_1,f_2}$.*

Claim 7 *Since $\bar{\Phi} \subseteq \Phi_1$ and $\bar{\Phi} \subseteq \Phi_2$ it follows from Claim 5 that $\succeq_1$ and $\succeq_2$ are partial orderings on $\bar{\Phi}$.*

Definition 12 *Let Φ_f be the set of fixed points of a coherent choice function f, and let $\Psi \subseteq \Phi_f$. We define the join $X \vee_{f,\Psi} Y$ of two elements of Ψ in the usual way as their least upper bound in Ψ:*

$C = X \vee_{f,\Psi} Y$ iff $C \in \Psi, C \succeq_f X, C \succeq_f Y$, and for any $D \in \Psi$ such that $D \succeq_f X, D \succeq_f Y$, we have $D \succeq_f C$.

Analogously we define the meet $X \wedge_{f,\Psi} Y$ as the greatest lower bound of X and Y in Ψ:

$C = X \wedge_{f,\Psi} Y$ iff $C \in \Psi, X \succeq_f C, Y \succeq_f C$, and for any $D \in \Psi$ such that $X \succeq_f D, Y \succeq_f D$, we have $C \succeq_f D$.

If the join and the meet exist for a pair of sets of contracts X and Y then, due to anti-symmetry of $\succeq_f$, they are unique. If the join $X \vee_{f,\Psi} Y$ and the meet $X \wedge_{f,\Psi} Y$ exist for any two elements X and Y of Ψ, then $(\Psi, \vee_{f,\Psi}, \wedge_{f,\Psi})$ is called a *lattice.*

5.2 The Best and the Worst Agreements

In this section we show that the stable agreement Z obtained by the iterative process described in Chapter 4 is the best stable agreement for side 1, and worst for side 2, i.e. $\forall W \in \bar{\Phi} : Z \succeq_1 W$ and $W \succeq_2 Z$.

Lemma 6 *Assume W is any stable agreement. If $W \subseteq Y_i$ (see Chapter 4 for notation), then*

1. $f_1(W \cup f_1(Y_i)) = f_1(Y_i)$,
2. $f_2(W \cup f_1(Y_i)) = W$, *and*
3. $W \subseteq Y_{i+1}$.

Proof:

1. Since $f_1(Y_i) \subseteq W \cup f_1(Y_i) \subseteq Y_i$, by Cumulativity, we have $f_1(W \cup f_1(Y_i)) = f_1(Y_i)$.
2. Since W is a stable agreement, we know that
$$f_1(W \cup f_1(Y_i)) \cap f_2(W \cup f_1(Y_i)) \subseteq W$$
Therefore $f_1(Y_i) \cap f_2(W \cup f_1(Y_i)) \subseteq W$. But, by Contraction,
$$f_2(W \cup f_1(Y_i)) - f_1(Y_i) \subseteq W$$
and therefore we may conclude that $f_2(W \cup f_1(Y_i)) \subseteq W$. We have:
$$f_2(W \cup f_1(Y_i)) \subseteq W \subseteq W \cup f_1(Y_i)$$
and, by Cumulativity:
$$f_2(W \cup f_1(Y_i)) = f_2(W) = W$$
3. $f_1(Y_i) \subseteq W \cup f_1(Y_i)$ and, by Sα,
$$f_1(Y_i) \cap f_2(W \cup f_1(Y_i)) \subseteq f_2(f_1(Y_i))$$
Therefore:
$$f_1(Y_i) \cap W \subseteq f_2(f_1(Y_i)) \subseteq Y_{i+1}$$
But $Y_i - f_1(Y_i) \subseteq Y_{i+1}$ and $W \subseteq Y_i$, therefore $W - f_1(Y_i) \subseteq Y_{i+1}$. We conclude that $W \subseteq Y_{i+1}$.

■

Corollary 3 *If W is a stable agreement, then $W \subseteq Y_n$.*

Proof: By Lemma 6, since $W \subseteq Y_0 = \Omega$. ■

Theorem 2 *If W is a stable agreement, then $f_1(W \cup Z) = Z$ and $f_2(W \cup Z) = W$.*

Proof: By Corollary 3 and Lemma 6 we have $f_1(W \cup f_1(Y_n)) = f_1(Y_n)$ and $f_2(W \cup f_1(Y_n)) = W$. But $Z = f_1(Y_n)$. ■

5.3 Lattice Structure

In this section we prove the following theorem:

Theorem 3 *Let $\bar{\Phi}$ be the set of all agreements in the context of two-sided matching. Then $(\bar{\Phi}, \vee_{f_1,\bar{\Phi}}, \wedge_{f_1,\bar{\Phi}})$ and $(\bar{\Phi}, \vee_{f_2,\bar{\Phi}}, \wedge_{f_2,\bar{\Phi}})$ are lattices.*

In order to prove the theorem we will need some additional constructions:

Definition 13 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. We define "wish list" function $\tilde{f} : 2^\Omega \to 2^\Omega$ as $\tilde{f}(A) = \{y \in \Omega |\ y \in f(A \cup \{y\})\}$.*

See Appendix C for some properties of $\tilde{f}()$.

Given two agreements F and G the natural candidate for their meet with respect to $\succeq_1$ is the set $H_0 = f_1(F \cup G)$. Unfortunately, in general case H_0 is not necessarily an agreement. A counter example may be found in [6].

In Lemma 7 we proceed by building a non-decreasing sequence $H_0, H_1, ...$ of upper bounds of F and G with respect to $\succeq_1$. We prove that the sequence converges to a fixed point $H = F \vee_{f1,\bar{\Phi}} G$.

Although the results of Lemma 7 are new for the general model we consider, they closely follow the results obtained by [6] for a more restricted model (that applies both to the statement of the lemma as well as to some ideas in the proof).

Certain limitation of the model considered by [6] allow for a straightforward proof that the fixed point is an agreement. In our case, however, in order to prove that H is an agreement we need to do some additional work. Part of that work is done in proving Lemma 15 and the rest is done in Theorem 3.

