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RATIONALIZABILITY IN CONTINUOUS GAMES

by

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Rationalizability in Continuous Games

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Abstract

Define a *continuous game* to be one in which every player's strategy set is a Polish space, and the payoff function of each player is bounded and continuous. We prove that in this class of games the process of sequentially eliminating "never-best-reply" strategies does not terminate after the first uncountable ordinal, and that this bound is tight. Also, we examine the connection between this process and common belief of rationality in the universal type space of Mertens and Zamir [11].

1 Introduction

A *never-best-reply* (NBR) strategy of a player in a game is one that is not a best reply to any distribution of strategies of the other players. In seminal papers, Pearce [12] and Bernheim [3] called a strategy *rationalizable* if it survives repeated elimination of NBR strategies. As Pearce put it: "For each game, rationalizability distinguishes those strategies that players could employ without violating the implications of the common knowledge they possess, from those that are patently unreasonable."

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This work explores rationalizability in *continuous* games, i.e., games in which the strategy set of each player is a Polish space, and the payoff function is bounded and continuous. Such games have been investigated since the dawn of game theory; see, e.g., Ville [14]. We shall see that in those games, the NBR elimination process does not necessarily terminate after the "first infinity;" a longer elimination process may be needed. That is, after ω rounds of eliminating NBR strategies, it may be that the remaining strategy set does not have the *self-best-reply* property, and more rounds of elimination are needed. So the elimination process continues to ordinals greater than the first infinite ordinal.

Lipman [10] also relates to this kind of phenomenon, and demonstrates that mutual knowledge of rationality of a finite but arbitrarily high order provides a poor approximation for the common knowledge assumption. Our first goal is to characterize how large the gap is. That is, we seek a bound for the number of rounds needed in order to get the rationalizable set. In Theorem 1 we show that the length of the NBR elimination process cannot exceed the first uncountable ordinal ω_1. Moreover, this bound is tight; in Subsection 3.2 we exhibit a continuous game in which exactly ω_1 rounds of NBR elimination are needed to get the rationalizable set.

Our second aim is to explore the connection between the NBR elimination process and mutual belief of rationality. We adopt the approach of Tan and Werlang [13] and transform such non-cooperative games into Bayesian games with the universal type space of Mertens and Zamir [11] as their underlying belief space. In doing so, we provide an epistemic characterization for the NBR elimination process. In particular, we show that for natural numbers k, a strategy s_i of player i survives k rounds of NBR elimination iff there exists a type for which s_i is a best reply and mutual belief of rationality of order k obtains in that type. Moreover, s_i is rationalizable iff there exists a type for which s_i is a best reply and common belief of rationality obtains in that type.

In two-player finite strategic games a strategy is a best reply with respect to a probability distribution over the other player's strategy set iff it is not strongly dominated by any mixture of strategies. In our case, however, only one direction is true; that is, there exists an NBR strategy that is not strongly dominated (see Example A.1). Therefore, every strategy that survives α rounds of NBR elimination survives also α rounds of strong domination elimination, but not vice versa.

The assumption of a Polish strategy space provides a natural generalization of the compact metric strategy space assumption. Much of the recent work concerning incomplete information games adopts this framework (e.g., [5]). The difficulty that arises in this generalized setup, in both the complete and the incomplete information case, is that the set of strategies surviving k rounds of NBR elimination need not be a Borel measurable (Example 3.1). The implication of our results to the incomplete information setup will be discussed in Section 5.2.

The paper proceeds as follows. In Section 2 we begin the formal treatment and formally present the motivation for Theorem 1. Section 3 is devoted to the proof of Theorem 1 and some related results. In Section 4 we introduce the concept of belief system and provide an epistemic framework for the probabilistic NBR elimination process. In Section 5 we generalize some of our results and sketch an example of a game with a non-Polish strategy space and continuous payoff function, in which Theorem 1 does not hold. In Appendix A we adduce an example of a continuous game for which an undominated strategy is not a best reply.

2 Preliminaries

Definition 2.1. A *continuous* game Γ is a triple $(N, (S_i)_{i\in N}, (U_i)_{i\in N})$, where N is a finite set of players and for every $i \in N$, S_i is a Polish space[1] and $U_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is bounded and continuous.

We start by assuming that N contains only two players, and deal first

[1] A Polish space is a separable, completely metrizable topological space.

with a non-probabilistic framework.[2] Let $\Gamma = (\{1,2\}, (S_i)_{i=1,2}, (U_i)_{i=1,2})$ be a two-player continuous game, where S_i are the strategy sets of player i.[3]

Definition 2.2. For $s_j \in S_j$, let $\mathrm{br}_i(s_j) = \{s_i \in S_i : s_i \in \mathrm{argsup}_{S_i} U_i(\cdot, s_j)\}$.

That is, $\mathrm{br}_i(s_j)$ is the set of strategies in S_i that are best replies with respect to s_j. Obviously, $\mathrm{br}_i(s_j)$ can be empty. We call a strategy of player i, s_i, *rational* if it is a best reply with respect to some strategy of player j, s_j. That is, s_i is rational if $\exists s_j \in S_j$ s.t. $s_i \in \mathrm{br}_i(s_j)$; otherwise we call s_i a *never-best-reply* (NBR) strategy. Consider the following example:

Example 2.1. Assume $S_1 = S_2 = \omega + 2 = \mathbb{N} \cup \{\beta_1, \beta_2\}$ is equipped with the discrete topology. Each player i gets a payoff of 1 if he chooses a number strictly greater than the one j chose; otherwise he gets 0. This game is obviously a continuous game. Consider the process of simultaneous elimination of NBR strategies (henceforth NBR elimination). After n rounds of NBR elimination we remain with the set $S_i \setminus \{1, 2, \ldots, n\}$ for each player i. After iterated NBR elimination (i.e., ω rounds of NBR elimination) we remain with the set $S_i = (\{\beta_1, \beta_2\})$, for each player i. But the set $\{\beta_1, \beta_2\} \times \{\beta_1, \beta_2\}$ is not a "fixed point" set, in the sense that it still contains NBR strategies, namely, β_1 for each player. Therefore, one more round of NBR elimination is needed to reach $S_i = \{\beta_2\}$, which is the desired fixed point set. This example demonstrate, a case where $\omega + 1$ rounds of iterated elimination is needed to reach the *rationalizable* set.

Note that the strategy β_1 is consistent with mutual knowledge of rationality of any finite order, but is inconsistent with common knowledge of rationality. A similar kind of example may be constructed for every countable ordinal α.

