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RATIONALIZABLE EXPECTATIONS

by

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Rationalizable Expectations[1]

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Abstract:

Consider an exchange economy with asymmetric information. What is the set of outcomes that are consistent with common knowledge of rationality and market clearing?

We propose the concept of $CKRMC$ as an answer to this question. The set of price functions that are $CKRMC$ is the maximal set F with the property that every $f \in F$ defines prices that clear the markets for demands that can be rationalized by some profile of subjective beliefs on F. Thus, the difference between $CKRMC$ and *Rational Expectations Equilibrium* (REE) is that $CKRMC$ allows for a situation where the agents do not know the true price function and furthermore may have different beliefs about it. We characterize $CKRMC$, study its properties, and apply it to a general class of economies with two commodities. $CKRMC$ manifests intuitive properties that stand in contrast to the full revelation property of REE. In particular, we obtain that for a broad class of economies: (1) There is a whole range of prices that are $CKRMC$ in every state. (2) The set of $CKRMC$ outcomes is monotonic with the amount of information in the economy.

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1 Introduction

We study the implications of the assuming common knowledge of rationality and market clearing in economies with asymmetric information.

The starting point is the concept of rational expectations equilibrium (REE). REE extends the classical concept of a competitive equilibrium to economies with asymmetric information (i.e., economies in which different agents might have different information). When each agent has only partial information on the value of a commodity or an asset he can deduce additional information from the prices because prices reflect the information that other agents have. REE is a solution concept that is based on the assumption that agents make these inferences. However, the concept of REE is based on an additional strong assumption that agents know (and therefore agree on) the function that specifies the prices in each state. (A state specifies the real variables of the economy, i.e., preferences and endowments.) This strong assumption leads to a strong result that in a generic economy with a finite number of states the only REE is a fully revealing equilibrium, i.e., an equilibrium in which each agent can infer from the prices all the information that any other agent has (Radner (1979).)

In the current research the assumption that players know the price function is relaxed, that is, we consider a situation where each agent may have a different theory on how the vector of prices which is observed has materialized and on what would have happened in other states. However, the assumption is maintained that each agent makes inferences from the observed prices and furthermore assumes that other agents are doing likewise. More precisely we are interested in characterizing and analyzing properties of the set of outcomes that are consistent with *common knowledge of rationality and market clearing* (henceforth, outcomes that are $CKRMC$).

The outline of the paper is as follows:

1. Definition and characterization of $CKRMC$.

The basic idea that underlies the solution concept that we propose is simple: Say that a set of price functions F is $CKRMC$ if every $f \in F$ defines prices which clear the market for demands that can be rationalized by some profile of subjective beliefs on F.

$CKRMC$ assumes a situation where all the agents have a common prior on the set of states of the economy. Consider now a different model where each agent, given a price p, may have any subjective belief on the states of the economy that is consistent with his private signal. Say that an outcome (s, p), where s is a state and p is a price, is $EXPR$ (Ex-Post Rationalizable) if a situation where the price p clears the market in the state s is consistent with common knowledge of rationality and market clearing in this model. The main result in this paper, theorem 1 , establishes that under a mild qualification , the set of $CKRMC$ outcomes[4] equals the set of $EXPR$ outcomes.

Theorem 1 is useful in two ways: First, it establishes that two different types of assumptions on the uncertainty that players face lead to similar predictions. Second, it simplifies the computation of the set of outcomes that are $CKRMC$.

2. Applications.

We use the characterization of theorem 1 to compute the set of prices that are $CKRMC$ in a general class of economies with two commodities. In this class of economies $CKRMC$ manifests several properties which stand in contrast to the full revelation property of REE. In particular, we obtain that :

(a) For a robust subset of these economies there is a whole range of prices that are consistent with all the possible states and therefore these prices do not reveal any information.

(b) Refining the knowledge of a positive measure of agents strictly shrinks the set of $CKRMC$ prices.

(c) Trade is consistent with $CKRMC$ even when there is common knowledge that there are no mutual gains from it.

3. Properties of $CKRMC$.

Two different issues are examined.

Theorem 2 establishes that the set of price functions that are $CKRMC$ can be obtained by a procedure in which functions that cannot be supported by any profile of beliefs are iteratively deleted. Theorem 3 addresses the following concern: the definition of $CKRMC$ does not refer explicitly to the beliefs of agents about other agents so one can ask in what sense is a function that is $CKRMC$ indeed consistent with common knowledge of rationality and market clearing. Theorem 3 establishes that it is possible to construct a model where each state contains a complete description of what each agent believes that any other agent believes (that any other agent believes...), in which common knowledge of rationality and market clearing is indeed equivalent to $CKRMC$.[5]

There is some previous work which examines the implications of the assumptions of rationality and market clearing in a situation where players may have different beliefs about the relationship between states of the economy and prices. However the solution concepts that are proposed in these papers are different from $CKRMC$. MacAllister (1990) and Dutta and Morris (1997) propose a solution concept, Belief Equilibrium, which is stronger than $CKRMC$ as it assumes that in addition to common knowledge of rationality and market clearing there is also common knowledge of the belief of each player on the set of price functions. As we show in section 3 this additional assumption

[4] An outcome (s,p) is $CKRMC$ if there exits a $CKRMC$ function f such that $f(s)=p$.(A function f is $CKRMC$ if there exists a set of functions F that is $CKRMC$ such that $f \in F$.)

[5] See Morris (1995) for such a complete model that justifies REE.

restricts in a significant way the set of possible outcomes. Desgranges and Guesnerie (2000)[6] examine iterative deletion of weakly dominated demand strategies in a simple example which is similar to example 1 in the current paper. The solution set that they obtain is equal to the set of $CKRMC$ and $EXPR$ outcomes that is obtained in the current paper. The closest to the current paper is Desgranges (2004). Desgranges also studies the implications of $EXPR$ (Desgranges calls it common knowledge equilibrium)[7]. The focus in his work is on determining conditions under which $EXPR$ implies the REE outcome. The main difference between Desgranges and the current paper is that our starting point is the concept of $CKRMC$ and the main interest is in determining properties of this concept.

The current paper is organized as follows: In section 2 the definition of an exchange economy with asymmetric information is reviewed and a simple example which motivates and demonstrates the concept of $CKRMC$ is presented. We then present the definition of $CKRMC$. In section 3 we define the concept of $EXPR$ and present theorem 1 which establishes that $CKRMC$ and $EXPR$ are 'almost' equivalent. In section 4 we characterize the set of $EXPR$ outcomes in a general class of economies with two commodities and then use theorem 1 to derive implications on $CKRMC$ outcomes. The solution that is obtained manifests the properties that were mentioned in item 2 above. Section 5 contains theorem 2 and theorem 3. Section 6 discusses two assumptions on the beliefs of the agents and section 7 concludes.

2 The Model.

In this section we review the definition of an exchange economy with asymmetric information and present a simple example which motivates the solution notion of $CKRMC$. We then define $CKRMC$.

An economy with asymmetric information is defined by:

1. $I = [0,1] -$ The set of players (consumers).
2. $X_1, ..., X_K -$ K commodities.
3. $S = \{s_1, ..., s_n\} -$ The set of states of nature.
4. $\alpha \in \triangle(S) - \alpha$ is a common prior on S .
5. $\Pi_i -$ A partition on S that describes the information of player i.
   $\Pi_i(s) \subseteq S$ is the information that player i gets at the state s.
6. $u_i : R^K \times S \to R -$ A V.N.M utility function for player i.
   $u_i(x, s)$ is the utility of player i from a bundle $x \in R^K$ in the state s.
7. $e_i : S \to R^K - e_i(s)$ is the initial bundle of player i at state s.
   We assume that e_i is measurable w.r.t Π_i and that

[6] See also Guesnerie (2002)

[7] A first draft of Desgranges paper was written before ours. We have developed the concept of $EXPR$ independently, before we learned of his work.

$\forall s \in S$, $\int_i e_i(s)-$ the aggregate supply in state $s-$exists.

A *price* p is a vector $p = (p_1, ...p_{K-1})$ where p_k is the price of X_k. The price of X_K is normalized to be 1.

A *price function* $f, f : S \rightarrow R^{K-1}$, assigns to every state s a price $f(s)$. We will sometimes think of a price function as a vector in $R^{S\times(K-1)}$.

We let L_i denote the set of signals of agent i. So, $L_i \equiv \{\Pi_i(s) : s \in S\}$.

A *demand strategy* for player i is a function z_i, $z_i : L_i \times R^{K-1} \rightarrow R^K$, such that $z_i(l_i, p)$ is in the budget set defined by the price p and the initial endowment $e_i(l_i)$. ($e_i(l_i)$ is well defined because e_i is measurable w.r.t Π_i.)

The standard solution notion for economies with asymmetric information is Rational Expectations Equilibrium, *REE*. A *REE* is a price function f such that for each state s the price $f(s)$ clears the market when every agent i makes a demand which is optimal w.r.t the price $f(s)$ and the information that is revealed by his private signal $\Pi_i(s)$ and the fact that the price is $f(s)$. Formally,

Definition: A price function f is a *REE* if there exists a profile of demand strategies, $\{z_i\}_{i\in I}$, that satisfies :

1. Rationality, $\forall s \in S$, $z_i(\Pi_i(s), f(s))$ is optimal w.r.t the price $f(s)$ and the posterior $\alpha(\cdot \mid \Pi_i(s) \cap f^{-1}(f(s)))$.

2. Market clearing, $\forall s \in S$, $\int_i z_i(\Pi_i(s), f(s)) = \int_i e_i(s)$.

A price function f is a *fully revealing REE* (*FREE*) if $f(s) \neq f(s')$ when $s \neq s'$.

We turn now to a simple example which demonstrates the difference between *REE* and consistency with common knowledge of rationality and market clearing.

Example 1:

There are two commodities in the economy, X and M (money).

The set of states is $S = \{1, 3\}$.

The probability of each state is 0.5.

The set of agents is the interval $[0, 1]$. There are two types of agents, I_1 and I_2. Agents in I_1 know the true state; agents in I_2 do not know it. $I_1 = [0, \delta]$ and $I_2 = (\delta, 1]$. All the agents have the same utility and the same initial bundle. The utility is:

(2.1) $$u(x, m, s) = s \times \log(x) + m$$

where x and m are the quantities of X and M respectively and s is the state.

The initial bundle consists of one unit of X and $\overline{m}$ units of M where $\overline{m} \geq 3$.

Let p be the price of a unit of X in units of M. It follows from the definition of the utility function in (2.1) that the demand for X of an agent who knows the true state is:

$$x = \frac{s}{p}$$

More generally, the demand of an agent i who assigns to the state s probability $\gamma(s)$ is

(2.2) $$x = \frac{\gamma(1)\times 1 + \gamma(3)\times 3}{p}$$

In this example for every $\delta > 0$ there is only one REE, f^* ,where $f^*(s) = s$. To see that we, first, note that if f is a REE then $f(1) \neq f(3)$. This follows because if $f(1) = f(3) = p$ then agents in I_2 do not obtain any information about the true state and therefore their demand in both states is the same:

$x = \frac{0.5\times 1 + 0.5\times 3}{p} = \frac{2}{p}$

However, the demand of agents from I_1 in state 1 is different than their demand in state 3 and therefore the aggregate demands are different as well. Since the aggregate amount of X is fixed this means that the market doesn't clear in at least one of the states and therefore f is not a REE. Thus, if f is a REE then $f(1) \neq f(3)$. In this case agents in I_2 infer the state from the price and it follows from (2.2) that $f(1) = 1$ and $f(3) = 3$. Thus, the only REE is a fully revealing equilibrium (henceforth, $FREE$) in which the price reveals the state. Indeed, Radner (1979) has shown that in a generic class of economies with a finite number of states the only REE is a $FREE$ in which the information that all the agents have together is revealed.

We now show that if we relax the assumption that players know the price function (and therefore agree on it) then there are other price functions which are consistent with common knowledge of rationality and market clearing. We call such price functions 'functions that are $CKRMC$'.

Assume that $\delta = \frac{1}{6}$. We will show that the following price functions are $CKRMC$:

$$\begin{array}{ll} f(1) = 2 & \qquad g(1) = 1 \\ f(3) = 3 & \qquad g(3) = 2 \end{array}$$

Suppose that a fraction β of the agents in I_2 assign probability $\frac{3}{4}$ to the event that f is the price function and a probability $\frac{1}{4}$ to the event that g is the price function, call this belief 'theory A'. Assume that the other agents in I_2 think that g is more likely, they assign probability $\frac{1}{4}$ to the event that f is the price function and probability $\frac{3}{4}$ to the event that the price function is g, call this belief 'theory B'.

What are the beliefs of different agents in I_2 about the true state when they observe the price 2 ?

Since the prior assigns probability 0.5 to each state it is easy to see that agents in I_2 who believe in theory A assign probability $\frac{3}{4}$ to the state 1 and probability $\frac{1}{4}$ to the state 3[8]. Similarly, agents who believe in theory B assign probability $\frac{1}{4}$ to the state 1 and probability $\frac{3}{4}$ to the state 3.

It follows from (2.2) that the demand for X at price 2 of agents who believe in theory A is $(\ \frac{3}{4}\times 1 + \frac{1}{4}\times 3\ )/2 = \frac{3}{4}$ while the demand of agents who believe in theory B is $(\frac{3}{4}\times 3 + \frac{1}{4}\times 1)/2 = \frac{5}{4}$.

[8] Let $P_A(s\,|p = 2)$ denote the posterior that an agent who believes in theory A assigns to the state s upon observing the price 2. Then $P_A(1\,|p = 2) = \frac{0.75\cdot\alpha(1)}{0.75\cdot\alpha(1)+0.25\cdot\alpha(3)} = 0.75$

Let $x(\beta, s, p)$ denote the aggregate demand for X in state s at price p when a proportion β of the agents in I_2 believe in theory A and the rest of I_2 believe in theory B. We have

$x(\beta, 1, 2) = (1-\delta) \times \beta \times \frac{3}{4} + (1-\delta) \times (1-\beta) \times \frac{5}{4} + \delta \times \frac{1}{2}$

$x(\beta, 3, 2) = (1-\delta) \times \beta \times \frac{3}{4} + (1-\delta) \times (1-\beta) \times \frac{5}{4} + \delta \times \frac{3}{2}$

Let β_f and β_g be the numbers which equate demand and supply at price 2 in the states 1 and 3 respectively, that is,

$x(\beta_f, 1, 2) = 1$ and $x(\beta_g, 3, 2) = 1$.For $\delta = \frac{1}{6}$ we obtain $\beta_f = 0.3$ and $\beta_g = 0.7$.

Now we observe that when β_f of the agents in I_2 believe in theory A and $1-\beta_f$ of them believe in B then the function f specifies prices which clear the market. We have just seen that the price 2 clears the market in $s = 1$ and when the price is 3 everyone assigns probability 1 to the state 3 and therefore the price 3 clears the market. Similarly, when β_g of the agents in I_2 believe in theory A (and the rest in B) the function g specifies prices which clear the market.