Lemma 7 *Let $f_1, f_2 : 2^\Omega \to 2^\Omega$ be coherent choice functions. Let $F, G \in \bar{\Phi}_{f_1,f_2}$. Consider the sequence $\{H_i\}_{i\in\mathbb{N}}$ defined recursively as follows:*

$$H_0 = f_1(F \cup G)$$

$$H_{i+1} = f_1(f_2(\tilde{f}_1(H_i)))$$

Then $\forall i \in \mathbb{N}$:

1. $H_i \succeq_1 F$, $H_i \succeq_1 G$
2. $\forall E \in \bar{\Phi} : (E \succeq_1 F$ *and* $E \succeq_1 G) \Rightarrow E \succeq_1 H_i$
3. $H_i \subseteq f_2(\tilde{f}_1(H_i))$
4. $H_i \subseteq \Phi_1 \cap \Phi_2$

Proof: Induction on i.

[Induction basis: i = 0]

1. $f_1(H_0 \cup F) = f_1(f_1(F \cup G) \cup F) =$ (by Path Independence) $f_1((F \cup G) \cup F) = f_1(F \cup G) = H_0$. Therefore $H_0 \succeq_1 F$. Analogously, $H_0 \succeq_1 G$.

2. Suppose E is an agreement, $E \succeq_1 F$, $E \succeq_1 G$ (i.e. $f_1(E \cup F) = f_1(E \cup G) = E$). We have: $f_1(E \cup H_0) = f_1(E \cup f_1(F \cup G)) =$ (by Path Independence) $f_1(E \cup (F \cup G)) = f_1((E \cup F) \cup (E \cup G)) =$ (by Path Independence)
$f_1(f_1(E \cup F) \cup f_1(E \cup G)) = f_1(f_1(E)) = f_1(E)$. So, $E \succeq_1 H_0$.

3. Since $H_0 \succeq_1 F$, by Claim 10, $\tilde{f}_1(H_0) \subseteq \tilde{f}_1(F)$. Applying $S\alpha$:

$$\tilde{f}_1(H_0) \cap f_2(\tilde{f}_1(F)) \subseteq f_2(\tilde{f}_1(H_0))$$

Since F is an agreement, $f_2(\tilde{f}_1(F)) = F$ by Lemma 15. Therefore:

$$\tilde{f}_1(H_0) \cap F \subseteq f_2(\tilde{f}_1(H_0))$$

Since $H_0 = f_1(H_0)$, by Claim 11, $H_0 \subseteq \tilde{f}_1(H_0)$. Therefore:

$$\tilde{f}_1(H_0) \cap F \subseteq f_2(\tilde{f}_1(H_0)) \Rightarrow H_0 \cap F \subseteq f_2(\tilde{f}_1(H_0))$$

Analogously, $H_0 \cap G \subseteq f_2(\tilde{f}_1(H_0))$. Therefore $H_0 \cap (F \cup G) \subseteq f_2(\tilde{f}_1(H_0))$. However, $H_0 = f_1(F \cup G) \subseteq (F \cup G)$, and we obtain:

$$H_0 = H_0 \cap (F \cup G) \subseteq f_2(\tilde{f}_1(H_0))$$

4. $H_0 \in \Phi_1$ is immediate. Now, by Claim 11 $H_0 \subseteq \tilde{f}_1(H_0)$. Applying $S\alpha$:

$$H_0 \cap f_2(\tilde{f}_1(H_0)) \subseteq f_2(H_0)$$

Combining that with $H_0 \subseteq f_2(\tilde{f}_1(H_0))$ (as we have shown), we obtain $H_0 \subseteq f_2(H_0)$. However, $f_2(H_0) \subseteq H_0$. Therefore $H_0 = f_2(H_0)$ and thus $H_0 \in \Phi_1 \cap \Phi_2$.

[Induction step]

1. $f_1(H_{i+1} \cup H_i) = f_1(f_1(f_2(\tilde{f}_1(H_i))) \cup H_i) =$ (by Path Independence) $f_1(f_2(\tilde{f}_1(H_i)) \cup H_i)$. Since by the induction hypothesis $H_i \subseteq f_2(\tilde{f}_1(H_i))$:

$$f_1(f_2(\tilde{f}_1(H_i)) \cup H_i) = f_1(f_2(\tilde{f}_1(H_i))) = H_{i+1}$$

Therefore $H_{i+1} \succeq_1 H_i$. Since by the induction hypothesis $H_i \succeq_1 F$ and $H_i \succeq_1 G$, we obtain $H_{i+1} \succeq_1 F$ and $H_{i+1} \succeq_1 G$.

2. Suppose E is an agreement, $E \succeq_1 F$, $E \succeq_1 G$. By the induction hypothesis $E \succeq_1 H_i$. Therefore, by Claims 11 and 10 $E \subseteq \tilde{f}_1(E) \subseteq \tilde{f}_1(H_i)$. Since E is an agreement, we have:

$$f_2(E \cup f_2(\tilde{f}_1(H_i))) \cap f_1(E \cup f_2(\tilde{f}_1(H_i))) \subseteq E \tag{1}$$

Now, $f_2(E \cup f_2(\tilde{f}_1(H_i))) =$ (by Path Independence) $f_2(E \cup \tilde{f}_1(H_i))$. Since $E \subseteq \tilde{f}_1(H_i)$:

$$f_2(E \cup \tilde{f}_1(H_i)) = f_2(\tilde{f}_1(H_i)) \supseteq f_1(f_2(\tilde{f}_1(H_i))) = H_{i+1}$$

So,

$$H_{i+1} \subseteq f_2(E \cup f_2(\tilde{f}_1(H_i))) \tag{2}$$

On the other hand,

(3) $f_1(E \cup f_2(\tilde{f}_1(H_i))) = (by\ Path\ Independence)\ f_1(E \cup f_1(f_2(\tilde{f}_1(H_i)))) = f_1(E \cup H_{i+1})$

So we have:

$$f_1(E \cup H_{i+1}) = f_1(E \cup f_2(\tilde{f}_1(H_i))) \tag{4}$$

Combining 1,2, and 4 we obtain:

$$H_{i+1} \cap f_1(E \cup H_{i+1}) \subseteq E$$

Since $f_1(E \cup H_{i+1}) \subseteq E \cup H_{i+1}$, it must be the case that $f_1(E \cup H_{i+1}) \subseteq E$. So, $f_1(E \cup H_{i+1}) \subseteq E \subseteq E \cup H_{i+1}$, and by Cumulativity $f_1(E \cup H_{i+1}) = f_1(E)$, which is equivalent to $E \succeq_1 H_{i+1}$.