Definition 2.3. Define by induction the NBR elimination process as follows:

- $S_i^1 = S_i$,

[2]For the general case see Sections 4 and 5.1.

[3]Henceforth, we assume that $i \neq j$ and i is either player 1 or 2.

- $S_i^2 = \{s_i : \exists s_j \text{ s.t. } s_i \in \mathrm{br}_i(s_j)\ \}$.
- For successor ordinal α, define $S_i^\alpha = \{\ s_i : \exists s_j \in S_j^{\alpha-1} \text{ s.t. } s_i \in \mathrm{br}_i(U_i, s_j)\ \}$,
- for limit ordinal η, define $S_i^\eta = \cap_{\alpha<\eta} S_i^\alpha$.

The assumption in the core of this process argues that if player i is rational, he will play a strategy that is a best reply with respect to some strategy of player j, i.e., a strategy in S_i^2. If, in addition, player i assumes that the other player is rational, his chosen strategy should be a best reply with respect to a rational strategy of player j, i.e., a strategy from S_i^3, and so on. As we saw, unlike the case where the S_i's are compact or finite (see [13]), in this case the elimination process can go beyond the first infinite ordinal ω.

For every continuous game Γ there exists a least ordinal α such that $S_i^\alpha = S_i^{\alpha+1} = S_i^\infty$ for $i = 1, 2$.[4] We denote this minimal ordinal by $|\Gamma|$ and call it *the rank of* Γ. Each $s_i \in S_i^\infty$ is called *rationalizable*. Note that the sets of rationalizable strategies $(S_i^\infty)_{i=1,2}$ have the best-reply property. I.e., for every $i = 1, 2$ and $j \neq i$, each $s_i \in S_i^\infty$ is a best reply with respect to some $s_j \in S_i^\infty$. Our first goal is to give an ordinal that bounds the NBR elimination process for continuous games. Theorem 1 argues that $|\Gamma|$ is bounded from above by the first uncountable ordinal, ω_1:

Theorem 1. For every continuous game Γ, $|\Gamma| \leq \omega_1$.

Without assuming any topological constraints on the strategy sets and the payoff functions of the players, one can deduce only that the NBR process would terminate at an ordinal with cardinality not exceeding the power of the strategy set. So if the strategy set of each player is Polish uncountable, then its power is a continuum ([9], Corollary (6.5)) and we can bound the length of the NBR process with the first ordinal with cardinality greater than the continuum. Theorem 1, however, provides a better bound for our

[4] S_i^∞ might be empty.

particular case. Moreover, we show that this bound is tight; i.e., there exists a continuous game Γ, for which $|\Gamma| = \omega_1$.

The next section is devoted to the proof of Theorem 1 and its consequences.

3 Proof of Theorem 1

Define the best-reply relation over the direct sum of S_1 and S_2 ($S_1 \bigoplus S_2$) as follows: $y \prec x$ if and only if $x \in S_i$, $y \in S_j$, $i \neq j$, and $x \in \mathrm{br}_i(y)$. One can identify $\prec$ with a subset of the space $(S_1 \bigoplus S_2)^2$. We begin by proving the following lemma:

Lemma 2. $\prec$ defines a closed subset in $Z = (S_1 \bigoplus S_2)^2$.

Proof. Let $\{(x^n, y^n)\}_{n=1}^{\infty}$ be a convergence sequence in Z and let (x, y) be its limit. Assume that for each $n \in \mathbb{N}$, $y^n \prec x^n$. We have to show that $y \prec x$. Without loss of generality we can assume that for every $n > N$, $x^n \in S_i$ and $y^n \in S_j$. So $x \in S_i$ and $y \in S_j$ and we claim that $x \in \mathrm{br}_i(y)$. To see this, note that by definition, $\forall t \in S_i$, $U_i(t, y^n) \leq U_i(x^n, y^n)$, and so by the continuity of U_i, $\forall t \in S_i$ $U_i(t, y) \leq U_i(x, y)$. $\square$

For $n \geq 1$, define $T_n = (S_i \times S_j)^n$ for even-numbered values of n, and $T_n = (S_i \times S_j)^{n-1} \times S_i$ for odd-numbered values of n. That is, $T_1 = S_i$ $T_n = T_{n-1} \times S_j$ for even n, and $T_n = T_{n-1} \times S_i$ for odd n. Define $T = \bigoplus_{n=1}^{\infty} T_n$ (note that T is Polish space as a countable direct sum of Polish spaces).

Definition 3.1. Define inductively a relation $\lhd$ over T as follows: for $r \in T_1$ $t \in T_2$, $t \lhd r$ iff $\mathrm{proj}_{T_1} t = r$ and $\mathrm{proj}_{S_j} t \prec r$. That is, if $t = (t_1, t_2)$, then $t_1 = r$ and $t_2 \prec r$. For general $r, t \in T$, $t \lhd r$ iff $r = (r_1, \ldots r_n) \in T_n$, $t = (t_1, \ldots, t_n, t_{n+1}) \in T_{n+1}$, $t_i = r_i$ $\forall i \leq n$ and $t_{n+1} \prec \ldots \prec t_1$.

Note that $\lhd$ defines a closed relation in $(T)^2$. To see this, let $r^m, t^m \in T$ such that $\forall m$, $t^m \lhd r^m$, and $(r^m, t^m) \to (r, t)$. We can deduce that $r^m \in T_n$ and $t^m \in T_{n+1}$ for every $n > N$. By definition $t_{n+1}^m \prec \ldots \prec t_1^m$ and

$\text{proj}_{T_n} t^m = r^m$ for all m. Since $\prec$ is closed and the projection function is continuous, we have that $t_{n+1} \prec \ldots \prec t_1$ and $\text{proj}_{T_n} t = r$, as desired.

Definition 3.2. Let F be a set and $<$ a binary relation on F. We define $<$ to be *well-founded* if there is no infinite descending chain $\cdots < f_2 < f_1 < f_0$. Otherwise, we call $<$ *ill-founded*.

For every $s \in S_i$, let $T(s) = \{r \in T : \text{proj}_{T_1} r = s\}$, and so $T(s)$ corresponds to the closed subset of T comprised of the elements $t \in T_n$ whose first coordinate is s. We can view $\lhd$ as defining a tree over $T(s)$, where the root of the tree is (s) and for $r, s \in T(s)$, r is a son of s iff for some n, $s \in T_n$ $r \in T_{n+1}$ $s \subset r$ and $r_{n+1} \prec \ldots \prec r_2 \prec r_1 = s$.