We have thus shown that when the assumption that players know the price function is relaxed then there is more than one price function that can be rationalized. Specifically, there exists a profile of beliefs (i.e., a belief for each player), $\overline{\beta_f}$, such that when each player makes a demand which is optimal w.r.t his beliefs the prices specified by f clear the market. Similarly, there is a profile of beliefs $\overline{\beta_g}$ which rationalizes g. Furthermore, since the beliefs of each player in the profiles $\overline{\beta_f}$ and $\overline{\beta_g}$ assign a positive probability only to f and g, which are functions that can be rationalized, f and g are consistent *not only* with rationality and market clearing but also with *common knowledge* of rationality and market clearing. Specifically, one can think of the theory A (B) not only as a theory which assigns probabilities to price functions, probability $\frac{3}{4}$ $(\frac{1}{4})$ to f and probability $\frac{1}{4}$ $(\frac{3}{4})$ to g, but as a richer theory which refers to the beliefs of agents as well. This extended theory A (B) assigns probability $\frac{3}{4}$ $(\frac{1}{4})$ to the event that the profile of beliefs in the population is $\overline{\beta_f}$ and probability $\frac{1}{4}$ $(\frac{3}{4})$ to the event that the profile of beliefs is $\overline{\beta_g}$. The proof of theorem 3 in section 5 provides a precise description of such a structure of beliefs.

We can now turn to the general definition of $CKRMC$.

We say that a set of functions $F, F \subseteq R^{n\times(K-1)}$, is $CKRMC$ if every $f \in F$ defines prices which clear the market for demands that can be rationalized by beliefs on F.

To provide a completely formal description we need some preliminary definitions:

Definition: A *belief* μ_i for player i on a set of Borel price functions F is a finite lexicographic sequence of probability measures, $\mu_i = (\mu_i^1, ..., \mu_i^m)$, on F.

We assume that the selection of the state of nature is independent of the selection of the price function and therefore the beliefs of player i on the set $S \times F$ is a product of his prior probability distribution on S, α,and his beliefs on F, μ_i. Specifically, the belief of an agent i on $S \times F$ is the lexicographic sequence of probabilities $\alpha \times \mu_i = (\alpha \times \mu_i^1, ...,$

$\alpha \times \mu_i^m$) where for $S' \subseteq S$ and $F' \subseteq F$, $\alpha \times \mu_i^k(S' \times F') = \alpha(S') \cdot \mu_i^k(F')$. The information that player i has when he makes a demand is his private signal $l_i \in L_i$ and the fact that a given price p has materialized. Given a set of price functions F we let (l_i, p) denote the event in $S \times F$ which is consistent with l_i and p. That is,

$(l_i, p) = \{(s, f) : s \in S,\ f \in F \ \ \Pi_i(s) = l_i \text{ and } f(s) = p\}$

We say that a belief $\mu_i, \mu_i = (\ \mu_i^1, ..., \mu_i^m)$, of player i is *consistent* with the event (l_i, p) if there exists k,$1 \leq k \leq m$, such that $\alpha \times \mu_i^k(l_i, p) > 0$. Given a belief μ_i and an event (l_i, p) which is consistent with it we allow for some abuse of notation and let $\mu_i(\ |(l_i, p))$ denote the marginal distribution of $\alpha \times \mu_i^k(\cdot\ |(l_i, p))$ on S, where k is the lowest index with the property that $\alpha \times \mu_i^k(l_i, p) > 0$. $\mu_i(\cdot\ |(l_i, p))$ is the posterior on S of a player i with a belief μ_i given the event (l_i, p).

We are now ready to give a formal definition of $CKRMC$.

Definition: A set of price functions F is $CKRMC$ if for every f in F there is a profile of demand strategies $\left\{z_i^f\right\}_{i \in I}$ and a profile of beliefs $\left\{\mu_i^f\right\}_{i \in I}$ on some Borel subset of F that satisfy:

1. Rationality: for every $i \in I$ and every (l_i, p) that is consistent with μ_i^f $\ z_i^f(l_i, p)$ is an optimal bundle at the price p w.r.t $\mu_i^f(\cdot\ |(l_i, p))$. For every $s \in S$ and $i \in I$ $(\Pi_i(s), f(s))$ is consistent with μ_i^f.

2. Market Clearing at the prices specified by f: for every $s \in S$ $\int_i z_i^f(\Pi_i(s), f(s)) = \int_i e_i(s)$.

We will say that profiles of beliefs and demand strategies $\mu^f \equiv \left\{\mu_i^f\right\}_{i \in I}$ and $z^f \equiv \left\{z_i^f\right\}_{i \in I}$ support f w.r.t F if μ^f and z^f satisfy conditions 1. and 2. in the definition of $CKRMC$. A profile of beliefs μ^f supports f w.r.t F if there exists a profile of demand strategies z^f such that μ^f and z^f support f w.r.t F.

Definition: A price function f is $CKRMC$ if there exists a set of price functions F such that $f \in F$ and F is $CKRMC$.

We let $FCKRMC$ denote the set of functions that are $CKRMC$.

Definition: An *outcome* (p, s), $p \in R^{K-1}$, $s \in S$, is a pair consisting of a price and a state.

An outcome (p, s) is $CKRMC$ if there exists a price function $f \in FCKRMC$ such that $f(s) = p$.

To demonstrate and clarify the definitions we note that:

1. A price function f is an REE iff the set $F = \{f\}$ is $CKRMC$. In particular, a price function f that is an REE is also a function that is $CKRMC$.

2. In example 1 the set $F = \{f, g\}$ is $CKRMC$. In particular, any profile of beliefs $\{\mu_i\}_{i \in I}$, $\mu_i = (\mu_i^1)$ (i.e. the beliefs of each player consist of just one probability distribution), where 0.3 of the agents in I_2 assign probability 0.75 to f and 0.25 to g and the

rest assign 0.25 to f and 0.75 to g supports demands which clear the market in the prices specified by f. Similarly, as we have seen in the discussion of the example, there are profiles of beliefs that support g. It follows that (1,1), (2,1), (2,3) and (3,3) are outcomes that are $CKRMC$.

3. The definitions of $CKRMC$ and $FCKRMC$ readily imply that $FCKRMC$ is the maximal set of functions that is $CKRMC$. Theorem 2 (section 5) establishes that $FCKRMC$ can be obtained by a procedure in which price functions that cannot be supported by any beliefs are iteratively deleted. Furthermore, the procedure terminates after a finite number of steps.

4. The belief μ_i^f is defined to be a lexicographic sequence of probabilities on F. There are two reasons for using this definition rather than just defining μ_i^f to be one probability measure on F. First, if μ_i^f was defined to be a single probability measure we would have to require that it assigns a positive probability to every event $(\Pi_i(s), f(s)), s \in S$. A model with lexicographic beliefs allows for a situation where an agent initially assigns probability zero to an event but if that event occurs the agent updates his assessment and optimizes w.r.t a new probability measure. Second, there is an equivalence between optimizing w.r.t lexicographic beliefs and choosing a demand strategy that is admissible (i.e. not weakly dominated.) The proof of this equivalence is long and therefore beyond the scope of this paper. We view this result as important because it means that two different criteria for rational behavior i.e. optimization w.r.t beliefs and not playing a dominated strategy are equivalent. Finally, we note that theorem 1 is valid in a model where there is just one probability distribution (the proof given in the next section applies). However, the proof of theorem 2 relies on the beliefs being a lexicographic sequence of probabilities. We do not know whether theorem 2 applies in the single probability model[9]. In section 7 we discuss the assumption that the belief of an agent consists of just a finite number of probability distributions and that the belief on S is independent of the belief on F. We show that the first assumption is w.l.o.g. and that with the obvious modifications all our results are valid in a model where the belief on S maybe correlated with the belief on F. The reason for assuming independence is that it seems to us a plausible assumption in this context and thus our point is that our results can be obtained when independence is assumed.

In the next section we will provide a characterization of outcomes that are $CKRMC$ and use this characterization to compute the whole set of outcomes that are $CKRMC$ in example 1.

[9] More precisely, we do not know whether the iterative deletion of functions that cannot be supported terminates after a countable number of steps.

3 A Characterization.

In this section we define the concept of $EXPR$[10] and present theorem 1 which establishes that under a mild qualification the set of $CKRMC$ outcomes equals the set of $EXPR$ outcomes. This result is useful in two ways: First, it establishes that two different types of assumptions about the uncertainty that players face determine similar sets of outcomes. Second, $EXPR$ is simpler to compute and hence the result facilitates the computation of the set of outcomes that are $CKRMC$.

Definition: A price p is $EXPR$ w.r.t to a set of states $\widehat{S} \subseteq S$ if for every $s \in \widehat{S}$ there exists a profile of probabilities on $\widehat{S}$ $\{\gamma_i^s\}_{i\in I}$, $\gamma_i^s \in \triangle(\widehat{S} \cap \Pi_i(s))$, and a profile of demands $\{x_i^s\}_{i\in I}$, $x_i^s \in R^K$, such that:
1. For every $i \in I$ x_i^s is an optimal bundle at the price p w.r.t γ_i^s.
2. Markets clear, that is, $\int_i x_i^s = \int_i e_i^s$.

We will say that the price p can be supported in the state s by the beliefs $\gamma^s = \{\gamma_i^s\}_{i\in I}$ on $\widehat{S}$ if there exists a profile of demands $x^s = \{x_i^s\}_{i\in I}$ such that the conditions 1. and 2. above are satisfied.

The idea is that if p is $EXPR$ w.r.t $\widehat{S}$ then $\widehat{S}$ is a set of states in which p could be a clearing price because for every $s \in \widehat{S}$ there is a profile of beliefs on $\widehat{S}$, $\{\gamma_i^s\}_{i\in I}$, which is consistent with the private information of the players and which rationalizes demands that clear the markets at p. (The belief γ_i^s,in turn, is possible for player i because p can be a clearing price in every $s \in \widehat{S}$.)

Definition: An outcome (p, s) is $EXPR$ (alternatively, p is $EXPR$ in s) if there exists a set of states $\widehat{S}$ such that $s \in \widehat{S}$ and p is $EXPR$ w.r.t $\widehat{S}$.

Let $S(p)$ be the set of all states such that (p, s) is $EXPR$. It is easy to see that p is $EXPR$ w.r.t $S(p)$ and that $S(p)$ is the maximal set w.r.t which p is $EXPR$.

It is useful to think of $EXPR$ in the following way: Consider a model where there is no common prior on the states and the belief of each agent is a subjective probability on S. Specifically, in a state s each agent may have any subjective probability on S that is consistent with his private information, $\Pi_i(s)$. It is easy to show that an outcome (p, s) is $EXPR$ iff (p, s) is consistent with common knowledge of rationality and market clearing in this model.[11]

We can now state our main result.

[10] As we have pointed out in the introduction the concept of $EXPR$ is also studied in Degranges (2004).

[11] A formal statement and proof of this proposition is done by defining a richer model in which a state of the world defines not only the preferences of the players but also their beliefs on S, their beliefs on the beliefs of other players and so forth. We define such a model in the proof of theorem 3 in section 5 so we do not repeat this here.

Theorem 1:

a. If (p, s) is $CKRMC$ then (p, s) is $EXPR$.

b. Let E be an economy in which there is a fully revealing REE, $\widehat{f}$. Let p be a price such that $p \notin \cup_{s \in S} \widehat{f}(s)$. Then (p, s) is $CKRMC$ iff (p, s) is $EXPR$.

Remarks:

1. The set of economies in which there exists a fully revealing REE is generic. For this set theorem 1 provides a characterization of $CKRMC$ outcomes modulo outcomes which involve prices that are in the range of every fully revealing REE.[12]

2. There can be $CKRMC$ outcomes which do not satisfy the condition formulated in part b. of the theorem. We do not have a general necessary and sufficient condition for an $EXPR$ outcome to be a $CKRMC$ outcome. It is clear that such a condition would be cumbersome. However, Proposition 3.1 in the appendix establishes that in an economy where in each state there are at least $2n$ $EXPR$[13] prices every $EXPR$ outcome is a $CKRMC$ outcome. The proposition also provides a complete characterization of the set of $CKRMC$ functions for this case.

Before demonstrating theorem 1 and proving it we make a few further comments on the relationship between $CKRMC$ and $EXPR$. There are two main differences between the two concepts. First, $CKRMC$ assumes that each player has a complete theory about what price might materialize in each state. This theory is represented by a belief on price functions. Second $CKRMC$ assumes a situation where all the agents have a common prior on S. Then, given a price p and a private signal, each player updates his probability distribution on S. By contrast, $EXPR$ assumes that given a price p each player i may have any probability distribution on $S(p)$ that is consistent with his private signal. There is no common prior, in fact, there are no priors and no updating at all. Player i does not assess the likelihood of a state s given the price p by asking himself what is the prior probability distribution on S and how likely is p in different states, rather, given p, he forms some probability on the states in which p can be a clearing price. In this sense he does not have a complete theory and his reasoning is Ex-Post.

We now use the result to solve the set of outcomes (p, s) that are $CKRMC$ in example 1. Let P_s , $s = 1, 3$, denote the set of prices that are $EXPR$ in s. We will compute P_s and conclude, using theorem 1, that P_s is also the set of prices that are $CKRMC$ in the state s. Let $P(\widehat{S})$ denote the set of prices that are $EXPR$ w.r.t the set of states $\widehat{S}$, $\widehat{S} \subseteq S$. It follows from the definitions that: $P_s = \cup_{\widehat{S}, s \in \widehat{S}} P(\widehat{S})$. In our example:

(3.1) $\quad P_1 = P(\{1\}) \cup P(\{1, 3\})$

(3.2) $\quad P_3 = P(\{3\}) \cup P(\{1, 3\})$

$P(\{1\}) = 1$ and $P(\{3\}) = 3$ because 1 and 3 are the prices which clear the markets in the states 1 and 3 respectively when everyone knows the state. We now compute

[12] There are economies in which the set of fully revealing REE is not a singleton.

[13] We remind that n is the number of states.

$P(\{1,3\})$. Let $P_s(\{1,3\})$ denote the set of prices that can clear the markets in state $s, s=1,3$, when players in I_2 may have any profile of beliefs on $\{1,3\}$. It follows from the definition of $P(\{1,3\})$ that

(3.3) $\quad P(\{1,3\}) = P_1(\{1,3\}) \cap P_3(\{1,3\})$.