3. We have shown above that $H_{i+1} \succeq_1 H_i$. Therefore by Claim 10, $\tilde{f}_1(H_{i+1}) \subseteq \tilde{f}_1(H_i)$. Applying S$\alpha$:

$$\tilde{f}_1(H_{i+1}) \cap f_2(\tilde{f}_1(H_i)) \subseteq f_2(\tilde{f}_1(H_{i+1}))$$

Now, by Claim 11, $H_{i+1} \subseteq \tilde{f}_1(H_{i+1})$. Also,

$$H_{i+1} = f_1(f_2(\tilde{f}_1(H_i))) \subseteq f_2(\tilde{f}_1(H_i)).$$

Therefore:

$$H_{i+1} \subseteq \tilde{f}_1(H_{i+1}) \cap f_2(\tilde{f}_1(H_i)) \subseteq f_2(\tilde{f}_1(H_{i+1}))$$

4. The proof of $H_{i+1} \subseteq \Phi_1 \cap \Phi_2$ based on $H_{i+1} \subseteq f_2(\tilde{f}_1(H_{i+1}))$ is the same as for H_0 above. ∎

Corollary 4 *$\forall i \in \mathbb{N}$: $H_{i+1} \succeq_1 H_i$.*

Proof: See the proof of Lemma 7 (induction step part 1). ∎

Corollary 5 *The sequence $\{H_i\}_{i \in \mathbb{N}}$ as defined in Lemma 7 converges to a fixed point H, i.e. $\exists i$ s.t. $H_i = H_{i+1} = H$.*

Proof: Since $\succeq_1$ induces a partial ordering on Φ_1, the sequence is drawn from Φ_1, non-decreasing in terms of the ordering (as follows from Corollary 4), and Φ_1 is finite, therefore the sequence must converge to a fixed point. ∎

We are now ready to prove Theorem 3.

Theorem 3 *Let $\bar{\Phi}$ be the set of all agreements in the context of two-sided matching. Then $(\bar{\Phi}, \vee_{f_1,\bar{\Phi}}, \wedge_{f_1,\bar{\Phi}})$ and $(\bar{\Phi}, \vee_{f_2,\bar{\Phi}}, \wedge_{f_2,\bar{\Phi}})$ are lattices.*

Proof: We prove the theorem for f_1. The proof for f_2 is symmetrical.
We need to demonstrate that for any pair of agreements $F, G \in \bar{\Phi}$ their join $F \vee_{f_1,\bar{\Phi}} G$ and meet $F \wedge_{f_1,\bar{\Phi}} G$ always exist.

We start with the join $F \vee_{f_1,\bar{\Phi}} G$. According to Definition 12 we need to show that:

$$\forall F, G \in \bar{\Phi}\ \exists H \in \bar{\Phi}\ s.t.$$

$$H \succeq_1 F, H \succeq_1 G \ and\ \forall E \in \bar{\Phi}((E \succeq_1 F\ and\ E \succeq_1 G) \Rightarrow E \succeq_1 H)$$

We will show that the fixed point H of the sequence $\{H_i\}_{i\in\mathbb{N}}$ as defined in Lemma 7 satisfies the required conditions.

Since H satisfies the conditions of Lemma 7 the only requirement left to be shown is that $H \in \bar{\Phi}$. By definition of $\{H_i\}_{i\in\mathbb{N}}$ and the fact that H is its fixed point we have: $H = f_1(f_2(\tilde{f}_1(H)))$. Suppose $y \in f_2(\tilde{f}_1(H))$. According to Lemma 7 $H \subseteq f_2(\tilde{f}_1(H))$. Therefore we have:

$$H \subseteq H \cup \{y\} \subseteq f_2(\tilde{f}_1(H))$$

Substituting for H we obtain:

$$f_1(f_2(\tilde{f}_1(H))) \subseteq H \cup \{y\} \subseteq f_2(\tilde{f}_1(H))$$

By Cumulativity:

$$H = f_1(f_2(\tilde{f}_1(H))) = f_1(H \cup \{y\})$$

Since $y \in f_2(\tilde{f}_1(H)) \subseteq \tilde{f}_1(H)$ we have $y \in f_1(H \cup \{y\}) = H$. So, $y \in H$. We have demonstrated that $y \in f_2(\tilde{f}_1(H)) \Rightarrow y \in H$, which is equivalent to $f_2(\tilde{f}_1(H)) \subseteq H$. Thus:

$$H \subseteq f_2(\tilde{f}_1(H)) \subseteq H \Rightarrow f_2(\tilde{f}_1(H)) = H$$

and by Lemma 15, $H \in \bar{\Phi}$. So we have: $H = F \vee_{f_1,\bar{\Phi}} G$.

Now, by the same process of Lemma 7, only with the roles of f_1 and f_2 reversed, we build the join $L = F \vee_{f_2,\bar{\Phi}} G$. But by Claim 15, $F \wedge_{f_1,\bar{\Phi}} G = L$, completing the proof. ∎

6 Conclusion and open questions

By abstracting from particulars of individual matching models we have demonstrated some properties essential to two-sided matching in general: the nature of stability in terms of Individual Rationality and Stability, polarization of interests between the two sides, and the lattice structure of the set of stable matchings.

The general consensus in the literature is that coherence (or the substitutability condition) seems to be necessary in order to assure existence of stable matchings. When that condition is removed the set of agreements may be empty (which is not the same as to say that there is an agreement, but it contains no contracts). An interesting practical example is provided in [21]: when doctors' choice of hospitals depends of choices of other doctors (for example, two doctors who are married want to work in the same geographical area), a stable matching between doctors and hospitals may fail to exist. A careful examination of that example when modeled as a two-sided matching problem shows that doctors' preferences are not coherent.