Obviously $T(s)$ is a closed set of T. The following lemma determines whether $s \in S_i^\infty$. Denote by $\lhd^s$ the reduction of $\lhd$ to $T(s)$.

Lemma 3. $\lhd^s$ is ill-founded iff $s \in S_i^\infty$.

Proof. ($\Leftarrow$): Assume that $s \in S_i^\infty$. It follows that there exists $s_1 \in S_j^\infty$ such that $s_1 \prec s$. Since also $s_1 \in S_j^\infty$, there exists $s_2 \in S_i^\infty$ such that, $s_2 \prec s^1$. Thus we can construct an infinite chain $\cdots s_3 \prec s_2 \prec s_1 \prec s = s_0$. This chain induces a descending chain in $T(s)$ with respect to $\lhd^s$, i.e., $\ldots (s, s_1, s_2) \lhd^s (s, s_1) \lhd^s s$.

($\Rightarrow$): Assume there exists a descending chain in $T(s)$ $\ldots \lhd^s s^2 \lhd s^1$ such that, $s^1 \in T_n$ and $s^k \in T_{n+k-1}$. Let $s^1 = (s_0, \ldots, s_{n-1})$ since $s^1 \in T(s)$, $s_0 = s$. From $s_1 \lhd^s s_2$ we have $\text{proj}_{T_n} s^2 = s^1$. Therefore, $s^2 = (s_0, \ldots, s_{n-1}, s_n)$ and $s_n \prec s_{n-1} \prec \ldots \prec s_0 = s$. Inductively, we construct two sets $\tilde{S}_i = \{s_0, s_2, s_4, \ldots\} \subseteq S_i$, $\tilde{S}_j = \{s_1, s_3, s_5, \ldots\} \subseteq S_j$ such that $\ldots \prec s_2 \prec s_1 \prec s_0 = s$. Using a simple transfinite induction one can show that $\tilde{S}_i \subseteq S_i^\alpha$, $\tilde{S}_j \subseteq S_j^\alpha$ for every ordinal α and, in particular, $s \in S_i^\infty$. □

Given a well-founded binary relation $<$ over a set F, we define the rank function $\rho_< : F \to ORD$ as follows:
For minimal f, $\rho_<(f) = 0$ recursively,

$$\rho_<(f) = sup\{\rho_<(g) + 1 : g < f\}.$$

$\rho_<$ maps F onto some ordinal η denoted by $\rho(<)$, which we define as the *rank* of the relation $<$.

For $s_i \in S_i \setminus S_i^\infty$, denote by $|s_i|$ the unique ordinal α such that $s_i \in S_i^\alpha \setminus S_i^{\alpha+1}$. We have the following correspondence between the rank of $\lhd^s$ and $|s|$:

Proposition 4. For $s \notin S_i^\infty$, we have $|s| = \rho(\lhd^s)$.

Proof. First note the following properties of S_i^η:

a. For successor ordinal η, $s \in S_i^\eta$ iff $\exists s_j \in S_j^{\eta-1}$ such that $s_j \prec s$,

b. For limit ordinal η, $s \in S_i^\eta$ iff for every $\beta < \eta$ $\exists s_j \in S_j^\beta$ such that, $s_j \prec s$.

We prove the result using transfinite induction; for s such that $|s| = 0$ the result is immediate. Assume the theorem is obtained for every α such that $\alpha < \eta$, where η is any ordinal, and assume $|s| = \eta$. Given the definition of a rank, we obtain

$$\rho(\lhd^s) = sup\{\rho_{\lhd^s}((s,r)) + 1 : r \prec s\} = sup\{\rho_{\lhd^r} + 1 : r \prec s\}.$$

The second equality is due to the natural embedding of $(T(r), \lhd^r)$ in $(T((s,r)), \lhd^s)$. The theorem obtains from properties a, b, and the induction hypothesis. $\square$

Proof of Theorem 1. Let $s \in S_i$ be such that $s \notin S_i^\infty$. $T(s)$ is Polish, and from Lemma 4 $\lhd^s$ is a closed relation over $T(s)$ (and, in particular an analytic relation; see Definition 3.3). By Lemma 3 $\lhd^s$ well-founded, so we can use the fact that the rank of any well-founded analytic relation is countable (see [9], Theorem 31.1) and deduce that $|s| = \rho(\lhd^s) < \omega_1$. Therefore,

$$|\Gamma| = sup\{\ |s| \mid s \notin S_i^\infty\ \} \leq \omega_1.$$

$\square$

3.1 Characterization of The Sets S_i^α

In this section we characterize the sets S_i^α and demonstrate that they may not be Borel sets.

Definition 3.3. For Polish space X, a set $A \in X$ is called *analytic* if there exists a Polish space Y, and a continuous function $f : Y \to X$ such that $f(Y) = A$.

The class of analytic sets for an uncountable Polish space is strictly inclusive of all Borel sets ([9] Theorem 14.2) and closed under continuous projection and countable unions or intersections.

Example 3.1. Let X be an uncountable Polish space and let $A \subseteq X$ be an analytic set that is not Borel. Let Y be Polish and $f : Y \to X$ be continuous such that $f(Y) = A$. Then the set $graf(f) = \{(x, y) : x = f(y)\}$ is closed in $X \times Y$. Let Γ be a game where $S_1 = X$, $S_2 = Y$ and let d be any compatible metric on $X \times Y$ such that $d((x, y), (x', y')) \leq 1 \ \forall (x, y), (x', y') \in X \times Y$. Define a continuous payoff function

$$U_1(x, y) = -d((x, y), graf(f)).$$

We argue that $S_i^1 = A$; to see this note that for every $x \notin A$ and every $y \in Y$, $U_1(x, y) = d((x, y), graf(f)) < 0$, because $graf(f)$ is a closed set. For every $y \in Y$ there exists $x \in X$ such that $U_1(x, y) = 0$ (namely, $x = f(y)$). Therefore, x is a best reply with respect to some y iff $x \in A$ and $y = f(x)$.

Proposition 5. S_i^α are analytic sets for every ordinal α.