We claim that $P_1(\{1,3\}) = [1, 3-2\cdot\delta]$. This follows because the price 1 clears the market when every agent in I_2 assigns probability 1 to the state 1(every agent in I_1 knows that the state is 1.) Clearly, the aggregate demand for X and therefore its price are minimal when everyone assigns the state 1 probability 1. Similarly, the price 3-2$\cdot\delta$ clears the market when every agent in I_2 assigns probability 1 to the state 3 and therefore the maximal point in $P_1(\{1,3\})$ is $3-2\cdot\delta$. It is easy to see that for every $1 \leq p \leq 3-2\cdot\delta$ there is a probability $\gamma(p)$ such that if every agent in I_2 assigns probability $\gamma(p)$ to the state 3 then p clears the market. The set $P_3(\{1,3\})$ is computed in a similar way. When each agent in I_2 assigns the state 1 probability 1 the clearing price is $1+2\cdot\delta$. When agents in I_2 assign the state 3 probability 1 the clearing price is 3. It follows that $P_3(\{1,3\}) = [1+2\cdot\delta, 3]$. From (3.3) we obtain that for $\delta \leq 0.5$ $P(\{1,3\}) = [1+2\cdot\delta, 3-2\cdot\delta]$. For $\delta > 0.5$ $P(\{1,3\}) = \emptyset$. From (3.1) and (3.2) we have that for $\delta \leq 0.5$ $P_1 = \{1\} \cup [1+2\cdot\delta, 3-2\cdot\delta]$ and $P_3 = \{3\} \cup [1+2\cdot\delta, 3-2\cdot\delta]$ and for $\delta > 0.5$ $P_1 = \{1\}$ and $P_3 = \{3\}$. It follows from theorem 1 that the difference between the set P_s and the set of prices that are $CKRMC$ in s, $s=1,3$, is at most the price s. Now, s is the REE price in the state s and therefore s is a price that is $CKRMC$ in the state s. It follows that the sets P_s, $s=1,3$, that we have computed are the sets of prices that are $CKRMC$ in the respective states.

The solution of the example is interesting in several ways: First, when δ is smaller than 0.5 there is a whole range of prices that are $CKRMC$ in both states. Second, the set of $CKRMC$ outcomes (i.e., P_1 and P_3) depends on δ (the fraction of agents who know the true state) in an intuitive way. As δ increases the set P_s shrinks and when more than 0.5 of the population is informed ($\delta > 0.5$) the only price function that is $CKRMC$ is the REE. Thus, when $\delta > 0.5$ the assumption of rationality and knowledge of rationality is sufficient to select the REE[14]. (Without assuming that the price function is known a-priori.)

Consider now the case where all the agents have the same initial endowment and $\delta < 0.5$. In this case $CKRMC$ allows for trade despite the fact that it is common knowledge that there are no gains from trade (all the agents have the same utility and the same initial endowment) and furthermore it is common knowledge that trade benefits agents in I_1 at the expense of some of the agents in I_2. The point is that when agents may have different beliefs about the price function and when the fraction of agents who are uninformed is high enough common knowledge of rationality does not preclude the

[14] Since there are just two states in our example the set of outcomes that is consistent with (just) rationality and knowledge of rationality equals the set of outcomes that are consistent with common knowledge of rationality. In particular, $P_s(\{1,3\})$ is the set of prices that are consistent with rational behavior in state s. When $\delta > 0.5$ $P_1(\{1,3\})$ and $P_3(\{1,3\})$ are disjoint and therefore an agent who knows that all the other agents are behaving rationally can infer the state from the price.

possibility that each uninformed agent is optimistic and believes that he is making a profit at the expense of other uniformed agents. The result that speculative trade is consistent with common knowledge of rationality hinges on the following two properties of $CKRMC$: (1) Different agents may have different beliefs on the set of price functions. (2) Each agent does not know the probability distributions of the other agents. Property (2) distinguishes $CKRMC$ from the solution concept that is studied by MacAllister(1990) and Dutta and Morris(1997) and which is based on the assumption that the beliefs of the players are common knowledge. To appreciate the importance of property (2) we note that when $\delta > 0$ it is impossible to obtain trade, even with different beliefs, if these beliefs are common knowledge. The reason for this impossibility is that the beliefs of the uninformed agents determines their demands. So if an agent i in I_2 knows these beliefs he knows the aggregate demand of the uninformed agents. Since the aggregate amount of X is known the agent i can infer the aggregate demand of the informed agents. However, the aggregate demand of the informed agents reveals the state. Thus, if an uninformed agent i knows the beliefs of the other agents and observes the price p he can infer the true state and if everyone infers the true state there is no trade. Indeed for every $\delta > 0$ the REE is the only solution in the models of M and DM.

We now turn to the proof of theorem 1.

Proof: Start with part 1. If $(p, \widehat{s})$ is $CKRMC$ then there exists a set of price functions F that is $CKRMC$ and a function $\widehat{f} \in F$ such that $\widehat{f}(\widehat{s}) = p$. Define

$\widehat{S} \equiv \{s : \exists f \in F \text{ s.t. } f(s) = p\}$. Clearly, $\widehat{s} \in \widehat{S}$. We now show that p is $EXPR$ w.r.t to $\widehat{S}$. So let $\overline{s} \in \widehat{S}$ we have to show that there exists a profile of probabilities $\{\gamma_i^{\overline{s}}\}_{i \in I}$, $\gamma_i^{\overline{s}} \in \triangle(\widehat{S} \cap \Pi_i(\overline{s}))$, and a profile of demands $\{x_i^{\overline{s}}\}_{i \in I}$ s.t $x_i^{\overline{s}}$ is optimal for player i w.r.t $\gamma_i^{\overline{s}}$ at the price p and $\int_i x_i^{\overline{s}} = \int_i e_i(\overline{s})$. Let $f \in F$ be a function such that $f(\overline{s}) = p$. Since F is $CKRMC$ there exists a profile of beliefs on F, $\left\{\mu_i^f\right\}_{i \in I}$, and a profile demands $\left\{z_i^f(\Pi_i(s), f(s)) : s \in S\right\}_{i \in I}$ such that in every state s the aggregate demand equals the aggregate supply and $z_i^f(\Pi_i(s), f(s))$ is optimal w.r.t $\mu_i^f(\cdot\ |\Pi_i(s), f(s))$. In particular, these properties are satisfied in the state $\overline{s}$. Now, $\mu_i^f(\cdot\ |\Pi_i(\overline{s}), f(\overline{s})) = \mu_i^f(\cdot\ |\Pi_i(\overline{s}), p) \in \triangle(\widehat{S} \cap \Pi_i(\overline{s}))$ and therefore by defining $\gamma_i^{\overline{s}} = \mu_i^f(\cdot\ |\Pi_i(\overline{s}), p)$ and $x_i^{\overline{s}} = z_i^f(\Pi_i(\overline{s}), p)$ we have defined probabilities and demands which satisfy the requirements in the definition of $EXPR$.

We turn now to the proof of the second part of the theorem. Let $\overline{f}$ be a fully revealing REE and let $\overline{p}_s \equiv \overline{f}(s)$. Let $(p, \widehat{s})$ be an outcome which is $EXPR$ where $p \neq \overline{p}_s$ for every $s \in S$. We want to show that there exists a price function $f_{\widehat{s}}$that is $CKRMC$ such that $f_{\widehat{s}}(\widehat{s}) = p$. To prove this we now define a price function f_s for every $s \in S(p)$ as follows:

$$(3.4) \quad f_s(s') \equiv \begin{cases} p & s' = s \\ \overline{p}_{s'} & s' \neq s \end{cases}$$

We will show that the set $F \equiv \{f_s :\ s \in S(p)\}$ is $CKRMC$. Since $f_{\widehat{s}} \in F$ this will complete the proof. So let $s \in S(p)$ we need to show that there exists a profile of beliefs

$\left\{\mu_i^{f_s}\right\}_{i\in I}$ on F and demands $\left\{z_i^{f_s}(\Pi_i(s'), f(s')) : s' \in S\right\}_{i\in I}$ such that the demands clear the market and are optimal w.r.t the beliefs. First, we observe that for any probability distribution μ_i on F and for any state $s' \in S$ $\mu_i(\ \cdot\, |\Pi_i(s'), \overline{p}_{s'})$ assigns probability 1 to the state s' (because for every $f \in F$ $f(\widetilde{s}) = \overline{p}_{s'}$ implies $\widetilde{s} = s'$.) Therefore, if for every $s' \neq s$ we define the demand $z_i^{f_s}(\Pi_i(s'), f(s')) = z_i^{f_s}(\Pi_i(s'), \overline{p}_{s'})$ to be the optimal bundle for player i at the price $\overline{p}_{s'}$ in the state s' then we have satisfied the requirements for rationality and market clearing in state s' for any belief μ_i on F.(Market clearing follows because $\overline{p}_{s'}$ is the clearing price in state s' in the fully revealing REE $\overline{f}$.) So the only question is how to define the demands $z_i^{f_s}(\Pi_i(s), f(s))$ (which is $z_i^{f_s}(\Pi_i(s), p)$) and the beliefs $\mu_i^{f_s}$ so that the requirements of market clearing and rational choice are satisfied in the state s. Since (p, s) is $EXPR$ there exists a profile of probabilities, $\{\gamma_i^s\}_{i\in I}$, $\{\gamma_i^s\} \in \triangle(S(p) \cap \Pi_i(s))$, and a profile of demands $\{x_i^s\}_{i\in I}$ such that x_i^s is an optimal choice for player i w.r.t γ_i^s at the price p and such that the aggregate demand equals the aggregate supply. We now define $z_i^{f_s}(\Pi_i(s), f(s)) = x_i^s$ and establish the result by showing that we can define probabilities $\left\{\mu_i^{f_s}\right\}_{i\in I}$ on F so that

(3.5)
$$\mu_i^{f_s}(\ \cdot\, |\Pi_i(s), p) = \gamma_i^s$$

To show this we rely on the following lemma which is proved in the appendix.

Lemma 1.1: Let $\alpha_1, ..., \alpha_m$ be m positive numbers and let $\gamma = (\gamma_1, ..., \gamma_m)$ be a probability vector. There exists a probability vector $\delta = (\delta_1, ..., \delta_m)$ which solves the following system of equations:

$$\gamma_k = \frac{\alpha_k \cdot \delta_k}{\sum_{j=1}^m \alpha_j \cdot \delta_j} \quad k = 1, ..., m$$

We map the lemma to our proof as follows: Suppose that $S(p) \cap \Pi_i(s)$ is the set $\{1, .., m\}$. Define $\alpha_j \equiv \alpha(j)$, the prior probability of the state j and $\gamma_k \equiv \gamma_i^s(k)$, the probability of the state k according to γ_i^s. The lemma says that if we define $\mu_i^{f_s}(f_k)$, the probability of the price function f_k to be δ_k (and the probability of a price function different from $f_1, ..., f_m$ to be zero) then the equation (3.5) is satisfied. This follows because Bayesian updating implies that for $k = 1, ..., m$

$$\mu_i^{f_s}(k\ |\Pi_i(s), p) = \frac{\alpha(k) \cdot \mu_i^{f_s}(f_k)}{\sum_{j=1}^m \alpha(j) \cdot \mu_i^{f_s}(f_j)}$$[15]

and therefore $\mu_i^{f_s}(k\ |\Pi_i(s), p) = \gamma_i^s(k)$.

This completes the proof of the theorem.

[15] The fact that Bayesian updating is given by the equation above relies on the assumption that $p \neq \overline{p_s}$ for every $s \in S$. This is the only point where this assumption is used.

We now present an example which shows two things: First, the possibility of non-existence of a price function that is $CKRMC$ (and therefore non-existence of an outcome (p, s) which is $CKRMC$.) Second, the possibility of a difference between the set of outcomes that are $CKRMC$ and the set of outcomes that are $EXPR$. The example is similar to examples of non-existence of REE that were given by Kreps(1977) and Allen(1986). However, the argument which establishes non-existence of $CKRMC$ is a bit more involved. In particular, we will demonstrate later on that non-existence of REE does not imply non-existence of price functions that are $CKRMC$.

Example 2: The example is a simple variation on example 1.

There are two states, $S = \{1, 2\}$. The probability of each state is 0.5. The set of agents is $I = [0, 1]$ where agents in $I_1 = [0, \delta]$ know the true state and agent in $I_2 = (\delta, 1]$ don't know it. The utility of an agent in I_1 is $u_1(x, m, s) = a_s \cdot \log(x) + m$. The utility of an agent in I_2 is $u_2(x, m, s) = b_s \cdot \log(x) + m$. The aggregate amount of X is 1 and the number of units of M that each agent has exceeds $\text{Max}\{a_s, b_s : \ s = 1, 2\}$. All this implies that if p is the price of X in units of M then the demand for X of an agent in I_1 in state s is $\frac{a_s}{p}$ and the demand of an agent in I_2 who assigns probability $\gamma(s)$ to the state s is $\frac{\gamma(1) \cdot b_1 + \gamma(2) \cdot b_2}{p}$.

We make the following assumption:

(3.6) $\quad a_1 > a_2 \quad$ and $b_1 < b_2$

(3.7) $\quad$ There exists a number $\widehat{p}$ such that

$$a_1 \cdot \delta + b_1(1 - \delta) = a_2 \cdot \delta + b_2(1 - \delta) = \widehat{p}$$

We claim that under these assumptions $FCKRMC = \emptyset$.

To prove this we compute, first, the set of outcomes that are $EXPR$. Let $\gamma = \{\gamma_i\}_{i \in I_2}$ be a profile of probabilities on S, (agents in I_1 assign probability 1 to the true state), and let $x_s^p(\gamma)$ denote the aggregate demand for X in the state s at the price p when the profile is γ. Since $b_1 < b_2$ the demand of each agent in I_2 is increasing in the probability which he assigns to the state 2. It follows that for every profile γ

(3.8) $\ x_1^p(\gamma) \geq \frac{a_1 \cdot \delta + b_1(1-\delta)}{p}$ and

(3.9) $\ x_2^p(\gamma) \leq \frac{a_2 \cdot \delta + b_2(1-\delta)}{p}$.

Now we claim that (3.7)-(3.9) and the fact that the aggregate supply of X is 1 imply that the only outcomes that are $EXPR$ are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$. To see that we, first, observe that $\widehat{p}$ is the clearing price in state s when every agent in I_2 assigns the state s probability 1 and therefore $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$ are $EXPR$. Now, assume by contradiction that there exists $p \neq \widehat{p}$ such that $(p, 1)$ is $EXPR$. It follows from (3.8) that if $p < \widehat{p}$ then for every profile of probabilities γ $x_1^p(\gamma) > 1$,but this is impossible because the aggregate amount of X is 1. If $p > \widehat{p}$ then (3.9) implies that for every profile γ $x_2^p(\gamma) < 1$ which means that p cannot be a clearing price in state 2. It follows that p cannot be $EXPR$ w.r.t the set S. Clearly, p cannot be $EXPR$ w.r.t $\{1\}$ and therefore we have obtained a contradiction. A similar argument establishes that $\widehat{p}$ is the only price that is $EXPR$ in the state 2. It follows from this and from part 1 of theorem 1 that the only price function that could possibly be $CKRMC$ is the function $\widehat{f}$ where $\widehat{f}(1) = \widehat{f}(2) = \widehat{p}$, but it is impossible to support $\widehat{f}$, because when each agent in I_2 assigns $\widehat{f}$ probability 1(which he must because there is no other function that is $CKRMC$) his posterior on the states (upon observing the price $\widehat{p}$) is the prior, probability 0.5 for each state, but with such a posterior the aggregate demand does not equal the aggregate supply. It follows that $\widehat{f}$ is not $CKRMC$ and therefore $FCKRMC = \emptyset$.