However, maybe there is another (perhaps strictly weaker) condition that will ensure existence of stable matchings? And if there is, does it preserve the lattice structure? The first challenge here is how exactly to formulate the problem in technical terms. One can always come up with specific preference setups where coherence does not hold, but the set of agreements is not empty. So in the strictest sense it is not a necessary condition. One possibility is to define natural extensions of preference setups to families of preference setups. Then show that without coherence there will always be a setup with no agreements in every such family. The techniques developed in this work may help in finding the right approach to this open question.

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A Classical Matching Models

In this section we formally define two matching models that we use as examples.

A.1 The Marriage Problem

Definition 14 *A **marriage market** $(M, W, (\succeq_i)_{i \in M \cup W})$ consists of*

- *A finite set of men M*
- *A finite set of women W that is disjoint from M*

- *For each man $m \in M$ his linear (reflexive, transitive, anti-symmetric, and total) preference ordering $\succeq_m$ on the set $W \cup \{\emptyset\}$*
- *For each woman $w \in W$ her linear preference ordering $\succeq_w$ on the set $M \cup \{\emptyset\}$*

Inclusion of the empty set into the set of preferences is used to capture the player's preference of staying single. For example, $\{w\} \succ_m \emptyset$ means that m prefers w to being single.

In order to make the formal definition of the marriage model and its translation into the 2-SAP model less cumbersome, we naturally translate the definitions of $\succeq_m$ and $\succeq_w$ to the sets of singletons over W and M respectively. That is, every $\succeq_m$ is translated to be over the set $\{x| \ x = \{w\} \ where \ w \in W\} \cup \{\emptyset\}$. Similarly, every $\succeq_w$ is translated to be over the set $\{x| \ x = \{m\} \ where \ m \in M\} \cup \{\emptyset\}$. We retain the same notation for the translations, since it is always clear which set we are talking about.

Definition 15 *A **matching** in the marriage market $(M, W, (\succeq_i)_{i \in M \cup W})$ is a function $\mu : \ M \cup W \to 2^{M \cup W}$ such that:*

1. *$|\mu(p)| \leq 1$*
2. *If $m \in M$ then $\mu(m) \subseteq W$*
3. *If $w \in W$ then $\mu(w) \subseteq M$*
4. *$q \in \mu(p) \Rightarrow p \in \mu(q)$*

The meaning of $\mu(x) = \{y\}$ is that x is matched to y. The meaning of $\mu(x) = \emptyset$ is that x remains single.

Definition 16 *We say that matching μ is **blocked by an individual** x if $\emptyset \succ_x \mu(x)$. If μ is not blocked by any individual we call it **individually rational**. We say that matching μ is **blocked by a pair** $(m \in M, w \in W)$ if $\{w\} \succ_m \mu(m)$ and $\{m\} \succ_w \mu(w)$.*

Definition 17 *A matching μ is called **stable** if it is not blocked by any individual or pair.*

A.1.1 Existence of a Stable Matching

A stable matching always exists. One can always be found by the iterative process described in this section.

Definition 18 *Let $\hat{W} \subseteq W$, $m \in M$. A* choice function *of m is a function $C_m : 2^W \rightarrow W \cup \{\emptyset\}$ such that:*

1. *Either $C_m(\hat{W}) \in \hat{W}$ or $C_m(\hat{W}) = \emptyset$*
2. *$\forall w \in \hat{W} \quad C_m(\hat{W}) \succeq_m w$.*

In other words, C_m chooses one of the most preferred for m members of $\hat{W}$. Clearly, if $\succeq_m$ is strict, then C_m is unique. Otherwise every C_m imposes some strict ordering on W that is consistent with $\succeq_m$. We can similarly define choice functions C_w for the women's preferences. In terms of choice functions the algorithm that achieves a stable matching can be described as follows:

```
menFirst()
begin
  for every m ∈ M do Ŵ_m := W; μ(m) := ∅ endfor;
  for every w ∈ W do M̂_w := ∅; μ(w) := ∅ endfor;
  while changes to μ occur do
    for every m ∈ M do
      μ(m) := setify(C_m(Ŵ_m));
      if μ(m) ≠ ∅ then M̂_w := M̂_w ∪ μ(m);
    endfor;
    for every w ∈ W do μ(w) := setify(C_w(M̂_w)) endfor;
    for every m ∈ M do
      if (μ(m) ≠ ∅) and ({m} ≠ μ(μ(m))) then
        Ŵ_m := Ŵ_m − μ(m);
      endif;
    endfor;
  endwhile;
return(μ);
end;


setify(u)
begin
  if u = ∅ then
```

return ($\emptyset$)
else
return ($\{u\}$)
endif;
end;

In plain words the algorithm can be described as follows: every man proposes to the most desirable woman on his list. Then every woman gives a tentative 'yes' to the most desirable man among those who have proposed to her, and a final 'no' to the others among those who have proposed. Then all rejected men propose to the next desirable woman on their lists, and so forth. The algorithm terminates when there are no more rejections. All tentative 'yes' answers that still stand become final.

Clearly an algorithm can be modified symmetrically such that women propose and men accept or reject. We'll call such an algorithm womenFirst.

Theorem 4 ***([12])*** *menFirst (womenFirst) algorithm always terminates and returns a stable matching.*

A.1.2 Duality and Structure of the Set of Stable Matchings

We can define an aggregate preference relation on the set of matchings that reflects if the set of men as a whole prefers one matching over another.

Definition 19 *Given two matchings μ and ν we say that $\mu \succeq_M \nu$ (or men weakly prefer μ over ν) if $\forall m \in M \quad \mu(m) \succeq_m \nu(m)$. Moreover, we say that $\mu \succ_M \nu$ (or men strictly prefer μ over ν) if $\mu \succeq_M \nu$ and for at least one $m \in M \quad \mu(m) \succ_m \nu(m)$. We similarly define an aggregate preference relation $\succeq_W$ on the set of matchings that reflects preferences of women as a whole.*

Theorem 5 ***Duality ([12])*** *Let μ and ν be two stable matchings. Then $\mu \succeq_M \nu$ if and only if $\nu \succeq_W \mu$. Moreover, $\mu \succ_M \nu$ if and only if $\nu \succ_W \mu$.*

Theorem 6 ***Best and Worst Matchings ([12])***
Suppose all preferences $\succeq_m$, $\succeq_w$ are strict. Then matching μ achieved by menFirst algorithm is the most preferred matching with respect to $\succeq_M$ and the least preferred matching with respect to $\succeq_W$. The symmetric result hods for $\succeq_W$ due to duality.