Proof. The set $R_i = \{(s_i, s_j) \in S_i \times S_j : s_i \in \mathrm{br}_i(s_j)\}$ is closed; therefore $S_i^1 = \mathrm{proj}_{S_i^1} R$ is an analytic set as a projection of a closed set. We proceed inductively: $S_i^{\alpha+1} = \mathrm{proj}_{S_i} R_i \cap (S_i \times S_j^\alpha)$ and $R_i \cap S_i \times S_j^\alpha$ is an analytic set as an intersection of analytic sets and so $S_i^{\alpha+1}$ is analytic as a projection of an analytic set. For countable limit ordinal α, the result is immediate since a countable intersection of analytic sets is analytic. So we just have to prove that $S_i^\infty = S_i^{\omega_1}$ is analytic. As a consequence of Lemma 3 we know that

$s \in S_i^\infty$ iff there exists a sequence $\{s^n\}_{n=0}^\infty$ such that $s^0 = s$, $s^n \in S_i$ for n even, $S^n \in S_j$ for n odd, and $\ldots \prec s^1 \prec s^0$. The set of all such sequences can be viewed as a closed subset M of $(S_i \times S_j)^{\mathbb{N}}$, and S_i^∞ is the projection of M to the first coordinate and hence analytic. $\square$

We call a space *analytic* if it is homeomorphic to an analytic subset of a Polish space. Theorem 1 remains valid also for the case where the strategy set of each player is an analytic space. To see this, assume that S_i is analytic for $i = 1, 2$, and let $s_i \notin S_i^\infty$; then the space $T(s)$ is an analytic space and $\lhd^s \subseteq [T(s)]^2$ is a closed relation in an analytic space hence analytic well-founded relation. We can use again the fact ([9] Theorem 31.1) that the rank of a well-founded analytic relation is countable in order to deduce that $|s| < \omega_1$.

3.2 Tightness of Theorem 1

The question that remains unanswered is whether equality can be obtained in Theorem 1. That is, does there exist a continuous game Γ for which $|\Gamma| = \omega_1$? In this subsection we'll answer this question in the affirmative and demonstrate the existence of a two-player game Γ for which $|\Gamma| = \omega_1$.

Let $\Gamma = (\{1, 2\}, (S_i)_{i=1,2}, (U_i)_{i=1,2})$, where $S_1 = S_2 = TO \times \mathcal{N}$. $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is the Baire space of all functions $f : \mathbb{N} \to \mathbb{N}$ and $TO \subset 2^{\mathbb{N}^2}$ is the set of total orderings over N we identify each element of $2^{\mathbb{N}^2}$ with a binary relation on $\mathbb{N}$, and TO represents those relations that are total orderings. The topology on $\mathcal{N}$ is the usual product topology, where $\mathbb{N}$ is equipped with the discrete topology and the topology on TO is induced from $2^{\mathbb{N}^2}$. TO is a closed set in $2^{\mathbb{N}^2}$, and therefore the S_i's are Polish spaces.

Let $F \subseteq (TO \times \mathcal{N})^2$ be the following set:

$$\{((x, f), (y, g)) : \forall m, n \;\; m <_y n \Rightarrow f(m) <_x f(n) \text{ and } \forall n f(n) <_x 1\},$$

where $m <_x n$ stands for $x(m, n) = 1$ and $x(n, m) = 0$. That is, $((x, f), (y, g)) \in F$ iff the relation y is embedded via f in the preface of all the elements in N

that are smaller than 1 according to x. We argue that F is a closed set. To see this, note that for each pair m, n the set

$$F_{m,n} = \{((x, f), (y, g)) : m <_y n \Rightarrow f(m) <_x f(n)\}$$

is closed. And the set $H = \{((x, f), (y, g)) : \forall n f(n) <_x 1\}$, as a countable intersection of a closed set, is closed. But $F = \cap_{m,n} F_{m,n} \cap H$ and is, therefore, a closed set.

Let d be any complete bounded metric over $S_1 \times S_2$, define the payoff function for each player i as follows:

$$U_i((x, f), (y, g)) = -d(((x, f), (y, g)), F),$$

where $(x, f) \in S_i$ and $(y, g) \in S_j$. Denote by WO, the subset of all total well-orderings on $\mathbb{N}$. For every $x \in WO$, let $1(x) = \{m : m <_x 1\}$ be the set of elements that are smaller than 1 according to x. For every countable or finite ordinal α, let D_α be the following set:

$$D_\alpha = \{x \in WO : \text{ [the rank of } x \text{ reduced to the set } 1(x)] \geq \omega + \alpha\}.$$

To prove that $|\Gamma| = \omega_1$ we prove the following proposition:

Proposition 6.

1. For every $n \in \mathbb{N}$ and $x \in WO$, $x \in \text{proj}_{TO} S_i^{n+2}$ iff $x \in D_n$.
2. For every countable ordinal α, $x \in \text{proj}_{TO} S_i^{\alpha}$ iff $x \in D_{\alpha+1}$.

Proof. We prove part 1 first.
$n = 0$:
$(\Leftarrow)$: Let $x \in WO$, and assume that $x \in D_0$, so that the rank of the reduction of x to $1(x)$ is at least ω. Therefore, if, for example, $y =\leq$ is the natural ordering on $\mathbb{N}$, then there exists a function $f : \mathbb{N} \to 1(x)$ such that $m < n \Rightarrow f(m) <_x f(n)$ and obviously $f(m) <_x 1 \ \forall m$. So $((x, f), (y, g)) \in F \ \forall g \in \mathcal{N}$ and, therefore, (x, f) is a best reply with respect to (y, g) for any $g \in \mathcal{N}$.

$(\Rightarrow)$: Assume that $1(x)$ is finite. If by contradiction $(x, f) \in S_i^2$ for some f, then by definition there exists (y, g) such that

$$d(((x, f), (y, g)), F) = \min_{(z,h)\in TO\times\mathcal{N}} d(((z, h), (y, g)), F). \tag{3.1}$$

It is not hard to see that $((z', h'), (y, g)) \in F$ for some (z', h'); hence the right-hand side of 3.1 is 0. Since F is a closed set, $((x, f), (y, g)) \in F$. By definition of F, we deduce that the range of f is $1(x)$ and for every m, n, $m <_y n$ iff $f(m) <_x f(n)$. But we assumed $1(x)$ is finite so there exists m, n such that $f(m) = f(n)$ since x is a total order, and this is obviously a contradiction.

Proceed inductively; assume 1 is true for $n - 1$, we prove it for n:

$(\Leftarrow)$: Let $x \in WO$, such that $x \in D_n$. By definition, the rank of the reduction of x to $1(x)$ is at least $\omega + n$. Take $y \in D_{n-1} \setminus D_n$, with 1 as a maximal element, i.e., $1(y) = \mathbb{N} \setminus \{1\}$. Then again there exists $f : \mathbb{N} \to 1(x)$ such that $m < n \Rightarrow f(m) <_x f(n)$. So $((x, f), (y, g)) \in F$ $\forall g \in \mathcal{N}$ and, therefore, (x, f) is a best reply with respect to (y, g) for any $g \in \mathcal{N}$. By the induction hypothesis, $\exists g$ s.t. $(y, g) \in S_i^n$, and so $(x, f) \in S^{n+1}$.