Example 2 is an example where a REE does not exist and where the set of outcomes that are $EXPR$ is different from the set of outcomes that is $CKRMC$. Neither one of these properties implies the other. In the appendix we present two examples, examples 3 and 4, which demonstrate this point. In example 3 there is a fully revealing REE with a price $\widehat{p}_1$ in state 1. The price $\widehat{p}_1$ is $EXPR$ in the two other states- states 2 and 3-but it is not $CKRMC$ in these states. Example 4 is an example of an economy with two states in which there is no REE and yet there is a segment of prices that are $CKRMC$ in both states. Furthermore, the set of outcomes that are $CKRMC$ equals the set of outcomes that are $EXPR$.

4 $CKRMC$ in economies with two commodities.

In this section we characterize the set of $EXPR$ outcomes in a class of economies with two commodities, X and M, in which the utility function of each agent is quasi-linear w.r.t M. We then apply theorem 1 and proposition 3.1 to obtain implications regarding the set of $CKRMC$ outcomes. The class of economies that are studied includes the examples in sections 2 and 3. Here, however, we allow for any finite number of types of agents where a type is characterized by a utility function and an information partition. We provide a characterization of the set of prices, P_s,that are $EXPR$ in each state $s \in S$. This characterization is useful in several ways. First, it extends the qualitative results which were obtained for example 1 to this more general class of economies. In particular, the characterization implies that for a robust[16] class of economies there is a whole segment of prices that are $EXPR$ in every state and therefore the observation of a price in this segment does not exclude any state. Second, we derive a corollary on the effect of refining the knowledge of agents on the set of $EXPR$ prices in a given state. Finally, the characterization result makes it possible to solve the system, i.e., compute P_s for every $s \in S$, by a simple procedure which involves (only) n^2 calculations of Walrasian equilibrium prices in complete information economies, where n is the number of states.

[16] A set of economies is robust if it contains an open set. In the appendix we define a metric on the space of economies.

We turn now to the formal description.

The commodities are denoted by X and M.

The set of states is $S = \{1, ..., n\}$.

The set of agents is $I = [0, 1]$. There are L types of agents, so $I = \cup_{l=1}^{L} I_l$ and $I_j \cap I_k = \emptyset$ for $j \neq k$. We let λ_l denote the measure of the set I_l. All the agents in I_l, $1 \leq l \leq L$, have the same utility function $\overline{u}_l(x, m, s) = u_l(x, s) + m$, the same initial bundle $e_l = (\overline{x}_l, \overline{m}_l)$, and the same information partition Π_l. We make the following assumptions:

(1) The function u_l, as a function of x, is strictly monotonic, strictly concave, and twice continuously differentiable. For every $s \in S$ $\lim_{x \to 0} u_l'(x, s) = \infty$ and $\lim_{x \to \infty} u_l'(x, s) = 0$. Also, $\overline{m}_l > 0$ for every $l = 1, ...L$.

(2) For every $x \in R$, $s, s' \in S$, such that $s > s'$ $u'(x, s) > u'(x, s')$. That is, the marginal utility from X increases in s.

(3) The elements of Π_l are segments of states. That is if $\pi \in \Pi_l$ then there exist $\overline{s}$ and $\underline{s}$, $\overline{s} > \underline{s}$, such that $\pi = \{s : \underline{s} \leq s \leq \overline{s}\}$.

Let $p \in R$ denote the price of a unit of X in terms of units of M.

For every $\widehat{s} \in S$ and $\widehat{S} \subseteq S$ such that $\widehat{s} \in \widehat{S}$ define $P(\widehat{s}, \widehat{S})$ to be the set of all the prices p with the following property: there exists a profile of probabilities $\gamma = (\gamma_i)_{i \in I}$, $\gamma_i \in \triangle(\Pi_i(\widehat{s}) \cap \widehat{S})$, such that p is an equilibrium price w.r.t γ. Thus, $P(\widehat{s}, \widehat{S})$ is the set of equilibrium prices that can be generated in $\widehat{s}$ when the support of the probability distribution of an agent i is contained in $\widehat{S}$ and in his information set in the state $\widehat{s}$. Define:

$P(\widehat{S}) \equiv \cap_{\widehat{s} \in \widehat{S}} P(\widehat{s}, \widehat{S})$ and

$P_{\widehat{s}} \equiv \cup_{\widehat{s} \in \widehat{S}} P(\widehat{S})$

Thus, $P(\widehat{S})$ is the set of prices that are $EXPR$ w.r.t the set $\widehat{S}$ and $P_{\widehat{s}}$ is the set of prices that are $EXPR$ w.r.t the state $\widehat{s}$.

We now characterize the set $P(\widehat{s}, \widehat{S})$.

Define $\overline{p}(\widehat{s}, \widehat{S})$ to be the equilibrium price implied by the profile $\overline{\gamma} = \{\overline{\gamma}_i\}_{i \in I}$ where $\overline{\gamma}_i$ assigns probability 1 to the maximal state in the set $\Pi_i(\widehat{s}) \cap \widehat{S}$[17]. Similarly, we define $\underline{p}(\widehat{s}, \widehat{S})$ to be the equilibrium price implied by the profile $\underline{\gamma} = \left\{\underline{\gamma}_i\right\}_{i \in I}$ where $\underline{\gamma}_i$ assigns probability 1 to the minimal state in the set $\Pi_i(\widehat{s}) \cap \widehat{S}$.

Proposition 4.1: Let $\widehat{s} \in S$ and $\widehat{S} \subseteq S$ such that $\widehat{s} \in \widehat{S}$ then $P(\widehat{s}, \widehat{S}) = \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$.

The intuition behind this result is very simple. Since the marginal utility from X increases with s (assumption 2) then for any price p the demand of each agent i for X increases when i assigns a higher probability to a higher state. It follows that for any price p the maximal aggregate demand for X, in the state $\widehat{s}$ when beliefs are restricted to the set $\widehat{S}$, is obtained when each agent i assigns probability 1 to the maximal state in $\Pi_i(\widehat{s}) \cap \widehat{S}$. It follows that $\overline{p}(\widehat{s}, \widehat{S})$ is the maximal price in $P(\widehat{s}, \widehat{S})$. Similarly $\underline{p}(\widehat{s}, \widehat{S})$ is the minimal price in $P(\widehat{s}, \widehat{S})$. The result that every price p, $\underline{p}(\widehat{s}, \widehat{S}) \leq p \leq \overline{p}(\widehat{s}, \widehat{S})$, can be obtained as an equilibrium price for some profile of probabilities γ_p follows from the

[17] The existence and uniquness of $\overline{p}(\widehat{s}, \widehat{S})$ is proved in the sequel.

continuity of the equilibrium price in the beliefs. The formal proof of the proposition is given in the appendix.

Let $\underline{s}$,$\overline{s} \in S$, $\underline{s}<\overline{s}$. Define $[\underline{s}, \overline{s}] \equiv \{s : \underline{s} \leq s \leq \overline{s}\}$.

The following proposition is a simple consequence of proposition 4.1.

Proposition 4.2:

(a) Let $\underline{s}$,$\overline{s} \in S$, $\underline{s}<\overline{s}$. $P([\underline{s}, \overline{s}]) = \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$ [18]

In words, $P([\underline{s}, \overline{s}])$ is the segment of prices where the lowest (highest) price is the lowest (highest) equilibrium price in the maximal (minimal) state in $[\underline{s}, \overline{s}]$ when the support of the probability of agent i is contained in $\Pi_i(\overline{s}) \cap [\underline{s}, \overline{s}]$ ($\Pi_i(\underline{s}) \cap [\underline{s}, \overline{s}]$).

(b) $P_s = \cup_{s \in [\underline{s}, \overline{s}]} P[\underline{s}, \overline{s}]$.

The proof of the proposition is in the appendix.

The characterization of $EXPR$ outcomes in proposition 4.2 has several implications for $CKRMC$ outcomes: First, it follows from theorem 1 that for a generic set of economies proposition 4.2 provides a characterization of $CKRMC$ outcomes modulo outcomes which involve the REE equilibrium prices. Second, it follows from proposition 4.2 that P_s contains a segment of prices whenever there exist states $\underline{s}$ and $\overline{s}$, $\underline{s} \leq s \leq \overline{s}$, such that $\overline{p}(\underline{s}, [\underline{s}, \overline{s}]) > \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. In the appendix we use this result to prove that the set of economies in which there is a segment of prices that are $EXPR$ in every state is robust. Proposition 3.1 implies that for an economy in this subclass the set of $EXPR$ outcomes equals the set of $CKRMC$ outcomes without any qualification.

The characterization that is obtained in proposition 4.2 extends the qualitative properties of the solution of example 1. First, as we have pointed out, there exists a robust subclass of economies in which there is a whole segment of prices that are $EXPR$ in every state, the observation of a price in this segment does not exclude any state. Another implication of proposition 4.2 has to do with the effect of refining the knowledge of agents on the set of $EXPR$ prices. To describe this implication we need to introduce some additional notation. Given a subset of agents I' we want to consider the economy $E^{I'}$ which is obtained from the original economy E by refining the knowledge of agents in I' so that each agent in I' has complete information on S. We will denote different terms which refer to $E^{I'}$ by adding a superscript I'. In particular, the set of prices that are $EXPR$ in a state s in the economy $E^{I'}$ will be denoted by $P_s^{I'}$.

Claim 4.3: Let $s \in S$ be a state such that P_s strictly contains the (singleton) set $P(\{s\})$. For any number $\mu > 0$ there exists a set of agents I' of a measure that is smaller than μ such that $P_s^{I'}$ is strictly contained in P_s. [19]

The proof of the claim is in the appendix.

Finally, the characterization in proposition 4.2 implies that it is possible to solve the economy, i.e., compute P_s for every $s \in S$, by a procedure which involves only n^2 calculations of Walrasian equilibrium prices. To see this we note that it follows from

[18] If $\underline{p}(\overline{s}, [\underline{s}, \overline{s}]) < \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$ then the RHS is the empty set.

[19] It is easy to see that weak containement is implied by any refinement of the knowledge of agents.

the first part of proposition 4.2 that for every $\underline{s}, \overline{s} \in S$, $\underline{s} < \overline{s}$, the computation of the set $P([\underline{s}, \overline{s}])$ involves two calculations of Walrasian equilibrium prices (i.e., $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ and $\overline{p}(\underline{s}, [\underline{s}, \overline{s}])$). In addition for every $s \in S$ we calculate $P(\{s\}) = P([s, s])$, that is, the WE price in the complete information economy that is defined by the state s. All this involves $2 \times \frac{n(n-1)}{2} + n = n^2$ calculations of WE prices. Now, by the second part of proposition 4.2 every set P_s is just a union of the sets $P([\underline{s}, \overline{s}])$ for $\underline{s}, \overline{s} \in S$ such that $s \in [\underline{s}, \overline{s}]$.

5 Properties of CKRMC

In this section we present two properties of $CKRMC$. Theorem 2 establishes that the set of $CKRMC$ functions, $FCKRMC$, is obtained by a procedure in which functions that cannot be supported by any profile of beliefs are iteratively deleted. Furthermore, the procedure terminates after a finite number of steps.

The second result, theorem 3, addresses the following concern: Our solution notion does not include a description of what one player knows, or believes, about another player so the reader might ask in what sense is a function that is $CKRMC$ indeed consistent with common knowledge of rationality and market clearing. On an intuitive level if a function f can be supported by beliefs on functions that are consistent with rational behavior and if each one of these functions, in turn, can be supported by such beliefs and so forth then f is consistent with common knowledge of rationality and market clearing. Theorem 3 establishes that this argument can be made precise by embedding our model in a richer model which includes beliefs of players about the beliefs of other players.

1. $FCKRMC$ as the result of iterative deletion of functions that cannot be supported.

Let F be a set of price functions. We let $J(F)$ denote the set of price functions that can be supported w.r.t F. Define F^k, $k = 0, 1, 2, ...$ inductively as follows: $F^0 = R^{n \times (K-1)}$ and $F^{k+1} = J(F^k)$. Define $F^\infty = \cap_{k=0}^{\infty} F^k$.

Theorem 2:
$(a.)$ $F^\infty = FCKRMC$.
$(b.)$ There exists a number M such that $F^\infty = F^M$.
$(c.)$ Let $f \in FCKRMC$. There exists a finite set of functions $F(f)$ such that $f \in F(f)$ and $F(f)$ is $CKRMC$.

The proof of the theorem is given in the appendix.

2. Common Knowledge of Rationality and Market clearing.

We construct a model where each state contains a complete description of the system including the price function, the demand functions, the beliefs of each player, his beliefs about the beliefs of other players and so forth. Theorem 3 establishes that a price function is $CKRMC$ iff it is materialized in a state of some model that is consistent with common knowledge of rationality and market clearing.

Say that an abstract measurable space (Ω, β)[20] is a *model* for a given economy E if each state $\omega \in \Omega$ specifies :

1. A price function f^ω.

2. A profile of demand strategies $\{z_i^\omega\}_{i \in I}$.

3. A profile of beliefs $\{\mu_i^\omega\}_{i \in I}$ where $\mu_i = (\mu_i^{\omega,1}, ..., \mu_i^{\omega,m})$ is a finite lexicographic sequence of probabilities on Ω such that for every $i \in I$ and for every index $k, 1 \leq k \leq m$, $\mu_i^{\omega,k}$ assigns probability 1 to states in which the demand strategy and the belief of player i are z_i^ω and μ_i^ω respectively. (This requirement reflects the assumption that player i knows his demand and belief.) In addition the transformation T_f which associates with each state $\omega \in \Omega$ the price function that is specified in ω is measurable and $F(\Omega) \equiv \{T_f(\omega) : \omega \in \Omega\}$ is a Borel set.