Theorem 7 ***Lattice Structure ([16] — attributed to John Conway)*** *Suppose all preferences $\succeq_m$, $\succeq_w$ are strict. Then the aggregate preference relation $\succeq_M$ imposes a* *lattice structure on the set of stable matchings. I.e. for any two stable matchings there is a stable matching that is the unique greatest lower bound* $(\mu \wedge \nu)$ *with respect to* $\succeq_M$ *called the* the meet *of* μ *and* ν *and there is a stable matching that is the unique least upper bound* $(\mu \vee \nu)$ *with respect to* $\succeq_M$ *called the* the join *of* μ *and* ν.

Moreover, the lattice is distributive, *meaning that the join and the meet have the following properties:*

$$(\mu \wedge \nu) \vee \lambda = (\mu \vee \lambda) \wedge (\nu \vee \lambda),$$

$$(\mu \vee \nu) \wedge \lambda = (\mu \wedge \lambda) \vee (\nu \wedge \lambda).$$

Moreover, due to duality $\succeq_W$ *imposes exactly the same structure on the set of stable matchings as* $\succeq_M$ *if we reverse the direction of preferences.*

A.2 Responsive Preferences

The one-to-one marriage problem can be extended to the more general one-to-many college admissions problem. Since agents on one of the sides (colleges) can utilize more than one resource on the other side (i.e. admit more than one student) we need to solve the problem of how to generalize preferences over individuals to preferences over subsets of individuals. For that purpose [23] introduces the idea of *responsive preferences*. In other aspects the definition of college admissions model closely follows that of the marriage model. We present here the case where preferences over individuals are strict.

Let $\succeq$ be a linear (i.e. total, reflexive, antisymmetric) preference relation on $S \cup \{\emptyset\}$ (think of S as the set of students; the empty set is introduced in order to specify the students that the college is not willing to admit at all). Let q be the *quota*, i.e. the maximum number of students a college is willing to admit. We first define an auxiliary relation $\succeq^r$ on subsets of S as follows:

Definition 20 *Let* $U, T \subseteq S$.

1. *If* $|U| > q$ *and* $|T| \leq q$ *then* $T \succeq^r U$.
2. *If* $T = U \cup s$ *and* $|T| \leq q$, *then* $T \succeq^r U$ *iff* $s \succeq \emptyset$.
3. *If* $T = U \cup s$ *and* $|T| \leq q$, *then* $U \succeq^r T$ *iff* $\emptyset \succeq s$.
4. *If* $T = Q \cup t$ *and* $U = Q \cup u$, *then* $T \succeq^r U$ *iff* $t \succeq u$.

[23] shows that the transitive closure of $\succeq^r$ does not contain cycles. So we can now extend $\succeq$ to preferences over subsets as follows:

Definition 21 *A linear preference relation* $\succeq^R$ *over subsets of* S *is called* responsive *to the preference relation* $\succeq$ *over elements in* $S \cup \{\emptyset\}$ *with quota* q *if* $\succeq^R$ *contains* $\succeq^r$ *as its subset (in other words* $\succeq^R$ *is consistent with* $\succeq^r$ *).*

Note that there may exist more than one preference relations that are responsive to a given preference relation over individuals.

Responsive preferences are coherent:

Claim 8 *Let* $\succeq^R$ *be responsive to the strict linear order* $\succeq$ *on* $S \cup \{\emptyset\}$ *with quota* q. *Set* $W = \{s \in S | \ s \succ \emptyset\}$. *For any* $X \subseteq S$ *define* $f(X)$ *to contain the* q *most preferred elements of* $X \cap W$ *if this set contains at least* q *elements and the whole of* $X \cap W$ *otherwise. Then* f *is the choice function for* $\succeq^R$ *and is coherent.*

Proof: The proof is left to the reader. ▮

A.3 One-to-many Job Matching

This is a more general two-sided matching model. There exist several variations on it in the literature. Our formalization follows [6] with some simplifications [16].

Definition 22 *A* ***job market***

$$M = (F, W, S, (L_f)_{f \in F}, (L_w)_{w \in W}, (\succ_f)_{f \in F}, (\succ_w)_{w \in W})$$

consists of

- *A finite set* F *of firms*
- *A finite set* W *of workers*
- *A finite set* S *of salaries (or "job descriptions")*
- *For every* $f \in F$ *a set* L_f *of* ***feasible*** *subsets of* $W \times S$ *that includes the empty set*
- *For every* $w \in W$ *a set* L_w *of* ***feasible*** *subsets of* $F \times S$ *that includes the empty set, satisfying the following condition: if* $A \in L_w$ *then* $|A| \leq 1$

[16] [6] considers a many-to-many model where firms and workers are completely symmetric (a worker may take employment with several firms). Our model generalizes that model as well. However, in order not to get bogged down in the technical details that shed little light on the underlying principles we simplify the model and do not allow a worker to be employed by more than one firm.

- *For every $p \in F \cup W$ a total strict preference ordering $\succ_p$ over L_p, such that $\emptyset$ is the minimal element of L_p according to $\succ_p$*

Notation: *For convenience we will often refer to the set $P = F \cup W$ of players, rather then to F and W separately.*

The interpretation of the feasible subsets is as follows: for a firm every subset represents a set of workers which the firm may be willing to hire in principle at the specified salary levels at the same time. Only those combinations are included that are preferred by the firm to not hiring any workers. Thus $\emptyset$ is the least preferred element of L_f as required by the definitions. The feasible subsets for workers have the same interpretation with the additional condition: a worker can't work for more than one firm, thus all feasible subsets for workers are either empty or singleton.