$(\Rightarrow)$: Let $x \in D_{n-1} \setminus D_n$, and assume by contradiction that $x \in S_i^{n+2}$, again using 3.1 $(x, f, y, g) \in F$ for some $(y, g) \in S_i^{n+1}$. y is determined via f since $\forall m, n$ $m <_y n \Leftrightarrow f(m) <_x f(n)$. In particular, we have that $y \in WO$. By the induction hypothesis, $y \in D_{n-1}$, and so the rank of y is at least $\omega + n$, a contradiction since y is embedded via f in a well-ordered set, $1(x)$, with a rank of $\omega + n - 1$.

We assume part 1 and prove part 2 for the case $\alpha = \omega$.

$(\Leftarrow)$: Let $x \in D_{\omega+1}$, and choose $y \in WO$ such that 1 is y-maximal and the rank of y reduced to A_y is exactly $\omega + \omega$. Choose a function $f : \mathbb{N} \to 1(x)$ such that $(x, f, y, g) \in F$ $\forall g \in \mathbb{N}$; such an f exists since the rank of y over $\mathbb{N}$ is $\omega + \omega + 1$, which is not higher than the rank of x over $1(x)$. From the finite case $\forall n$ $y \in proj_{TO} S_i^n$, so $\forall n \exists g_n$ s.t. $(y, g_n) \in S_n^i$, in particular $U_i((x, f), (y, g_n)) = 0$ $\forall n$, and so $(x, f) \in S_i^{\omega}$.

$(\Rightarrow)$: We show that if $x \in D_\omega \setminus D_{\omega+1}$, then $x \notin \mathrm{proj}_{TO} S_i^{\omega}$. If by contradiction there exists f such that $(x, f) \in S_i^{\omega}$, then for every $n \in \mathbb{N}$

there exists (y_n, g_n) such that $((x, f), (y_n, g_n)) \in F$ and $(y_n, g_n) \in S_i^n$. But since $((x, f), (y_n, g_n)) \in F$, the relation y_n is uniquely determined by f, i.e., $m <_{y_n} n \Leftrightarrow f(m) <_x f(n)$, and so $\forall n\ y_n = y$. The relation y is embedded via f to the reduction of x to $1(x)$, therefore y is well-ordered with a rank less than or equal to $\omega + \omega$. I.e., the rank of y reduced to $1(y)$ is strictly less then $\omega + \omega$, e.g., $\omega + k$ so $y \in D_k \setminus D_{k+1}$. But from the finite case we conclude that $y \notin \mathrm{proj}_{TO} S_i^{k+3}$, a contradiction. The proof of 2 readily follows by using transfinite induction and applying the same kind of arguments that were used in the proof of part 1 and in the $\alpha = \omega$ case. $\square$

For every ordinal $\alpha < \omega_1$, we can find a relation x such that the rank of x reduced to $1(x)$ is $\omega + \alpha$. Using proposition 6 we deduce that for every countable ordinal α there exists $x \in WO$ such that $x \in \mathrm{proj}_{2^{\mathbb{N}^2}} S_i^\alpha \setminus \mathrm{proj}_{TO} S_i^{\alpha+1}$; therefore, $S_i^\alpha \subsetneq S_i^{\alpha+1}$. This proves that $|\Gamma| \geq \omega_1$ and together with Theorem 1 we get the desired equality.

4 Knowledge, Belief, and Rationality

In this section, we adopt the approach of Tan and Werlang [13] and transform a noncooperative game into a Bayesian decision problem. The uncertainty faced by the player is the strategy choice of the other players. We assume that every player forms a subjective probability distribution over the strategy set of the other players, and explore the connection between the NBR elimination process that obtain for this case and mutual belief of rationality in the universal type space.

In establishing the connection between an n-order NBR elimination and mutual belief of rationality of order n, one faces the following mathematical difficulty: Suppose s_i is a best reply with respect to an assessment[5] $\mu \in \Delta(S_j)$ that gives probability 1 to "j plays a rational strategy." The question is whether we can embed this belief into the universal type space and find a type that has a marginal distribution on S_j equal to μ and, in addition,

[5] $\Delta(S_j)$ is the set of probability distributions over S_j.

ascribes probability 1 that j plays rationally. Friedenberg [7] faced a problem of extending probability measures from lower to higher belief-hierarchies, subject to similar constraints. She, however, dealt with this problem by using results of extending probability measures to a larger σ-algebra. Here we adopt a different approach and use uniformization functions.

The following definition is equivalent to Definition 5.1 in [13].

Definition 4.1. Given a continuous game $\Gamma = (S_1, S_2, U_1, U_2)$, define the probabilistic NBR elimination process as follows:

- $S_i(1) = S_i$.
- For successor ordinal α, $s_i \in S^i_\alpha$ iff $\exists \mu \in \Delta(S_j^{\alpha-1})$ such that $s_i \in \mathrm{br}_i(\mu)$,[6]
- for limit ordinal α, $S_i(\alpha) = \cap_{\beta<\alpha} S_i(\beta)$.

In [13] Tan and Werlang assume that the strategy set of each player is metric compact and in that case $S_i(\infty) = S_i(\omega)$.

As in the non-probabilistic case, denote by $|\Gamma|$ the first ordinal α such that, $S_1(\alpha) = S_1(\alpha + 1)$ and $S_2(\alpha) = S_2(\alpha + 1)$; also let $S_i(\infty) = S_i(|\Gamma|)$.

4.1 Rationality

Definition 4.2. Given a two-player continuous game $\Gamma = (S_1, S_2, U_1, U_2)$, a type space for Γ is a tuple $\mathrm{T} = (T_1, T_2, g_1, g_2)$. T_i is a Polish space and $g_i : T_i \to \Delta(S_j \times T_j)$ is a continuous function, for $i = 1, 2$ and $j \neq i$ where $\Delta(S_j \times T_j)$ is equipped with the weak* topology.

A type space is called *universal* if for every i, g_i is a homeomorphism. Using techniques introduced in Brandenburger and Dekel [4] (henceforth BD), and generalizing [11], it can be seen that for continuous games a universal type space may always be constructed.

For a given game Γ and type space T, we denote $\Omega_i = S_i \times T_i$, and define the set of states of the world as $\Omega = \Omega_1 \times \Omega_2$. That is, every state of the world

[6] Where in this case $s_i \in \mathrm{br}_i(\mu)$ iff $s_i \in \mathrm{argsup}_{s' \in S_i} \int_{S_j} U_i(s', s_j) d\mu(s_j)$.