We assume that the belief of each player i on Ω is independent of his belief α on S.(α is the common prior on S). Thus, the (ex-ante) belief of a player i at a state ω on the space $S \times \Omega$ is $\alpha \times \mu_i^\omega \equiv (\alpha \times \mu_i^{\omega,1}, ..., \alpha \times \mu_i^{\omega,m})$[21]. We now follow a notation which is similar to the one we have used in definition of $CKRMC$ in section 2. We let $[l_i, p]$ denote the event in $S \times \Omega$ which is consistent with l_i and p. That is,

$[l_i, p] = \{(s, \omega) : s \in S,\ \omega \in \Omega\ \ \Pi_i(s) = l_i \text{ and } f^\omega(s) = p\}$

We say that a belief $\mu_i, \mu_i = (\ \mu_i^1, ..., \mu_i^m)$, of player i is *consistent* with the event $[l_i, p]$ if there exists k,1$\leq\ k \leq m$, such that $\alpha \times \mu_i^k(l_i, p) > 0$. Given a belief μ_i and an event $[l_i, p]$ which is consistent with it we allow for some abuse of notation and let $\mu_i(\cdot\ |[l_i, p])$ denote the marginal distribution of $\alpha \times \mu_i^k(\cdot\ |[l_i, p])$ <u>on S</u>, where k is the lowest index with the property that $\alpha \times \mu_i^k\,[l_i, p] > 0$. $\mu_i(\cdot\ |[l_i, p])$ is the posterior on S of a player i with a belief μ_i given the event $[l_i, p]$.

Say that a model (Ω, β) is *consistent with common knowledge of rationality and market clearing*, henceforth $CKRMC$ if for each state $\omega \in \Omega$ the following two conditions are satisfied:

1.Rationality: for every $i \in I$ and (l_i, p) that is consistent with μ_i^ω $z_i^\omega(l_i, p)$ is an optimal bundle at the price p w.r.t $\mu_i^\omega(\cdot\ |[l_i, p])$. For every $i \in I$ and for every $s \in S$ $[\Pi_i(s), f^\omega(s)]$ is consistent with μ_i^ω. (just like in the definition of $CKRMC$.)

2. Market Clearing, for every $s \in S$ $\quad \int_i z_i^\omega(\Pi_i(s), f^\omega(s)) = \int_i e_i(s)$.

Conditions 1. and 2. are similar to the conditions in the definition of $CKRMC$. Here these conditions say that in every state $\omega \in \Omega$ every player is making a rational choice and markets clear .Now the point is that here the belief of a player i, μ_i, is on the <u>space Ω</u>, (that is, $\mu_i^k \in \triangle(\Omega), k = 1, ..., m$.) So each state ω describes the beliefs of each <u>player i</u> on the set of states in the model, which in turn describe the beliefs of each other

[20] β is the $\sigma-$algebra of measurable subsets of Ω.

[21] The complete set of states is $\overline{\Omega} \equiv S \times \Omega$ and the (ex-ante) belief of a player i in a state $(s, \omega) \in \overline{\Omega}$ is $\alpha \times \mu_i^\omega \equiv (\alpha \times \mu_i^{\omega,1}, ..., \alpha \times \mu_i^{\omega,m})$. Since this belief (being an ex-ante belief) does not depend on the state s it is more convenient to suppress the set of states S and work with the space Ω keeping in mind that the beliefs on Ω determine beliefs on $\overline{\Omega}$.

player on the set of states and so forth. In particular, each state ω describes what each player i believes about the beliefs of any other player j about the beliefs of any other player k and so forth. Since rationality and market clearing are satisfied in <u>every</u> state ω every proposition of the type, player i knows that player j knows that .... player k knows that everyone is rational and markets clear, is true in every state ω and therefore there is common knowledge of rationality and market clearing in every state $\widehat{\omega} \in \Omega$.

In the appendix we prove the following theorem.

<u>Theorem 3:</u> A price function f is $CKRMC$ iff there exists a model (Ω, β) that is consistent with common knowledge of rationality and market clearing and a state $\omega \in \Omega$ such that f is the price function that is specified in ω.

Theorem 3 makes precise the sense in which $FCKRMC$ is the set of price functions that are consistent with common knowledge of rationality and market clearing.

6 Beliefs

In this section we discuss two assumptions concerning the beliefs of the agents: (1) A belief of an agent consists of just a finite number of probability distributions.(Henceforth, finite beliefs.) (2) The belief on S is independent of the belief on F.

We explain why assumption 1 is in fact w.l.o.g. and point out how all our results (with obvious modifications) are valid in a model where the beliefs on S and F maybe correlated. We have assumed independence because it seems to us a plausible assumption in this context and thus our point is that our results can be obtained when it is assumed.

1. <u>Finite Beliefs.</u>

We are interested in a concept of rationality which requires an optimal choice given any realization of a signal and a price. That is, for any event (l_i, p) agent i should choose optimally w.r.t. some probability measure on price functions that are consistent with (l_i, p). Since there is a continuum number of such events a lexicographic belief of an agent i should in principle consist of a continuum number of probability measures. More precisely, it seems that a lexicographic belief μ_i for agent i should be a well ordered set of a continuum number of probability measures such that for every event (l_i, p) there exists some measure in the set μ_i which assigns it a positive probability. Rationality w.r.t. μ_i requires that the demand of agent i in (l_i, p) is optimal w.r.t. the minimal probability measure (according to the well-order on μ_i) which assigns (l_i, p) a positive probability. However, since S is finite then given a function f the number of events $(\Pi_i(s), f(s))$ is finite and therefore the number of probability measures in a given belief μ_i that are relevant ('activated') when f is supported is finite. It follows that the assumption that beliefs are finite is w.l.o.g.

2. <u>Independence.</u>

Let $FCCKRMC$ denote the set of price functions that are consistent with common knowledge of rationality and market clearing in a model where the belief of an agent on F maybe correlated with his belief on S. Specifically, a belief of an agent i in this "correlated model" can be any lexicographic sequence of probability measures on $S \times F$, $\mu_i = (\mu_i^1, ..., \mu_i^m)$, such that the marginal of each $\mu_i^k, 1 \leq k \leq m$, on S is the common prior α. Clearly, $FCKRMC \subseteq FCCKRMC$ and there are economies in which the inclusion is strict. However, all our results are valid in the correlated model as well. In particular, it is easy to see that the argument in the proof of part (a) of theorem 1 establishes that every $CCKRMC$ outcome (p, s) in the correlated model[22] is an $EXPR$ outcome. Since $FCKRMC \subseteq FCCKRMC$ it follows that theorem 1 and proposition 3.1 are true in the correlated model as well. Similarly, the arguments in the proof of theorem 2 apply in the correlated model as well. Regarding theorem 3, it is easy to see that when the belief of an agent on Ω can be correlated with his belief on S the same type of construction that was used in the proof of the theorem applies in the correlated model.

7 Conclusion

This research was motivated by the following question: What is the set of outcomes that are consistent with common knowledge of rationality and market clearing in an exchange economy with asymmetric information ? We have proposed the concept of $CKRMC$ as an answer to this question. The main difference between $CKRMC$ and REE is that $CKRMC$ allows for a situation where different agents have different beliefs on the price function. The main result, theorem 1, establishes that under a mild qualification the set of $CKRMC$ outcomes equals the set of $EXPR$ outcomes. Specifically, theorem 1 establishes that the set of outcomes that are consistent with common knowledge of rationality and market clearing in a model where each agent has a subjective probability on price functions that is independent of a common prior on S, is equal (under a mild qualification) to the set of outcomes that are consistent with common knowledge of rationality and market clearing in a model where each agent may have any subjective belief on S and furthermore his belief may be correlated with the state. Theorem 1 was used to characterize the set of $CKRMC$ outcomes in a general class of economies with two commodities. We have pointed out several properties of $CKRMC$ that stand in contrast to the full revelation property of REE. In particular, propositions 4.1 and 4.2 imply that in a robust class of economies, (1) There is a whole range of prices that are $CKRMC$ in every state (and therefore do not reveal the true state). (2) The set of $CKRMC$ outcomes is sensitive to the amount of information in the economy. (3) Trade is consistent with $CKRMC$ even when there is common knowledge that there are no mutual gains from it. Finally, we have shown that $FCKRMC$ can be obtained by a procedure in which price functions that cannot be supported by any profile of beliefs are

[22] An outcome (p, s) is $CCKRMC$ if there exists $f \in FCCKRMC$ such that $f(s) = p$.

iteratively deleted (theorem 2) and that $CKRMC$ can be supported in a model where there is a complete description of the beliefs of each agent on the beliefs of other agents (theorem 3).

References :

1. B. Allen,"General Equilibrium with Rational Expectations", chapter 1 in *Bayesian Models in Economic Theory*,(ed) M. Boyer and R. Kihlmstrom, North-Holland, 1986.

2. G. Desgranges,"Common Knowledge Equilibrium and Asymmetric Information in a Competitive Economy", mimeo, THEMA, July 2004.

3. G. Desgranges and R. Guesnerie, "Common Knowledge and the Information revealed through prices: some conjectures", in "Price expectations:new theoretical and empirical results", F. Gardes and G. Prat editors, Elgar, 52-62, 2000.

4. J. Dutta and S. Morris, "The Revelation of Information and Self-Fulfilling Beliefs", Journal of Economic Theory,1997,73,231-244.

5. R. Guesnerie, "Anchoring Economic Predictions in Common Knowledge", Econometrica, 2002, 70, 439-480.

6. D. Kreps, "A note on Fulfilled Expectations Equilibria", Journal of Economic Theory, 1977, 14, 32-43.

7. P. MacAllister, "Rational Behavior and Rational Expectations", Journal of Economic Theory, 1990, 52,332-363.

8. S. Morris, "Justifying Rational Expectations", CARESS working paper 1995.

9. R. Radner, "Rational Expectations Equilibrium: Generic existence and the Information revealed by prices", Econometrica, 1979, 47, 655-678.

Appendix

Section 3

Proof of Lemma 1.1:

First, we assume w.l.o.g that $\gamma_k > 0$ for every k because if this is not the case we define $\delta_j = 0$ if $\gamma_j = 0$ and proceed to prove the lemma for the set $\{k : \gamma_k > 0\}$.

Second, multiplying the equations by the denominator and subtracting the RHS from the LHS gives a system of m homogeneous linear equations in $\delta_1, .., \delta_m$ and therefore there exists a solution to this system, $\overline{\delta} = (\overline{\delta_1}, ..., \overline{\delta_m})$.

Third, if $\overline{\delta}$ is a solution and c is a constant then $c \cdot \overline{\delta}$ is also a solution.

Finally, since $\gamma_k > 0$ for all $k = 1, ..., m$ then if $\overline{\delta}$ is a solution then $\overline{\delta_1}, ..., \overline{\delta_m}$ all have the same sign which is the sign of the denominator.

It follows from all this that there is a solution $\widehat{\delta}$ to the system that is a probability vector because if $\overline{\delta}$ is some solution there is a constant c such that $c \cdot \overline{\delta}$ is a probability vector.

Proposition 3.1: Let E be an economy such that in each state $s \in S$ there are at least $2n$ $EXPR$ prices. A price function f is $CKRMC$ iff for every $s \in S$ there exist profiles of probabilities $\gamma = \{\gamma_i^s\}_{i \in I}, \gamma_i^s \in \Delta(\Pi_i(s) \cap S(f(s)))$ and demands $x = \{x_i^s\}_{i \in I}$ such that

(A.3.1) For every $i \in I$ x_i^s is an optimal bundle for player i w.r.t. the price $f(s)$ and the probability γ_i^s.

(A.3.2) For every $s \in S$ markets clear, that is, $\int_i x_i^s = \int_i e_i(s)$.

(A.3.3) If $s' \in \Pi_i(s)$ and $f(s) = f(s')$ then $x_i^s = x_i^{s'}$ and $\gamma_i^s = \gamma_i^{s'}$.

Furthermore, if f is $CKRMC$ there exists a finite set of functions $F(f)$ such that $f \in F(f)$ and $F(f)$ is $CKRMC$.

Proof of proposition 3.1 : Part (a) of theorem 1 establishes that if f is $CKRMC$ then for every $s \in S$ $(s, f(s))$ is an $EXPR$ outcome. This means that f satisfies conditions (A.3.1)-(A.3.2). Since the demand and posterior of each agent at a state s depends only on his private signal and the price at s f must satisfy condition (A.3.3) as well.

We will establish the second direction by proving proposition 3.2.

Proposition 3.2: Let $\mathcal{P} \equiv \{\overline{P}_s\}_{s \in S}$ be n set of prices such that each set $\overline{P}_s$ contains at least $2n$ prices that are $EXPR$ in s. Assume, also, that if $p \in \overline{P}_s$ then $p \in \overline{P}_{s'}$ for every $s' \in S(p)$. Define $F(\mathcal{P}) \equiv \{f : f(s) \in \overline{P}_s$ and f satisfies conditions (A.3.1)-(A.3.30$\}$. The set $F(\mathcal{P})$ is $CKRMC$.

The proof of proposition 3.1 from proposition 3.2 is immediate. Given a function f that satisfies conditions (A.3.1)-(A.3.3) and given the assumption that in each state s there are at least $2n$ $EXPR$ prices it is straightforward to define finite sets of prices

$\mathcal{P} = \{\overline{P}_s\}_{s \in S}$ which satisfy the provisions of proposition 3.2 and such that $f \in F(\mathcal{P})$.[23] It follows that f is $CKRMC$ and furthermore it belongs to a finite set of functions that is $CKRMC$. We now turn to the proof of proposition 3.2.

Proof of proposition 3.2: To make things as simple as possible it would be useful to consider, first, price functions which assign to each state a different price. Define,

$\widetilde{F}(\mathcal{P}) \equiv \{f : f \in F(\mathcal{P})\ f(s) \neq f(s') \text{ for } s \neq s'\}$. We will show now that $\widetilde{F}(\mathcal{P})$ is $CKRMC$. The idea behind the proof of the lemma is similar to the idea behind the proof of the second part of theorem 1. However, the construction here is a bit more involved. Let $f \in \widetilde{F}(\mathcal{P})$ we need to construct a profile of beliefs $\left\{\mu_i^f\right\}_{i \in I}$ and demand strategies $\left\{z_i^f\right\}_{i \in I}$ which support f w.r.t $\widetilde{F}(\mathcal{P})$. Assume $S = \{1, ..., n\}$ and define $p_s \equiv f(s)$.We have that for every $s \in S$ p_s is $EXPR$ in s and therefore there exist profiles of probabilities $\{\gamma_i^s\}_{i \in I}, \gamma_i^s \in \Delta(S(p_s) \cap \Pi_i(s))$, and demands $x_i^s, x_i^s \in R^K$, that support p in s. For every $s \in S$ we will define a probability measure $\mu_{s,i}^f$ on $\widetilde{F}(\mathcal{P})$ with the property that $\mu_{s,i}^f(|\Pi_i(s), p_s) = \gamma_i^s$. These measures will be defined in such a way so that their average measure, $\mu_i^f \equiv \frac{1}{n}\sum_{s \in S} \mu_{s,i}^f$,will satisfy

(A.3.4) $\mu_i^f(\cdot\,|\Pi_i(s), p_s) = \gamma_i^s$ for every $s \in S$.