Definition 23 *Given set $A \subseteq P \times S$ and player $p \in P$, the **choice set** $C_p(A)$ is a subset of A feasible for p that p prefers the most. Since $L_p \cap 2^A$ is not empty (it always contains the empty set), finite, and the preference relation $\succ_p$ over L_p is total and strict, C_p always exists and is unique.*

Definition 24 Legality condition. *If $(i, s) \in C_j(A)$ and $(i, t) \in C_j(A)$, then $s = t$.*

Definition 25 Substitutability condition. *If $(j, s) \in C_i(A)$, then $C_i(A) - \{(j, s)\} \subseteq C_i(A - \{(j, s)\})$.*

When talking about One-to-many Job Matching model we assume that the job market M satisfies the legality[17] and substitutability conditions. Note that for the workers' choice functions the substitutability condition is satisfied trivially, since if $(j, s) \in C_w(A)$, then $C_w(A) - \{(j, s)\} = \emptyset$.

Definition 26 *Let M be a job market as described in Definition 22 and let $P = F \cup W$. A **matching** in M is a function $\nu : P \rightarrow 2^{P \times S}$ such that:*

1. *If $(i, s) \in \nu(j)$ then $(j, s) \in \nu(i)$*
2. *If $p \in P$ then $\nu(p) \in L_p$*

Intuitively if $f \in F$ then $(w, s) \in \nu(f)$ means that firm f hires worker w at salary s.

Definition 27 *We define matching ν to be stable iff for all $i, j \in P, s \in S$:*

[17] In [6] the legality condition is not mentioned explicitly. A careful reading, however, reveals that it is implied in the proof of existence of a stable matching

1. $C_i(\nu(i)) = \nu(i)$

2. *If* $(j,s) \in C_i(\nu(i) \cup \{(j,s)\})$, *then* $C_j(\nu(j) \cup \{(i,s)\}) = \nu(j)$.

The first of those conditions says that no player wants to discard any of his partners in ν. This condition is similar to the individual rationality condition in the Marriage Problem.

If $(j,s) \in \nu(i)$ the second condition follows immediately from the definition of a matching and the first condition. If $(j,s) \notin \nu(i)$ the second condition means that if i wants (j,s) then j does not want (i,s). This condition is similar to the non-blocking by pairs condition in the Marriage Problem.

B Some Properties of Coherent Functions

Lemma 8 *Let f be a choice function that satisfies*

- *(Local Monotonicity)* $\forall X, Y \subseteq \Omega$ *if* $f(X) \subseteq Y \subseteq X$, *then* $f(Y) \subseteq f(X)$
- *(Sα)* $\forall X, Y \subseteq \Omega$ *if* $X \subseteq Y$, *then* $X \cap f(Y) \subseteq f(X)$.

Then for any $X, Y \subseteq \Omega$, f satisfies:

- *(Cumulativity) if* $f(X) \subseteq Y \subseteq X$, *then* $f(Y) = f(X)$, *and*
- *(Idempotence)* $f(f(X)) = f(X)$.

Proof: Suppose $f(X) \subseteq Y \subseteq X$. By Local Monotonicity $f(Y) \subseteq f(X)$, and by Sα $f(X) = Y \cap f(X) \subseteq f(Y)$. So $f(X) = f(Y)$.

By Contraction $f(X) \subseteq f(X) \subseteq X$, so by Cumulativity: $f(f(X)) = f(X)$. ∎

Lemma 9 *If f is coherent, then* $f(X \cup Y) \subseteq f(X) \cup f(Y)$.

Proof: By Contraction: $f(X \cup Y) = (X \cap f(X \cup Y)) \cup (Y \cap f(X \cup Y))$. By S$\alpha$, $X \cap f(X \cup Y) \subseteq f(X)$ and $Y \cap f(X \cup Y) \subseteq f(Y)$. ∎

Lemma 10 (Plott [20]) *A function f is coherent iff it satisfies Contraction and, for any $X, Y \subseteq \Omega$:*

- *(Path Independence)* $f(X \cup Y) = f(f(X) \cup Y)$.

Lemma 11 (Aizerman and Malishevski [2]) *A function f is coherent iff there is a finite set of binary relations $>_i$ on Ω such that for any X, $f(X)$ is the set of all elements of X that are maximal in X for at least one of the $>_i$'s.*

Lemma 12 *If $\leq$ is a partial order, i.e., reflexive and transitive, on Ω, then function f defined for every $X \subseteq \Omega$ as:*

$$f(X) = \{x \in X \mid \forall y \in X, y \geq x \Rightarrow x \geq y\}$$

is coherent.

Proof: Note that $f(X)$ is the subset of maximal elements of X. Contraction is obvious.

Local Monotonicity: suppose $f(X) \subseteq Y \subseteq X$. Assume by way of contradiction that $y \in f(Y)$, but $y \notin f(X)$, meaning there is a maximal element y of Y that is not maximal in X. Due to the finiteness of Ω, since $y \in X$ there must be some maximal element x of X such that $x \geq y$. Since $x \in f(X)$ it follows that $x \in Y$, so $y \geq x$, since y is maximal in Y. Therefore y must be maximal in X as well. Contradiction.

Sα: suppose $X \subseteq Y$ and $x \in X \cap f(Y)$. Since x is maximal in Y, it is also maximal in any subset of Y in which it is contained. So, since it is contained in X, it is maximal in it, meaning that $x \in f(X)$. ∎

Typically, individual preferences are supposed to be linear orders and therefore Lemma 12 applies to them.

Assume $\Omega' \subseteq \Omega$. If f is a coherent choice function on Ω its restriction to Ω' is obviously a coherent choice function on Ω'. More interesting is the fact that there exists a natural way to extend a coherent choice function on Ω' to Ω.

Definition 28 *If $\Omega' \subseteq \Omega$ and f is a choice function on Ω', its* natural extension *f_Ω to Ω is defined by $f_\Omega(X) = f(X \cap \Omega') \cup (X - \Omega')$ for any $X \subseteq \Omega$.*

Lemma 13 *If $\Omega' \subseteq \Omega$, then a choice function f on Ω' is coherent iff its natural extension to Ω is coherent.*

Proof: Suppose f_Ω is coherent. Since f is the restriction of f_Ω to Ω' it is also coherent.

Suppose f is coherent. We need to show that f_Ω is coherent. Contraction is immediate.