$\omega \in \Omega$ includes a strategy and an epistemic type for each player. For every type t_i, let $h_i(t_i) = \mathrm{marg}_{S_j} g_i(t_i)$, we say that player i is rational at state of the world ω if $s_i \in \mathrm{br}_i(h_i(t_i))$. That is, player i is rational if the strategy he plays maximizes his payoff with respect to his belief. The event corresponds to rationality of player i, denoted R_1^i, can be considered a subset of Ω_i.

For $n \geq 2$, define inductively the event corresponds to rationality and mutual belief of rationality of order n, $R_1^n \times R_2^n \subseteq \Omega$, as follows:

$$R_i^n = \{\ (s_i, t_i) \mid (s_i, t_i) \in R_{n-1}^i \text{ and } g_i(t_i)(R_{n-1}^j) = 1\}.$$

Lemma 7. *R_i^n is a closed subset of Ω_i for every $n \geq 1$.*

Proof. For $n = 1$ the proof is similar to Lemma 2. Assume the lemma is true for $n-1 \geq 1$. Therefore, by definition, $R_n^i = R_{n-1}^i \cap \{\ (s_i, t_i) \mid g_i(t_i)(R_j^{n-1}) = 1\}$. Using the portmantau theorem for a closed set $M \subseteq (S^j \times T^j)$, the set

$$B_i(M) = \{\mu \mid \mu \in \Delta(S_j \times T_j) \text{ s.t. } \mu(B) = 1\ \},$$

is closed. Therefore, using the continuity of g_i, we get that $R_i^m = R_i^{m-1} \cap g_i^{-1}(B_i(R_j^{m-1}))$ is closed. $\square$

Parts 1 and 2 of the following theorem establish the connection between the NBR elimination process described in Definition 4.1 and the mutual belief reduction. Part 3 is the generalization of Theorem 1.

Theorem 8. *For a continuous game Γ and the appropriate universal type space T, we have that for every player i,*

1. $\mathrm{proj}_{S_i} R_n^i = S_i(n+1)$ *for every n.*

2. $R_\infty^i = \mathrm{proj}_{S_i} \cap_{n \geq 1} R_n^i = S_i(\infty)$.

3. $S_i(\infty) = S_i(\omega_1)$.

Theorem 8 reveals a gap between mutual belief of rationality of a finite order and common belief of rationality. For finite $n \in \mathbb{N}$ we have a natural correspondence between strategies that survive n rounds of NBR elimination and mutual belief of rationality of order n. In the common belief case we "skip" through all the ordinals from ω to ω_1 up to the fixed point set S_i^∞. The reason for this gap (as identified also in [10]) is the fixed point property of an events that are common beliefs. That is, if in state ω E is a common belief (or common knowledge), then everybody believes in ω that E is a common belief. So if in ω rationality is a common belief, then everybody believe in ω that rationality is common belief. This circularity explains the gap.

The game introduced in Section 3.2 has a rank of ω_1 also with respect to the probabilistic NBR elimination process introduced in 4.1. The proof is omitted since it is close to the one given in Section 3.2.

Proof of Part 1 of Theorem 8. For a Polish space X denote by $\Sigma_1^1(X)$ the class of all analytic subsets of X. Note that from Lemma 7, $\mathrm{proj}_{S_i} R_n^i$ are analytic for every n.

We prove part 1 using induction on n. For $n = 1$, let $s_i \in R_1^i$, and so by definition, s_i is a best reply w.r.t. $h_i(t_i) \in \Delta(S_i^j)$ for some type t_i and so, $s_i \in S(2)_i$. For the other direction, let $s_i \in \mathrm{br}_i(\mu)$. Using universality we can find a type t_i such that $h_i(t_i) = \mu$, and so $(s_i, t_i) \in R_1^i$. Assume that part 1 holds for $n - 1$, the first direction obtains directly from the induction hypothesis similar to the $n = 1$ case.

For the other direction we need the following lemma:

Lemma 9. Let X, Y be a Polish space, $A \subseteq X \times Y$ a Borel subset, and $\mu \in \Delta(X)$ such that $\mu(proj_X A) = 1$. There exists a probability measure ν such that $\nu(A) = 1$ and $\mu = proj_X \nu$.

Proof of Lemma 9. $\mathrm{proj}_X A$ is an analytic set and therefore universally measurable by [9] (Theorem 21.10).[7] By [1] (Theorem 18.22) every Borel set $A \subseteq X \times Y$ admits a $\sigma(\Sigma_1^1(X))$-uniformizing function, where $\sigma(\Sigma_1^1(X))$ is the sigma-algebra containing the Borel sigma-algebra, generated by the analytic sets in X. That is, there exists a $\sigma(\Sigma_1^1(X))$-measurable function

[7] A set $B \subseteq X$ is universally measurable if B is in the completion of every probability measure $\mu \in \Delta(X)$.

$f : proj_X A \to Y$ such that $graf(f) \subseteq A$. By [9] (Theorem 21.10), $\sigma(\Sigma_1^1(X))$ is universally measurable.

Define a probability measure $\nu \in \Delta(X \times Y)$ as follows:

$$\nu(B) = f \circ \mu(B) = \mu(f^{-1}(B)),$$

for every Borel set $B \subseteq X$. Since f is universally measurable, ν is well defined and for every Borel $M \in X$ we have

$$\begin{aligned} proj_X \nu(M) &= \mu(f^{-1}(proj_X^{-1}(M)) && (4.1)\\ &= \mu(f^{-1}(M \times Y)) && (4.2)\\ &= \mu(M \cap proj_X(A)) = \mu(M), && (4.3) \end{aligned}$$

where 4.3 follows from the fact that $\mu(proj_X(A)) = 1$. □

Let $s_i \in S_i(n+1)$. By definition we have a probability measure μ such that

1. $s_i \in \mathrm{br}_i(\mu)$,
2. $\mu(S_i(n-1)) = \mu(\mathrm{proj}_{S_i} R^i_{n-1}) = 1$,

where 2 is due to the induction hypothesis. From Lemma 7, R_i^{n-1} is closed subset of $\Omega_j = S_j \times T_j$. Therefore, by Lemma 9 we can extend μ to a probability measure $\nu \in \Delta(\Omega_j)$ such that $\nu(R_i^{n-1}) = 1$ and $\mathrm{proj}_{S^j} \nu = \mu$. Using universality of the type space we get that there exists a type $t_i \in T_i$ such that $g_i(t_i) = \nu$ and so $(s_i, t_i) \in R_i^n$, and this complete the proof of part 1. □

We sketch a proof of parts 2 and 3, despite some similarities with the proof of Theorem 1, we believe is of special interest.