Once we have such a measure μ_i^f we are basically done because we can then define a demand strategy for agent i, $z_i^f(l_i, p)$, as follows: If $(l_i, p) = (\Pi_i(s), p_s)$ for some s then $z_i^f(l_i, p) \equiv x_i^s$, otherwise we define $z_i^f(l_i, p)$ to be some bundle x which is optimal w.r.t $\mu_i^f(\cdot\,|l_i, p)$.[24] It is straightforward to check that the beliefs $\left\{\mu_i^f\right\}_{i \in I}$ and the demand strategies $\left\{z_i^f\right\}_{i \in I}$ support f w.r.t. $\widetilde{F}(\mathcal{P})$. We will now define the measures $\mu_{s,i}^f, s \in S$, so that the average measure μ_i^f satisfies (A.3.4). To do that we select n prices $\overline{p}_1, ..., \overline{p}_n$ so that for every $s \in S$ $\overline{p}_s \in \overline{P}_s$, for $s \neq s'$ $\overline{p}_s \neq \overline{p}_{s'}$, and for every $s', s \in S$ $\overline{p}_s \neq p_{s'}$.The assumption that for each state $s \in S$ $\overline{P}_s$ contains at least $2n$ prices ensures that such a selection is possible. Now, let $\widehat{s}$ be a particular state. To define $\mu_{\widehat{s},i}^f$ we define a price function f_s for every $s \in S(p_{\widehat{s}})$ as follows:

$$f_s(s') \equiv \begin{cases} p_{\widehat{s}} & s' = s \\ \overline{p}_{s'} & s' \neq s \end{cases}$$

The role of the prices $\overline{p}_1, ..., \overline{p}_n$ and the functions f_s $s \in S(p_{\widehat{s}})$ in the current proof is exactly similar to the role of the REE prices $\overline{p}_s$ $s \in S$ and the functions f_s $s \in S(p)$ in the proof of the second part of theorem 1. An argument which is identical to the one given in the proof of theorem 1 establishes that there exists a probability measure $\mu_{\widehat{s},i}^f$ on the set $\{f_s : s \in \Pi_i(\widehat{s}) \cap S(p_{\widehat{s}})\}$ such that $\mu_{\widehat{s},i}^f(\cdot\,|\Pi_i(\widehat{s}), p_{\widehat{s}}) = \gamma_i^{\widehat{s}}$. Furthermore, because

[23] Let $\widehat{P}_s, s \in S$, be n sets of prices such that each set $\widehat{P}_s$ contains $2n$ prices that are $EXPR$ in s. For each $\overline{s} \in \overline{S}$ define $\overline{P}_{\overline{s}} = \left\{p : p \in \cup_{s \in S}\widehat{P}_s \text{ and } p \text{ is } EXPR \text{ in } \overline{s}\right\}$. It is easy to see that $\left|\overline{P}_{\overline{s}}\right| \geq 2n$ and that $p \in \overline{P}_{\overline{s}}$ implies $p \in \overline{P}_{s'}$ for every $s' \in S(p)$.

[24] If $\mu_i^f(\cdot\,|l_i, p)$ is not defined then the demand can be diefined in an arbitrary way. (What is important is that $\mu_i^f(\cdot\,|l_i, p)$ is defined for every $(l_i, p) = (\Pi_i(s), p_s), s \in S$.)

for every $\widehat{s} \in S$ we have: (a) $p_{\widehat{s}} \neq p_{\widetilde{s}}$ for every $\widetilde{s} \neq \widehat{s}$ and (b) $p_{\widehat{s}} \neq \overline{p}_s$ for every $s \in S$, we obtain that the average measure, μ_i^f, satisfies $\mu_i^f(\cdot\,|\Pi_i(\widehat{s}), p_{\widehat{s}}) = \gamma_i^{\widehat{s}}$ for every $\widehat{s} \in S$.

We have, thus, shown that every price function f in $\widetilde{F}(\mathcal{P})$ can be supported w.r.t $\widetilde{F}(\mathcal{P})$ and therefore also w.r.t $F(\mathcal{P})$. Now let f be a price function such that $f \in F(\mathcal{P}) \backslash \widetilde{F}(\mathcal{P})$. It is easy to see that the same definition of the probabilities $\mu_{\widehat{s},i}^f, \widehat{s} \in S, i \in I$, with the additional condition that if $f(\widehat{s}) = f(s')$ and $s' \in \Pi_i(\widehat{s})$ then $\mu_{\widehat{s},i}^f = \mu_{s',i}^f$, works.

The proof of proposition 3.2 and with it the proof of proposition 3.1 are now complete.

Example 3 :

This is an example where a REE exists and yet the set of $CKRMC$ outcomes is different from the set of $EXPR$ outcomes. Specifically we construct an economy in which there is a fully revealing REE with a price $\widehat{p_1}$ in state 1. The price $\widehat{p_1}$ is $EXPR$ in the two other states- states 2 and 3-but it is not $CKRMC$ in these states.

There are three states in the economy, $S = \{1, 2, 3\}$ and two commodities X and M. There are three sets of agents $I_1 = [0, \delta]$, $I_2 = \left(\delta, \frac{1+\delta}{2}\right]$, $I_3 = \left(\frac{1+\delta}{2}, 1\right]$. Agents in I_1 know the true state. The others don't know anything. The prior on S, α,could be any probability distribution with full support. The utility of agent $i, u_i(x, m, s)$, is $a_s \log(x) + m$ if $i \in I_1$ it is $b_s \log(x) + m$ if $i \in I_2$ and it is $c_s \log(x) + m$ if $i \in I_3$. We assume that the aggregate amount of $X, \overline{X}$, is 1 and that each agent i has enough money, that is, $\overline{m_i} \geq \max\{a_s, b_s, c_s : s \in S\}$ (where $\overline{m_i}$ is the initial amount of money of agent i.)

We assume:

A.3.5 $a_1 > a_2 = a_3$; $b_1 < b_2 < b_3$; $c_1 < c_3 < c_2$

A.3.6 $\widehat{p_1} \equiv a_1 \cdot \delta + (b_1 + c_1) \cdot \frac{(1-\delta)}{2} > \widehat{p_2} \equiv a_2 \cdot \delta + (b_2 + c_2) \cdot \frac{(1-\delta)}{2} > \widehat{p_3} \equiv a_3 \cdot \delta + (b_3 + c_3) \cdot \frac{(1-\delta)}{2}$

A.3.7 $\widehat{p_1} = a_2 \cdot \delta + (b_3 + c_2) \cdot \frac{(1-\delta)}{2} = a_3 \cdot \delta + (b_3 + c_2) \cdot \frac{(1-\delta)}{2}$

The inequalities in A.3.6 imply that the price function $\widehat{f}(s) = \widehat{p_s}$ is a fully revealing REE. (This follows because the equations in A.3.6 imply that the aggregate demand for X in the state s at the price $\widehat{p_s}$ when everyone assigns probability 1 to s is equal to the aggregate supply, $\widehat{X} = 1$.) The equalities in A.3.7 imply that the price $\widehat{p_1}$ is $EXPR$ w.r.t S. In both states 2 and 3 $\widehat{p_1}$ is supported by the profile of probabilities $\widehat{\gamma} = \{\widehat{\gamma}_i\}_{i \in [0,1]}$ in which each agent in I_2 assigns probability 1 to the state 3 while each agent in I_3 assigns probability 1to the state 2. (Obviously, each agent in I_1 assigns probability 1 to the true state.)

We now show that $(\widehat{p_1}, 2)$ and $(\widehat{p_1}, 3)$ are not $CKRMC$ outcomes. First, we observe that the profile $\widehat{\gamma}$ which supports $\widehat{p_1}$ in states 2 and 3 generates the maximal aggregate demand for X in these states. Any other profile of probabilities will lead to a smaller demand and therefore to a lower price. It follows that any price that is $EXPR$ in state 2 or state 3 is smaller or equal to $\widehat{p_1}$. We now check which prices are $EXPR$ in state 1. Because b_1 is smaller than b_2 and b_3 and because c_1 is smaller than c_2 and c_3 the demand of an agent in $I_2 \cup I_3$ is minimal when he assigns the state 1 probability 1. It follows that if p_1 is a price that is $EXPR$ in state 1 then $p_1 \geq \widehat{p_1}$. Now since a price p_1 that is higher than $\widehat{p_1}$ is not $EXPR$ in states 2 or 3 we obtain that the only price that is $EXPR$ in state 1 is $\widehat{p_1}$.[25] It follows (Theorem 1, part 1) that $\widehat{p_1}$ is also the only price that is $CKRMC$ in state 1. This means that every price function that is $CKRMC$ receives the value $\widehat{p_1}$ in state 1. Therefore, an agent who has a probability distribution on functions that are $CKRMC$ and who observes the price $\widehat{p_1}$ will assign state 1 a (conditional) probability which is at least the prior probability of this state. However, as we have seen, a profile of probabilities that assign the state 1 a positive probability cannot support $\widehat{p_1}$ in the states 2 and 3 and therefore $(\widehat{p_1}, 2)$ and $(\widehat{p_1}, 3)$ are not $CKRMC$ outcomes.

Example 4:

This is an example of an economy with two states in which there is no REE and yet there is a segment of prices that are $CKRMC$ in both states.

There are two states, $S = \{1, 2\}$, and each one of them has a probability 0.5. There are three sets of agents : $I_1 = [0, \delta]$, $I_2 = \left(\delta, \frac{1+\delta}{2}\right]$, $I_3 = \left(\frac{1+\delta}{2}, 1\right]$. Agents in I_1 know the true state. The others don't know it. The utilities of the agents are similar to those defined in the previous example so $u_i(x, m, s)$ is $a_s \log(x) + m$ if $i \in I_1$ it is $b_s \log(x) + m$ if $i \in I_2$ and it is $c_s \log(x) + m$ if $i \in I_3$. Also, the aggregate amount of X is 1 and each agent has enough money.

We assume:

A.3.8 $a_1 > a_2$; $b_1 < b_2$; $c_1 > c_2$.

A.3.9 $\widehat{p} \equiv a_1 \cdot \delta + (b_1 + c_1) \cdot \frac{(1-\delta)}{2} = a_2 \cdot \delta + (b_2 + c_2) \cdot \frac{(1-\delta)}{2}$

The equality in A.3.9 implies non-existence of a REE. The argument is familiar: Full revelation would imply that the price which clears the market is $\widehat{p}$ in both states. However, if that is the case then $\widehat{p}$ does not reveal the true state. On the other hand there cannot be a non-revealing REE $f, f(1) = f(2)$ because the demands of agents in I_1 for X in states 1and 2 are different so the same price cannot clear the market in both states.

We now compute the set of prices that are $EXPR$ w.r.t S and then use proposition 3.1 to conclude that this is also the set of prices that are $CKRMC$ in both states. The computation is similar to the one in example 1 in the main text. Let $P_s, s = 1, 2$, be the set of prices that clear the market in state s when agents in $I_2 \cup I_3$ may have any profile of probabilities $\widehat{\gamma} = \{\widehat{\gamma}_i\}_{i \in I_2 \cup I_3}$ on S (agents in I_1 assign , of course, probability 1 to the true state.) We claim that :

[25] Let $p_1 > \widehat{p_1}$. Assume by contradiction that p_1 is $EXPR$ in state 1. Since p_1 is not $EXPR$ in states 2 and 3 p_1 must be $EXPR$ w.r.t the set $\{1\}$ However, when all agents assign probability 1 to the state1 the clearing price is $\widehat{p_1}$.

$P_1 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_1 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

$P_2 = \left[a_2 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

To see this we note that the extreme points in each set are clearly the lowest and highest prices in the respective states (for example, the demand of agents in $I_2 \cup I_3$ is minimal when agents in I_2 assign probability 1 to the state 1and agents in I_3 assign probability 1to the state 2. When these are the beliefs the clearing prices in states 1 and 2 are the respective minimal points in P_1 and P_2.) Any price p between these points can be obtained as a clearing price by having a fraction $\beta = \beta(p)$ of the agents in I_2 and I_3 assign probability 1 to the states 1 and 2 respectively and a fraction $1 - \beta(p)$ assign probability 1 to the states 2 and 1 respectively. The set of prices that are $EXPR$ w.r.t S is:

$P \equiv P_1 \cap P_2 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

It follows from A.3.8 and A.3.9 that this is a non-empty segment. Proposition 3.1 implies that P is also the set of prices that are $CKRMC$ in S.

Section 4

Proof of proposition 4.1**:**

The proof of proposition 4.1 relies on lemma 4.1.1 below. To state the lemma we need the following notation. Let τ be a number $0 \leq \tau \leq 1$. Denote by $\gamma(\tau)$, $\gamma(\tau) = \{\gamma(\tau)_i\}_{i \in I}$, the profile of probabilities where $\gamma(\tau)_i$ assigns probabilities τ and $1 - \tau$ respectively to the maximal and minimal states in the set $\Pi_i(\widehat{s}) \cap \widehat{S}$.

Lemma 4.1.1**:**

(a) For every $0 \leq \tau \leq 1$ there exists a single equilibrium price w.r.t $\gamma(\tau)$. Denote this price by $p(\tau)$.

(b) $p(\tau)$ is continuous in τ.

(c) Let $\gamma = (\gamma_i)_{i \in I}$, $\gamma_i \in \triangle(\Pi_i(\widehat{s}) \cap \widehat{S})$, be a profile of probabilities such that there exists a price p which is an equilibrium price w.r.t γ then $\underline{p}(\widehat{s}, \widehat{S}) \leq p \leq \overline{p}(\widehat{s}, \widehat{S})$.

Proof of lemma 4.1.1

Let $i \in I_l$ for some $l \in \{1, ..L\}$, $\gamma \in \triangle(S)$, and $p \in R_+$. We let $x_l(\gamma, p)$ denote the demand of i for X at the price p when the probability distribution of player i on S is γ. (We note that for a given γ and p all the agents in I_l have the same demand hence the notation $x_l(\gamma, p)$.) Our assumptions on the utility function u_l imply that $x_l(\gamma, p)$ is an internal solution and therefore satisfies the first order condition i.e.,

(4.1) $\sum_{s \in S} \gamma(s) \cdot u_l'(x_l(\gamma, p), s) = p$

Let $0 \leq \tau \leq 1$. Recall that $\gamma(\tau)$ is a profile of probabilities $\gamma(\tau) = \{\gamma_i(\tau)\}_{i \in I}$ where for $i \in I_l$ $\gamma_i(\tau)$ is the probability distribution that assigns probabilities τ and 1-τ, respectively, to the maximal and minimal states in $\Pi_l(\widehat{s}) \cap \widehat{S}$. Thus, in the profile $\gamma(\tau)$ all the agents of the same type have the same probability distribution on S. Therefore, for every price p there exists an aggregate demand $x(\gamma(\tau), p)$, which equals:

(4.2) $x(\gamma(\tau), p) = \sum_{l=1}^{L} \lambda_l \cdot x_l(\gamma_l(\tau), p)$

where $x_l(\gamma_l(\tau), p)$ is the demand of an agent in I_l. The FOC ,4.1, plus the assumption that u_l' has a negative derivative imply that for every agent $i \in I_l$ and every probability $\gamma \in \triangle(\Pi_l(\widehat{s}) \cap \widehat{S})$ the demand of i for X, $x_l(\gamma, p)$,is: 1. Continuous in p. 2. Strictly decreasing in p. 3. $\lim_{p\to\infty} x_l(\gamma, p) = 0$ and $\lim_{p\to 0} x_l(\gamma, p) = \infty$. It follows that for every τ the aggregate demand $x(\gamma(\tau), p)$ has properties 1., 2. and 3. as well. These properties imply that for every τ there exits a unique price $p, p = p(\tau)$, such that $x(\gamma(\tau), p) = p$. Thus, part (a) of the lemma is established.