Suppose $f_\Omega(X) \subseteq Y \subseteq X$. Then $f_\Omega(X) \cap \Omega' \subseteq Y \cap \Omega' \subseteq X \cap \Omega'$. Now, $f_\Omega(X) \cap \Omega' = f(X \cap \Omega')$. Therefore, by Cumulativity of f, $f(X \cap \Omega') = f(Y \cap \Omega')$. Therefore, $f_\Omega(Y) = (Y - \Omega') \cup f(Y \cap \Omega') \subseteq (X - \Omega') \cup f(X \cap \Omega') = f_\Omega(X)$. Thus f_Ω satisfies Cumulativity.

Suppose $X \subseteq Y$. Then $X \cap \Omega' \subseteq Y \cap \Omega'$. By coherence of f, $(X \cap \Omega') \cap f(Y \cap \Omega') \subseteq f(X \cap \Omega') \Rightarrow X \cap f(Y \cap \Omega') \subseteq f(X \cap \Omega')$. Taking the union of the last expression with

$X \cap (Y - \Omega') \subseteq (X - \Omega')$ we obtain $X \cap f_\Omega(Y) \subseteq f_\Omega(X)$, meaning that f_Ω satisfies Sα.
▮

Lemma 14 is essential in showing that aggregate preference relations in 2-SAPs corresponding to classical matching models are coherent.

Definition 29 *A collection $\{f_i\}_{i \in I}$ of choice functions is a* party*, i.e., a suitable collective for matching purposes, if the set Ω of contracts can be partitioned into disjoint subsets Ω_i, $i \in I$, in such a way that for any $i \in I$, f_i is the natural extension to Ω of its restriction to Ω_i, i.e., $f_i(X) = f_i(X \cap \Omega_i) \cup (X - \Omega_i)$.*

Given any collection of choice function, f_i for $i \in I$, one may define a collective choice function f_I in the following way:

$$f_I(X) = \bigcap_{i \in I} f_i(X). \tag{5}$$

Lemma 14 *Suppose a collection of coherent choice functions $\{f_i\}_{i \in I}$ is a party. Then the collective choice function f_I is also coherent.*

Proof: We first make the following simple observations:

$$f_i(X) \cap \Omega_i = f_i(X \cap \Omega_i)$$

$$if\ i{\neq}j,\ then\ f_i(X) \cap \Omega_j = X \cap \Omega_j$$

Therefore $f_I(X) \cap \Omega_i = (X \cap \Omega_i) \cap f_i(X \cap \Omega_i) = f_i(X \cap \Omega_i)$.

Cumulativity of f_I is trivial.

Suppose $f_I(X) \subseteq Y \subseteq X$. Then, for any $i \in I$, $f_I(X) \cap \Omega_i \subseteq Y \cap \Omega_i \subseteq X \cap \Omega_i$. Since $f_I(X) \cap \Omega_i = f_i(X \cap \Omega_i)$, by Cumulativity of f_i we obtain: $f_i(X \cap \Omega_i) = f_i(Y \cap \Omega_i)$. Therefore, using the fact that $\{\Omega_i\}_i \in I$ is a partition of Ω, we obtain:

$$f_I(X) = \bigcup_{i \in I}(f_I(X) \cap \Omega_i) = \bigcup_{i \in I} f_i(X \cap \Omega_i) = \bigcup_{i \in I} f_i(Y \cap \Omega_i) = \bigcup_{i \in I}(f_I(Y) \cap \Omega_i) = f_I(Y)$$

Therefore f_I satisfies Cumulativity.

Suppose $X \subseteq Y$. Then for any $i \in I$, $X \cap \Omega_i \subseteq Y \cap \Omega_i$. By coherence of f_i:

$$\begin{aligned}
(X\cap\Omega_i)\cap f_i(Y\cap\Omega_i)\subseteq f_i(X\cap\Omega_i) \quad &\Rightarrow \quad X\cap f_i(Y\cap\Omega_i)\subseteq f_i(X\cap\Omega_i)\\
&\Rightarrow \quad \bigcup_{i\in I}(X\cap f_i(Y\cap\Omega_i))\subseteq\bigcup_{i\in I} f_i(X\cap\Omega_i)\\
&\Rightarrow \quad X\cap\bigcup_{i\in I} f_i(Y\cap\Omega_i)\subseteq\bigcup_{i\in I} f_i(X\cap\Omega_i)\\
&\Rightarrow \quad X\cap\bigcup_{i\in I}(f_I(Y)\cap\Omega_i)\subseteq\bigcup_{i\in I}(f_I(X)\cap\Omega_i)\\
&\Rightarrow \quad X\cap f_I(Y)\subseteq f_I(X).
\end{aligned}$$

Meaning that f_I satisfies Sα. ▮

Notice that the intersection of coherent functions is not in general coherent. Local Monotonicity does not obtain in general: consider two contracts a and b, and two agents such that $f_1(X) = f_2(X) = X$ for every X different from $\Omega = \{a, b\}$, $f_1(\Omega) = \{a\}$ and $f_2(\Omega) = \{b\}$.

C Some Properties of $\succeq_f$ and $\tilde{f}$

In this section we demonstrate the properties of $\succeq_f$ and $\tilde{f}$ that we need to prove the main result.

Claim 9 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. Then* $y \in f(A\cup B) \Rightarrow y \in f(A\cup\{y\})$.