4.2 The Universal Type Space

Previous to the proof of parts 2 and 3 in Theorem 8, we shall sketch the construction of the universal type space, which will serve us in both proofs.

Given a two-player continuous game Γ, the hierarchies of beliefs associated with Γ, defined inductively as follows. $X_i^0 = S^j$ is the first order of uncertainty for player i, and so the nth-uncertainty order is $X_i^n = X_i^{n-1} \times \Delta(X_j^{n-1})$.

This construction of the universal type space is based on Brandenburger and Dekel [4] (henceforth BD). Let $T_0^i = \prod_{n=0}^{\infty} \Delta(X_i^n)$ be the space of all possible hierarchies of beliefs for i, which we call *types*. We say that a type $t = (\delta_i^1, \ldots, \delta_i^n)$ is coherent if for every $n \geq 2$ $proj_{X_i^{n-2}} \delta_i^n = \delta_i^{n-1}$. Denote by T_i^1 the set of coherent types for player i. Applying Kolmogorov's extension theorems to Polish spaces, BD proved the following theorem:

Proposition 10. There exists a natural homeomorphism $h_i : T^i :\to \Delta(S^j \times T^j)$.

The homeomorphism is natural in the sense that each coherent belief hierarchy $t_i = (\delta^0, \ldots, \delta^n, \ldots)$ is mapped to the unique probability distribution $\mu \in \Delta(S^j \times T^j)$ satisfying $proj_{X_i^n} h_i(t_i) = \delta^n$.

Define inductively the set of types that are coherent up to order $k \geq 2$, T_k^i as follows:

$$T_k^i = \{\ t \in T_1^i : h_i(t)(S_j \times T_{k-1}^j) = 1\}.$$

Let $T^i = \cap_{k \geq 1} T_k^i$ be the set of types for which common knowledge of coherency is obtained. BD show that T^i is homeomorphic to $\Delta(S^j \times T^j)$ via h_i.

Proof of Part 3 of Theorem 8. For $s \in S_i$ let $\{Z_n(s)\}_{n=1}^{\infty}$ be disjoint copies of the space T_i and let $Z(s)$ be the following direct sum:

$$Z(s) = s \bigoplus_{n=1}^{\infty} Z_n(s).$$

Note that $Z(s)$ is Polish as a countable direct sum of Polish spaces. Let $\prec^s$ be the following binary relation over $Z(s)$:

1. $t \prec s$ iff $(s, t) \in R_1^i$ for $t \in Z_1(s)$.

2. $t' \prec t$ iff $t \in Z_{n-1}(s)$, $t' \in Z_n(s)$, $(s, t') \in R_n^i$ and $proj_{X_{n-1}^i} h_i(t) = proj_{X_{n-1}^i} h_i(t')$.

We obtain the following lemma:

Lemma 11. For every strategy $s \in S^i$, we have

1. $\prec^s$ is a closed relation over $Z(s)$ ($\prec^s \subseteq [Z(s)]^2$).
2. $s \in S_i^\infty$ iff $\prec^s$ is ill-founded.
3. For $s \in S_i \setminus S_i^\infty$ we have $|s| = \rho(\prec^s)$.

Proof of Lemma 11. Part 1 follows from an argument similar to that of Lemma 2. For part 2, let $(s,t) \in R_\infty^i$; we have $\cdots \prec^s t \prec^s t \prec^s s$. For the other direction assume that $\prec^s$ is ill-founded and so there exists a sequence $\{t_k\}_{k=1}^\infty = \{(\delta_k^1, \cdots, \delta_k^n, \cdots)\}_{k=1}^\infty$ such that $t_1 \prec s$ and $t_{k+1} \prec t_k$ for $k \geq 1$. Let $t = (\delta_1^1, \cdots, \delta_n^n, \cdots)$. Firstly, inductively one can see that t is indeed coherent hierarchy; secondly, $\forall n \in \mathbb{N}$, $t \in R_n^i$, since $\mathrm{proj}_{X_i^n} t = \mathrm{proj}_{X_i^n} t_n$ and $t_n \in R_i^n$. That is, if two types have the same n-hierarchy of beliefs and one of them is in R_i^n, then so is the other. $\square$

The proof of part 3 is now obtained in a similar way to that of Lemma 4. $\square$

Proof of Part 2 of Theorem 8. We prove part 2 using transfinite induction. Specifically, $\mathrm{proj}_{S^i} R_\infty^i \subseteq S_\eta^i$ for every ordinal η, and so $\mathrm{proj}_{S^i} R_\infty^i \subseteq S_i^\infty$. For the other direction, if $s \notin R_\infty^i$ then $\prec^s$ is a closed, well-founded relation with $\rho(\prec^s) < \omega_1$. But by part 2 of Theorem 11, $|s| = \rho(\prec^s)$ and so by definition $s \notin S_i(|s|+1)$, and $s \notin S_i(\infty)$. $\square$

5 Generalization of the Results

5.1 The n-Player Case

The case where the number of players exceeds two is a bit different from the two-player case, although the same kind of results are obtained. When there are three or more players, we may want to impose the *independent rationality* *condition.* That is, a player's beliefs regarding the strategies of the other players should reflect the fact that the strategies of the players are chosen independently (see also [13], Definition 5.3). Let $\Gamma = (N, (S_i)_{i\in N}, (U_i)_{i\in N})$ be an n-person continuous game. Define the following elimination process:

Definition 5.1. Let $S_i^{uc}(1) = S_i$. For successor ordinal α, $S_i^{uc}(\alpha) = \{\ s_i \in S_i^{\alpha-1} : \exists\ \mu \in \otimes_{j\neq i}\Delta(S_i^{\alpha-1})$ s.t. $s_i \in \mathrm{br}(\mu, U_i)\ \}$,
and for limit ordinal α, $S_i^{uc}(\alpha) = \cap_{\beta<\alpha} S_i^{uc}(\beta)$.