Consider now part (b). We will show that $p(\tau)$ is differentiable. It follows from (4.2) and part (a) that for every $0 \leq \tau \leq 1$ we can write:

(4.3) $\sum_{l=1}^{L} \lambda_l \cdot x_l(\gamma_l(\tau), p(\tau)) - \overline{x} = 0$

where $\overline{x}$ is the aggregate amount of X in the economy. If each function x_l has continuous partial derivatives w.r.t τ and p and if $\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial p} \neq 0$ then we can apply the implicit function theorem and obtain that p is differentiable w.r.t τ. Specifically, $\frac{dp}{d\tau} = -\frac{\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial \tau}}{\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial p}}$

We will now show that $\frac{\partial x_l}{\partial p}$and $\frac{\partial x_l}{\partial \tau}$ indeed exist, are continuous, and $\frac{\partial x_l}{\partial p}$<0.

Equation (4.4) below is the FOC (4.1) of the optimization problem of an agent in I_l w.r.t the probability distribution $\gamma_l(\tau)$ and the price p,

(4.4) $\tau \cdot u_l'(x_l(\gamma_l(\tau), p), \overline{s}_l) + (1 - \tau) \cdot u_l'(x_l(\gamma_l(\tau), p), \underline{s}_l) - p = 0$

where $\overline{s}_l$ and $\underline{s}_l$ are the maximal and minimal states in the set $\Pi_l(\widehat{s}) \cap \widehat{S}$, respectively. Since u_l' has a continuous derivative and since u_l''<0 we can apply the implicit function theorem w.r.t the variables x_l and τ (holding p fixed) and obtain that:

(4.5) $\frac{\partial x_l(\gamma(\tau),p)}{\partial \tau} = \frac{u_l'(x_l(\gamma_l(\tau),p),\underline{s}_l) - u_l'(x_l(\gamma_l(\tau),p),\overline{s}_l)}{\tau \cdot u_l''(x_l(\gamma_l(\tau),p),\overline{s}_l) + (1-\tau) \cdot u_l''(x_l(\gamma_l(\tau),p),\underline{s}_l)}$

So $\frac{\partial x_l}{\partial \tau}$ exists and since u_l' and u_l'' are continuous it is continuous as well.

Using again the equation (4.4) and applying the implicit function theorem, this time, w.r.t the variables x_l and p (holding τ fixed) we obtain that:

(4.6) $\frac{\partial x_l(\gamma(\tau),p)}{\partial p} = \frac{1}{\tau \cdot u_l''(x_l(\gamma_l(\tau),p),\overline{s}_l) + (1-\tau) \cdot u_l''(x_l(\gamma_l(\tau),p),\underline{s}_l)}$

So $\frac{\partial x_l}{\partial p}$ exists, it is continuous, and since u_l''<0 it is different from zero.

This completes the proof that $p(\tau)$ is differentiable.

We turn now to part (c). Since $u_l'(x, s)$ is increasing in s and decreasing in x it is easy to see that (4.1) implies that $x_l(\gamma_l(1), p) > x_l(\gamma, p)$ for any $\gamma \in \triangle(\Pi_l(\widehat{s}) \cap \widehat{S}), \gamma \neq \gamma_l(1)$ and any p. It follows that for any profile of probabilities $\gamma = \{\gamma_i\}_{i\in I}$, $\gamma_i \in \triangle(\Pi_i(\widehat{s}) \cap \widehat{S})$, and any price p, if the aggregate demand $x(\gamma, p)$ exists it is smaller or equal to $x(\gamma(1), p)$. Since for every agent $i \in I_l$ the demand $x_l(\gamma, p)$ is strictly decreasing in p an equilibrium price w.r.t the profile $\gamma, p(\gamma)$, cannot be higher than $p(1)$. A similar argument establishes that $p(\gamma)$ cannot be smaller than $p(0)$. Thus, the proof of part (c) is complete.

The proof of proposition 4.1 from lemma 4.1.1 is simple. We have $p(1) = \overline{p}(\widehat{s}, \widehat{S})$ and

$p(0) = \underline{p}(\widehat{s}, \widehat{S})$. The continuity of $p(\tau)$ implies that for every p, $p(0) \leq p \leq p(1)$, there exists a τ such that $p = p(\tau)$. Therefore, $P(\widehat{s}, \widehat{S}) \supseteq \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$. On the other hand part (c) of lemma 4.1.1 implies that $P(\widehat{s}, \widehat{S}) \subseteq \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$.

Proof of proposition 4.2:

(a) By definition $P([\underline{s}, \overline{s}]) = \cap_{s\in[\underline{s},\overline{s}]} P(s, [\underline{s}, \overline{s}])$. Therefore, $p \in P([\underline{s}, \overline{s}]) \Rightarrow p \in P(s, [\underline{s}, \overline{s}])$ for every $s \in [\underline{s}, \overline{s}]$. Since by lemma 4.1.1 $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ is the minimal element in $P(\overline{s}, [\underline{s}, \overline{s}])$ we obtain that $p \geq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. In a similar way we obtain that $p \leq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$ and therefore $P([\underline{s}, \overline{s}]) \subseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$. For the other direction we observe that since the information sets of each agent are segments (assumption 3) then for every $s \in [\underline{s}, \overline{s}]$ and $i \in I$ the maximal state in the set $\Pi_i(s) \cap [\underline{s}, \overline{s}]$ is greater or equal to the maximal state in the set $\Pi_i(\underline{s}) \cap [\underline{s}, \overline{s}]$. It follows that for every $s \in S$ $\overline{p}(s, [\underline{s}, \overline{s}]) \geq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$. Similarly, we obtain that for every $s \in S$ $\underline{p}(s, [\underline{s}, \overline{s}]) \leq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. It follows that for every $s \in S$ $P(s, [\underline{s}, \overline{s}]) \supseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$ and therefore the result follows.

(b) By definition $P_s \equiv \cup_{s\in\widehat{S}} P(\widehat{S})$. Clearly, $\cup_{s\in[\underline{s},\overline{s}]} P([\underline{s}, \overline{s}]) \subseteq \cup_{s\in\widehat{S}} P(\widehat{S})$ because the union in the RHS is taken over a larger set of sets. We will establish the claim by showing that for every set $\widehat{S}$ $P(\widehat{S}) \subseteq P([\underline{s}, \overline{s}])$ where $\underline{s}$ and $\overline{s}$ are the minimal and maximal states respectively in $\widehat{S}$. To see this recall that $P(\widehat{S}) \equiv \cap_{s\in\widehat{S}} P(s, \widehat{S})$ and therefore, in particular, $P(\widehat{S}) \subseteq P(\overline{s}, \widehat{S})$ and $P(\widehat{S}) \subseteq P(\underline{s}, \widehat{S})$. It follows that $P(\widehat{S}) \subseteq \left[\underline{p}(\overline{s}, \widehat{S}), \overline{p}(\underline{s}, \widehat{S})\right]$. Now since $\widehat{S} \subseteq [\underline{s}, \overline{s}]$ we have $\underline{p}(\overline{s}, \widehat{S}) \geq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ and $\overline{p}(\underline{s}, \widehat{S}) \leq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$. It follows that $P(\widehat{S}) \subseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right] = P([\underline{s}, \overline{s}])$ where the last equality is established by part (a) of the proposition.

Proof of claim 4.3**:**

Since P_s strictly contains $P(\{s\})$ then either there exists $p<P(\{s\})$ such that $p \in P_s$ or there exists $p > P(\{s\})$ such that $p \in P_s$. Assume w.l.o.g the former case. Let $\underline{p}$ be the minimal element in P_s. Assume, first, that there is a single set $[\underline{s}, \overline{s}]$, $s \in [\underline{s}, \overline{s}]$, such that $\underline{p} \in P([\underline{s}, \overline{s}])$. Since $\underline{p}$ is the minimal element in $P([\underline{s}, \overline{s}])$, it follows from proposition (2.a) that $\underline{p} = \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. Consider the state $\overline{s}$. Clearly, there exists a set of agents I_l, $l \in \{1, .., L\}$, such that $\Pi_l(\overline{s}) \cap [\underline{s}, \overline{s}]$ strictly contains $\overline{s}$, because if that were not the case then $\underline{p}(\overline{s}, [\underline{s}, \overline{s}]) > \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$ implying that $P([\underline{s}, \overline{s}]) = \emptyset$ in contradiction to the assumption that $\underline{p} \in P([\underline{s}, \overline{s}])$. Let I' be a set of agents of a positive measure in I_l (the measure of I' can be as small as we wish.) Since the minimal state in the set $\Pi_l(\overline{s}) \cap [\underline{s}, \overline{s}]$, $\widetilde{s}$, is strictly smaller than $\overline{s}$ the demand of agents in I' at $\widetilde{s}$ is smaller than their demand at the state $\overline{s}$. Since the set I' has a positive measure it follows that $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ is strictly smaller than $\underline{p}^{I'}(\overline{s}, [\underline{s}, \overline{s}])$. (Recall that the superscript I' refers to the economy $E^{I'}$ that is obtained form the original economy E by refining the knowledge of agents in I'.) So the point is that $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ is an equilibrium price for a profile of probabilities $\widetilde{\gamma}$ in which agents in I' assign probability 1 to the state $\widetilde{s}$ while $\underline{p}^{I'}(\overline{s}, [\underline{s}, \overline{s}]$ is an equilibrium price for a profile of probabilities $\overline{\gamma}$ where agents in I' assign probability 1to $\overline{s}$. (Agents that do not belong to I' have the same beliefs in $\overline{\gamma}$ and $\widetilde{\gamma}$.) In particular, $\underline{p} = \underline{p}(\overline{s}, [\underline{s}, \overline{s}]) \notin P^{I'}([\underline{s}, \overline{s}])$. Since by assumption there was no segment of states $[\underline{s}^*, \overline{s}^*]$, $s \in [\underline{s}^*, \overline{s}^*]$, $[\underline{s}^*, \overline{s}^*] \neq [\underline{s}, \overline{s}]$, such that $\underline{p} \in P(\ [\underline{s}^*, \overline{s}^*])$ and since for any set $[\underline{s}^*, \overline{s}^*]$, $P^{I'}(\ [\underline{s}^*, \overline{s}^*]) \subseteq P([\underline{s}^*, \overline{s}^*])$[26], it follows that $\underline{p} \notin P_s^{I'}$ and therefore $P_s^{I'}$ is strictly contained in P_s. We have thus proved the claim for the case where $\underline{p}$ belongs to a single set $P[\underline{s}, \overline{s}]$. It is easy to see that if $\underline{p}$ belongs to a collection of sets we can find for each such set, $P(\ [\underline{s}^*, \overline{s}^*])$,a set of agents I', $I' = I'(\ [\underline{s}^*, \overline{s}^*])$, such that $\underline{p} \notin P^{I'}([\underline{s}^*, \overline{s}^*])$. In the economy which is obtained by *all* these refinements $\underline{p}$ is not a price that is $EXPR$ w.r.t s.

Robustness of economies in which there is a segment of prices that are $EXPR$ in every state.

We start with a definition of a metric on the space of economies.

Let $E(n, \mathcal{L}, \Pi)$ denote the set of economies where $S = \{1, .., n\}$ is the set of states, $\mathcal{L} = \{I_1, .., I_L\}$ is the set of types of agents, and $\Pi = (\Pi_1, ..\Pi_L)$ is the profile of information partitions of the different types. An economy E in $E(n, \mathcal{L}, \Pi)$ is characterized by a vector $(\lambda_l, \overline{m}_l, \overline{x}_l, u_l)_{l\in\mathcal{L}}$. The distance between two vectors E and E' is defined as the maximal distance among the $(4 \cdot L)$ coordinates of the two vectors where the distance between the different coordinates is defined as follows: The distance between two numbers is the absolute value of their difference. Since the demand function of an agent depends on the derivative of his utility we define the distance between utility functions u and v as follows:

$d(u, v) \equiv \max \{\|u - v\|, \|u' - v'\|, \|u'' - v''\|\}$

where $\|\cdot\|$ is the supremum norm, i.e.,

$\|u\| \equiv \sup \left\{u(x, s) : x \in R_+^K, s \in S\right\}$.

We now show that the set of economies in which there is a segment of prices that are $EXPR$ in every state is open. Let E be an economy such that $\underline{p}(n, [1, n]) < \overline{p}(1, [1, n])$. To economize in notation we let $\underline{p}$ and $\overline{p}$ denote the prices $\underline{p}(n, [1, n])$ and $\overline{p}(1, [1, n])$ respectively. Proposition 4.2 implies that (in the economy E) the segment $\left[\underline{p}, \overline{p}\right]$ is $EXPR$ in every state. We will show that a 'small' change in the utilities of the agents induces a 'small' change in the prices $\underline{p}$ and $\overline{p}$, so that if the change in the utilities is sufficiently small the inequality $\underline{p} < \overline{p}$ is maintained. It will then be easy to see that a small change in the other parameters of the economy $(\{\lambda_l, \overline{m}_l, \overline{x}_l, \}_{l\in\mathcal{L}})$ also leads to a small change in $\underline{p}$ and $\overline{p}$. This will establish the existence of a neighborhood of E, $N(E)$, such that for every $E' \in N(E)$ there exists a segment of prices that are $EXPR$ in every state.