Proof:

$$\begin{aligned}
y\in f(A\cup B) \quad &\Rightarrow \quad y\in A\cup B\Rightarrow A\cup\{y\}\subseteq A\cup B\\
(\textit{by } S\alpha) \quad &\Rightarrow \quad (A\cup\{y\})\cap f(A\cup B)\subseteq f(A\cup\{y\}).
\end{aligned}$$

Since $y \in (A\cup\{y\})\cap f(A\cup B)$, we conclude that $y \in f(A\cup\{y\})$ ▮

Claim 10 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. Let $C\succeq_f D$. Then $\tilde{f}(C) \subseteq \tilde{f}(D)$.*

Proof:

$$\begin{array}{rcl} y \in \tilde{f}(C) & \Leftrightarrow & y \in f(C \cup \{y\}) \\ \text{(by Path Independence)} & \Rightarrow & y \in f(f(C) \cup \{y\}) \\ \text{(by Definition 5.1)} & \Rightarrow & y \in f(f(C \cup D) \cup \{y\}) \\ \text{(by Path Independence)} & \Rightarrow & y \in f((C \cup D) \cup \{y\}) \\ & \Rightarrow & y \in f(C \cup (D \cup \{y\})) \\ \text{(by Claim 9)} & \Rightarrow & y \in f(D \cup \{y\}). \end{array}$$

Which is equivalent to $y \in \tilde{f}(D)$ ▮

Claim 11 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. Let $H \in \Phi_f$. Then $H \subseteq \tilde{f}(H)$.*

Proof: Immediately follows from $H = f(H)$ and the definition of $\tilde{f}$. ▮

Claim 12 *Let $f : 2^\Omega \to 2^\Omega$ be a coherent choice function. Let $H \in \Phi_f$, $C \in 2^\Omega$, $H \succeq_f C$. Then $H \subseteq \tilde{f}(C)$.*

Proof: Immediately follows from Claim 11 and Claim 10 ▮

Claim 13 *Suppose A is an agreement and $X \subseteq \Omega$. Then $f_1(A \cup X) \cap f_2(A \cup X) \subseteq A$.*

Proof: Suppose there exists $y \in \Omega$ such that $y \in f_1(A \cup X) \cap f_2(A \cup X)$, but $y \notin A$. Then by Claim 9, $y \in f_1(A \cup \{y\})$ and $y \in f_2(A \cup \{y\}$ and so

$$f_1(A \cup \{y\}) \cap f_2(A \cup \{y\} \nsubseteq A,$$

meaning that A is not an agreement. Contradiction. ▮

Claim 14 *Let $A, B \in \bar{\Phi}$ be agreements. Then $A \succeq_1 B \Leftrightarrow B \succeq_2 A$.*

Proof: Suppose $f_1(A \cup B) = A$. Since B is an agreement, by Claim 14,

$$f_1(A \cup B) \cap f_2(A \cup B) \subseteq B.$$

Therefore

$$A \cap f_2(A \cup B) \subseteq B \Rightarrow f_2(A \cup B) \subseteq B \subseteq A \cup B.$$

By Cumulativity we obtain: $f_2(A \cup B) = f_2(B) = B$. Thus $A \succeq_1 B \Rightarrow B \succeq_2 A$. The proof in the opposite direction is symmetrical. ▮

Claim 15 *Let $A, B \in \bar{\Phi}$ be agreements, then:*

$A \vee_{f_1,\bar{\Phi}} B$ exists if and only if $A \wedge_{f_2,\bar{\Phi}} B$ exists. In such a case $A \vee_{f_1,\bar{\Phi}} B = A \wedge_{f_2,\bar{\Phi}} B$.

Symmetrically, $A \vee_{f_2,\bar{\Phi}} B$ exists if and only if $A \wedge_{f_1,\bar{\Phi}} B$ exists. In such a case $A \vee_{f_2,\bar{\Phi}} B = A \wedge_{f_1,\bar{\Phi}} B$.

Proof: Follows directly from Definition 12 and Claim 14. ∎

Lemma 15 *Let $f_1, f_2 : 2^\Omega \to 2^\Omega$ be coherent choice functions. Then $(F \in \Phi_1$ and $F = f_2(\tilde{f}_1(F))) \Leftrightarrow F \in \bar{\Phi}$.*

Proof: ($\Leftarrow$) Suppose F is an agreement. It immediately follows from the definition of an agreement that $F = f_1(F)$ which is the same as $F \in \Phi_1$.

Suppose $y \in \tilde{f}_1(F)$ which is equivalent to $y \in f_1(F \cup \{y\})$. Since F is an agreement we have:

$$f_1(F \cup \{y\}) \cap f_2(F \cup \{y\}) \subseteq F$$

Therefore:

$$f_2(F \cup \{y\}) \subseteq F \subseteq F \cup \{y\}$$

and by Cumulativity:

$$f_2(F \cup \{y\}) = f_2(F) = F$$

Now, by Claim 11, $F \subseteq \tilde{f}_1(F)$, so:

$$F \cup \{y\} \subseteq \tilde{f}_1(F)$$

applying $S\alpha$:

$$(F \cup \{y\}) \cap f_2(\tilde{f}_1(F)) \subseteq f_2(F \cup \{y\}) = F$$

Thus we have $y \in \tilde{f}_1(F) \Rightarrow (F \cup \{y\}) \cap f_2(\tilde{f}_1(F)) \subseteq F$, and therefore:

$$(F \cup \tilde{f}_1(F)) \cap f_2(\tilde{f}_1(F)) \subseteq F$$

Since $F \subseteq \tilde{f}_1(F)$ and $f_2(\tilde{f}_1(F)) \subseteq \tilde{f}_1(F)$ the above reduces to $f_2(\tilde{f}_1(F)) \subseteq F$. So, $f_2(\tilde{f}_1(F)) \subseteq F \subseteq \tilde{f}_1(F)$, and by Cumulativity $f_2(\tilde{f}_1(F)) = f_2(F) = F$.

($\Rightarrow$) Suppose $F \in \Phi_1$ and $F = f_2(\tilde{f}_1(F))$. By Claim 11, $F \subseteq \tilde{f}_1(F)$. Thus

$$f_2(\tilde{f}_1(F)) \subseteq F \subseteq \tilde{f}_1(F)$$

and by Cumulativity it follows that $F = f_2(F)$.

Suppose $y \in \tilde{f}_1(F)$ which is equivalent to $y \in f_1(F \cup \{y\})$. Then:

$$f_2(\tilde{f}_1(F)) \subseteq F \cup \{y\} \subseteq \tilde{f}_1(F)$$

By Cumulativity: $f_2(F \cup \{y\}) = f_2(\tilde{f}_1(F)) = F$. Thus

$$y \in f_1(F \cup \{y\}) \Rightarrow f_2(F \cup \{y\}) = F$$

Which means that

$$\forall y \in \Omega : f_1(F \cup \{y\}) \cap f_2(F \cup \{y\}) \subseteq F$$

Meaning F satisfies all the requirements of the agreement definition. ∎