A hierarchy of beliefs is uncorrelated if it holds with common belief that players' assessments are uncorrelated. I.e., let $\mathfrak{T} = ((T_i)_{i\in N}, (g_i)_{i\in N})$ be the universal type space for Γ. Define inductively the uncorrelated hierarchy of belief at level k .
$T_i^1 = \{t \in T_i :\ g_i(t) = (\delta_i^1, \ldots, \delta_i^n, \ldots)$ s.t. $\delta_i^1 \in \otimes_{j\neq i}\Delta(S_j)\}$. That is T_i^1 is the set of coherent hierarchies whose first-order belief is uncorrelated.
$T_i^k = \{t \in T_{k-1} :\ g_i(t)(\prod_{j\neq i} T_j^{k-1} \times S_j) = 1\ \}$.
Finally, let $T_i^{uc} = \cap_{k\geq 1} T_i^k$.

Lemma 12. For every $k \geq 1$, T_i^k is closed and $g_i : T_i^{uc} \to \Delta(\prod_{j\neq i} S_j \times T_j^{uc})$ is a homeomorphism.

Proof. For $k = 1$, note first that if X, Y are Polish $\mu_k = \delta_k \otimes \beta_k \in \Delta(X) \otimes \Delta(Y)$ and $\mu_k \to \mu$, then $\mu \in \Delta(X) \otimes \Delta(Y)$. To see this, note that $\{\delta_k\}_{k=1}^{\infty}$, $\{\beta_k\}_{k=1}^{\infty}$ are convergent sequences (by the portmanteau theorem) to δ and β respectively; hence $\mu_k \to \delta \times \beta$. This shows that T_k^1 is closed, so by induction T_i^k is closed for every k. The second assertion is immediate. $\square$

A version of Theorem 8 is valid for this case as well. In particular, let T^{uc} be the universal type space of uncorrelated belief-hierarchies, then $(s_i, t_i) \in R_n^i$ iff $s_i \in S_i^{uc}(n + 1)$

5.2 The Incomplete Information Case

Battigalli and Siniscalchi, [2], followed by Dekel et al. [5], generalized Pearce's rationalizability process to incomplete information games. Our results can be easily generalized to that case. The natural way to do so is by treating the unknown parameter set as a strategy set of a dummy player with a trivial payoff function. In this way, we can obtain a version of Theorem 8 for the incomplete information setup.

5.3 Non-Polish Strategy Space

Some of our results are valid for the case where the strategy sets of each player are analytic spaces. However, Theorem 1 fails where the strategy set of each player is a separable metric space. For that case we construct a variant of the example introduced in Section 3.2 with separable metric strategy sets and continuous payoff functions for both players. In that game $\omega_1 + 1$ rounds of elimination are needed. We use the notation introduced in Section 3.2.

Example 5.1. Let $\tilde{\Gamma} = (\{1,2\}, (\tilde{S}_i)_{i=1,2}, (\tilde{U}_i)_{i=1,2})$ be the following game: The strategy set of player 2 remains as in the game Γ in Section 3.2, i.e., $\tilde{S}_2 = S_2$. The strategy set of player 1 is $\tilde{S}_1 = S_1 \cup \{p\}$, where p represents any point not in S_i. Let $\mathcal{T}$ be the topology over $2^{\mathbb{N}^2} \times \mathcal{N}$ obtained when we add the sets $\{S_i^\infty, (S_i^\infty)^c\}$ to the natural topology over S_i. This topology makes S_i a separable metric space but not a Polish space. Equip S_2 with the topology $\mathcal{T}$ and equip $\tilde{S}_1$ with the the topology obtained from the natural topology over S_1 when we add $\{p\}$ as an open set; i.e., p is an isolated point. The new payoff function for player 1 is:

$$\tilde{U}_1(s,t) = \begin{cases} U_1(s,t) & \text{if } s \neq p \\ -1 & \text{if } s = p \text{ and } t \in S_2^\infty \\ 0 & \text{if } s = p \text{ and } t \notin S_2^\infty. \end{cases}$$

$\tilde{U}_1$ is continuous since U_1 is continuous and the sets $\{p\}$ and S_2^∞ are simultaneously closed and open sets. The payoff function of player 2 is any continuous extension of U_2. In $\tilde{\Gamma}$ we have $\tilde{S}_1^{\omega_1} = S_i^\infty \cup \{p\}$, while $S_1^{\omega_1+1} = S_i^\infty$.

If we abandon the demand for a continuous payoff function and settle for a demand of a Borel measurable payoff function for each player, then part 1 of Theorem 8 remains valid, whereas it is not clear to us whether Theorem 1 remains so.

A Appendix

Assume that Γ is a two-player normal-form game. When the strategy sets are finite or the strategy sets and the payoff functions are nicely behaved, using separation theorems, one can prove that a strategy is a best-reply strategy iff it is not strongly dominated. In our case, however, as we shall show, this is not true. Given a two-player continuous game Γ, a strategy $s \in S_i$ is strongly dominated if there exists a mixed strategy that is strictly better than s, i.e., $\exists \pi \in \Delta(S_i)$ such that

$$U_i(\pi, s_j) > U_i(s, s_j) \forall\ s_j \in S_j.$$

Clearly if a strategy is strictly dominated it cannot be a best-reply. That is, if $s \in S_i$ is strictly dominated by π, then for every $\mu \in \Delta(S_j)$ there $s \notin \mathrm{br}_i(\mu)$. To see this, note that if $U_i(\pi, \mu) > U_i(s, \mu)$, then the set $\{s' \in S_i : U_i(s', \mu) > U_i(s, \mu)\}$ is not empty.

In the following example we exhibit a game and an undominated strategy for player 1, that is not a best-reply strategy.

Example A.1. We define a two-player continuous game as follows: $S^2 = \mathbb{N}$ and $S_1 = \mathbb{N} \cup \{\alpha\}$. Equivalently, $S_1 = \omega + 1$, and $S_2 = \omega$ equipped with the discrete topology. The payoff function for player 1 is

$$U_1(p, q) = \begin{cases} 1 & if\ p \neq \alpha \text{ and } p \geq q \\ \frac{1}{2} & if\ p = \alpha. \end{cases}$$

Note that α is not a best-reply strategy. Since for every $\mu \in \Delta(\mathbb{N})$ there exists a large enough n s.t., $\mu(\{1, 2, \ldots, n\}) > \frac{1}{2}$, the strategy n yields a payoff higher than $\frac{1}{2}$ and α is not a best-reply with respect to μ. To see that α is not strongly dominated, let $\pi \in \Delta(\mathbb{N} \cup \{\alpha\})$. We can assume that $\pi(\mathbb{N}) = 1$. For some m, $\pi(\{m, m{+}1, \ldots\}) < \frac{1}{2}$, and so $u(\pi, m) < \frac{1}{2} = u(\alpha, m)$. Therefore, α is not strongly dominated by π.

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