Consider the price $\overline{p}$ (the argument for the price $\underline{p}$ is similar.) The price $\overline{p}$ is generated by a profile of beliefs $\gamma \equiv \{\gamma_i\}_{i\in I}$ where all players of the same type assign probability 1

[26] Any refeinment of the knowledge of any set of agents can only shrink the set of prices that are $EXPR$ w.r.t a given set of states $\widehat{S}$. Therefore, $P^{I'}(\ [\underline{s}^*, \overline{s}^*])$ which is the set of prices that are $EXPR$ w.r.t $[\underline{s}^*, \overline{s}^*]$ in the economy $E^{I'}$ is contained, in the weak sense, in $P(\ [\underline{s}^*, \overline{s}^*]$.

to the same state. We fix the profile γ and examine the change in the clearing price when there is a small change in the utilities of the agents. Let u_l, $l \in \mathcal{L}$, denote the utility of an agent of type l in E. First, we note that for every $l \in \mathcal{L}$ and closed positive segment in which the parameter x may vary $\left|u_l^{''}(x,s)\right|$ is bounded from above and below by positive numbers M_l and m_l,respectively. Let v $=(v_1,...,v_L)$ be a vector of utilities such that $d(u_l, v_l) < \epsilon$ for every $l \in \mathcal{L}$. We have $m_l - \epsilon \leq |v_l''| \leq M_l + \epsilon$. Define $C \equiv \max_{l \in \mathcal{L}} \frac{M_l}{m_l - \varepsilon}$and let $\overline{p_v}$ denote the clearing price for the utilities v at the beliefs γ.

Claim: $|\overline{p_v} - \overline{p}| \leq 2C\varepsilon$.

Proof: Let u be a utility, l a type, and p a price. We let $x_l(p,u)$ denote the demand of an agent who has the belief $\gamma_l, \gamma_l \equiv \gamma_i$ for $i \in l$, and the utility u at the price p. We have

(4.7) $|x_l(\overline{p}, u_l) - x_l(\overline{p_v}, v_l)| \geq |x_l(\overline{p_v}, u_l) - x_l(\overline{p}, u_l)| - |x_l(\overline{p_v}, u_l) - x_l(\overline{p_v}, v_l)|$.

(4.8) $|x_l(\overline{p_v}, u_l) - x_l(\overline{p}, u_l)| \geq \frac{|\overline{p} - \overline{p_v}|}{M_l}$.

(4.9) $|x_l(\overline{p_v}, u_l) - x_l(\overline{p_v}, v_l)| \leq \frac{\varepsilon}{m_l - \epsilon}$.

The inequality (4.7) is the triangle inequality. Let s_l denote the state to which the belief γ_l assigns probability 1. To see (4.8) we note that the first-order condition implies that $|u_l'(x_l(\overline{p_v}, u_l), s_l) - u_l'(x_l(\overline{p}, u_l), s_l)| = |\overline{p_v} - \overline{p}|$. Since $\left|u_l^{''}(x,s)\right| \leq M_l$ (4.8) follows. The inequality (4.9) is established in a similar way; the first-order condition implies that $u_l'(x_l(\overline{p_v}, u_l), s_l) = v_l'(x_l(\overline{p_v}, v), s_l) = \overline{p_\nu}$. Since v_l'' is bounded from below by $m_l - \epsilon$ the fact that $\|u_l' - v_l'\| \leq \varepsilon$ as well implies (4.9). The inequalities (4.7)-(4.9) imply that if $\overline{p_v} - \overline{p} > 2C\varepsilon$ then $x_l(\overline{p}, u_l) > x_l(\overline{p_v}, v_l)$ for every $l \in \mathcal{L}$ which means that when the vector of utilities is v there is excess supply of X at the price $\overline{p_v}$ and hence it cannot be a clearing price . Similarly, $\overline{p_v}$ cannot be a clearing price if $\overline{p} - \overline{p_v} > 2C\varepsilon$. The claim follows.

We have, thus, shown that a small change in the utilities of the agents induces a small change in the prices $\overline{p}$ and $\underline{p}$. The argument that a small change in the other parameters of the economy also induces a small change in $\overline{p}$ and $\underline{p}$ is simpler. We omit the details.

Section 5:

Proof of theorem 2:

We start the proof of the theorem with two lemmas.

Lemma 2.1:

Part (b) of theorem 2 implies part (a).

Proof:

The definition of the operator J readily implies that if F and F' are two sets of price functions such that $F' \supseteq F$ then $J(F') \supseteq J(F)$. A simple induction implies that for each natural number k, $F^k \supseteq FCKRMC$. On the other hand if $F^k = F^{k+1}$ then F^k is $CKRMC$ and therefore $F^k \subseteq FCKRMC$ which implies $F^k = FCKRMC$. This establishes the lemma.

The proof of the theorem relies on the relationship between $CKRMC$ and $EXPR$ which was defined in section 3. For the proof it would be useful to think of $S(p)$-the maximal set of states w.r.t which p is $EXPR-$ as the result of an iterative process; define by induction a sequence of states $S^k(p)$ as follows:

$S^0(p) = S$

$$S^k(p) = \left\{ s : \begin{array}{c} p \text{ can be supported in } s \text{ by a profile of beliefs } \{\gamma_i^s\}_{i\in I} \text{ such that} \\ \gamma_i^s \in \Delta(\ S^{k-1}(p) \cap \Pi_i(s)) \end{array} \right\}^{27}$$

A simple induction establishes that for every k $S^{k+1}(p) \subseteq S^k(p)$. It follows that $S^n(p) = S^{n+1}(p) = S(p)$ where n is the number of states of nature.

For a set of states S' and a price p we let $B(S', p)$ denote the set of states in which the price p can be supported by a profile of probabilities on S'.In addition, for a set of price functions F and a price p define $S(F, p) \equiv \{s : \exists f \in F \text{ s.t } f(s) = p\}$.

We can now prove lemma 2 below.

Lemma 2.2: For $k \geq n$ $f \in F^k \Rightarrow \forall s \in S$ $f(s)$ is $EXPR$ in S.

Proof: We will show,

Claim 2.2.1: $\forall k$ $S(F^k, p) \subseteq S^k(p)$.

Since $S^k(p) = S(p)$ for $k \geq n$ the claim implies the lemma.

The proof of the claim relies on inclusion 6.1 below; Let F be a set of functions and let p be a price then

(6.1) $S(J(F), p) \subseteq B(S(F, p), p)$

The argument that establishes the inclusion 6.1 is identical to the argument that is used in the first part of theorem 1[28]. Given 6.1 claim 2.2.1 is proved by induction on k. For $k = 0$ we have $S(F^0, p) = S^0(p) = S$. Assume by induction that the claim has been proved for k and consider now the claim for $k + 1$. We have

(6.2) $S(F^{k+1}, p) = S(J((F^k), p) \subseteq B(S(F^k, p), p) \subseteq B(S^k(p), p) = S^{k+1}(p)$.

The first inclusion follows from (6.1) and the second one from the induction hypothesis. (The last equality is, simply, the definition of $S^{k+1}(p)$.) We have proved claim 2.2.1 and the lemma follows.

We can now proceed to the main part in the proof of theorem 2. We remind that for $s \in S$ P_s denotes the set of prices that are $EXPR$ in s. Now distinguish between the following three cases:

(1) For every $s \in S$ P_s is finite.

(2) For every $s \in S$ P_s is infinite.

(3) There is a set of states $\overline{S}, \varnothing \subsetneq \overline{S} \subsetneq S$, such that for $s \in \overline{S}$ P_s is infinite and for $s \notin \overline{S}$ P_s is finite.

Case 1:

[27] The definition of a price p supported in a state s by a profile of beliefs γ^s is given at the begining of section 3 right after the definition of $EXPR$.

[28] Let $f \in J(F)$ and let $\mu = \{\mu_i\}_{i\in I}$ and $z = \{z_i\}_{i\in I}$ be the profile of beliefs and demands that support f.If s is a state such that $f(s) = p$ then the profile of conditional probabilities on $S, \gamma = \{\gamma_i\}_{i\in I}, \gamma_i \equiv \mu_i(\cdot\,|P_i(s), p)$, supports p.(I.e., the pofile γ rationalizes the profile of demands $\{z_i(P_i(s), p)\}_{i\in I}$ which clear the market.) Clearly, for every $i \in I$, the support of the probability distribution γ_i is contained in $S(F, p)$, therefore, the inclusion in 6.1 follows.

Case (1) is simple, lemma 2.2 implies that F^n is finite, therefore there exists a number $m \leq |F^n|$ such that $F^{n+m} = F^{n+m+1}$. It follows that $F^{n+m} = F^\infty = FCKRMC$. Obviously, F^{n+m} is finite and therefore the proof of the theorem for this case is complete.

Case 2:

The proof of the theorem for case (2) is an immediate consequence of proposition 3.1 (section 3 in the appendix.)

Part (c) of the theorem is one of the claims in the proposition. Since part (a) of the theorem follows from part (b) (lemma 2.1 above) we only need to show part (b). Lemma 2.2 implies that if $f \in F^k$ where $k \geq n$ then $f(s)$ is $EXPR$ for every s, which means that f satisfies conditions (A.3.1) and (A.3.2) of proposition 3.1. Since the demand of each agent at a state s depends only on his private signal and the price at s f must satisfy condition (A.3.3) as well. Now, it follows from proposition 3.1 that for $k \geq n$ $F^k \subseteq FCKRMC$. Since for every k $FCKRMC \subseteq F^k$part (b) follows.

Case 3:

We will prove the theorem by providing an indirect characterization of the set of price functions that are $CKRMC$. Since this characterization is somewhat involved we start with an informal description. Let $\widehat{S}$ denote the set of states in which there is a finite number of prices that are $EXPR$. Thus, $\widehat{S} \equiv S \backslash \overline{S}$. Let $P \equiv \left\{ p : \ p \text{ is } EXPR \text{ in a state } s \in \widehat{S} \right\}$. To understand the characterization that will follow it is useful to think of every price function f as composed of two partial functions σ and g where σ is defined on states in which f receives values in P and g is defined on the rest of the states . More precisely, define $S(f,P) \equiv \{s : f(s) \in P\}$. For every $s \in S(f,P)$ define $\sigma_s \equiv f(s)$ and for every $s \in S \backslash S(f,P)$ $g(s) = f(s)$. A function f that is $CKRMC$ will be supported by a belief that is a composition of two beliefs, a belief μ^σ that supports σ on the states $S(f,P)$ and a belief μ^g that supports g on the states $S \backslash S(f,P)$. It follows from the definition of the set P that $S(f,P) \supseteq \widehat{S}$ and therefore $S \backslash S(f,P) \subseteq \overline{S}$. In the sequel we will use proposition 3.1 to characterize partial functions that are defined on a subset of $\overline{S}$ and are part of a $CKRMC$ function (such as g). The characterization of partial functions that receive values in P and are part of a $CKRMC$ function (such as σ) is indirect, these are partial functions that survive every finite number of steps in the iterative deletion procedure. We turn now to the formal description.

Define $\mathcal{S} \equiv \left\{ S' : \widehat{S} \subseteq S' \subseteq S \right\}$ and $\Sigma \equiv \{\sigma : \exists S' \in \mathcal{S} \text{ s.t } \sigma = (\sigma_s)_{s \in S'}, \sigma_s \in P\}$. Thus, Σ is the set of partial functions that receive values in P. However, not all of these functions are part of some $CKRMC$ function. For $\sigma \in \Sigma$ let $S(\sigma)$ denote the set of states on which σ is defined. For each $\sigma \in \Sigma$ define a set of price functions F_σ as follows,

$F_\sigma \equiv \{f : \text{ for } s \in S(\sigma) \ f(s) = \sigma_s; \text{ for } s \notin S(\sigma) \ f(s) \notin P\}$.

Define $\Sigma^\infty \equiv \{\sigma : \sigma \in \Sigma, \ F_\sigma \cap F^\infty \neq \varnothing\}$. Since Σ is a finite set there exists a number M such that for every $m \geq M$ and every $\sigma \in \Sigma \backslash \Sigma^\infty$ $F_\sigma \cap F^m = \varnothing$. We will show (lemma 2.3 below) that Σ^∞ is precisely the set of partial price functions that receive values in P and are part of a $CKRMC$ function. Let $E_{\overline{S}}$ denote the subeconomy where the set of states is $\overline{S}$. Define $G \equiv \left\{g : g \text{ is a } CKRMC \text{ function in } E_{\overline{S}} \text{ and } \forall s \in \overline{S}\ g(s) \notin P\right\}$. Since P is the set of all prices that are $EXPR$ in some $s \in \widehat{S}$ it follows that for every $s \in \overline{S}$ and $p \in P_s \backslash P$ $S(p) \subseteq \overline{S}$. (We remind that P_s is the set of prices that are $EXPR$ in the state s and that $S(p)$ is the set of all states in which p is$EXPR$.) Since P is finite and since for $s \in \overline{S}$ P_s is infinite it follows that $P_s \backslash P$ is infinite as well. It is now easy to see that the proof of proposition 3.1 (section 3 in the appendix) can be used to establish that $g \in G$ iff for every $s \in \overline{S}$ there exist profiles of probabilities $\gamma = \{\gamma_i^s\}_{i \in I}, \gamma_i^s \in \Delta(\Pi_i(s) \cap S(f(s)))$ and demands $x = \{x_i^s\}_{i \in I}$ such that

(6.3) For every $i \in I$ x_i^s is an optimal bundle for player i w.r.t. the price $f(s)$ and the probability γ_i^s.

(6.4) For every $s \in \overline{S}$ markets clear, that is, $\int_i x_i^s = \int_i e_i(s)$.

(6.5) If $s' \in \Pi_i(s)$ and $f(s) = f(s')$ then $x_i^s = x_i^{s'}$ and $\gamma_i^s = \gamma_i^{s'}$.

(6.6) For every $s \in \overline{S}$ $g(s) \notin P$.

For $\sigma \in \Sigma$ and $g \in G$ define the function $\sigma \nabla g$ as follows:

(6.7) $\sigma \nabla g(s) \equiv \begin{cases} \sigma_s & s \in S(\sigma) \\ g(s) & \text{otherwise} \end{cases}$

Define now

(6.8) $\widehat{F} \equiv \{f : \exists \sigma \in \Sigma^\infty \text{ and } g \in G \ \text{ s.t } f = \sigma \nabla g\}$

Lemma 2.3 below establishes theorem 3 for case 3 and shows that $\widehat{F}$ is the set of $CKRMC$ functions.

Lemma 2.3: Let $K \equiv \max\{n, M\}$.

(a) For $k \geq K$ $\widehat{F} = F^k = FCKRMC$.

(b) For every $\widehat{f} \in \widehat{F}$ there exists a finite set of $CKRMC$ functions, $\widehat{F}(\widehat{f})$, such that $\widehat{f} \in \widehat{F}(\widehat{f})$ and $\widehat{F}(\widehat{f}) \subseteq \widehat{F}$.

Proof:

First, we claim that part (b) of the lemma implies part (a). Clearly, part (b) implies that $\widehat{F}$ is $CKRMC$ and therefore $\widehat{F} \subseteq FCKRMC$. Now, let $k \geq K$ and let $f \in F^k$. Since $k \geq M$ there exists $\sigma \in \Sigma^\infty$ s.t. $f \in F_\sigma$. Since $k \geq n$ $f(s) \in P_s \backslash P$ for $s \in S \backslash S(\sigma)$. It follows that conditions (6.7)-(6.10) are satisfied for every $s \in S \backslash S(\sigma)$ and therefore $f \in \widehat{F}$[29]. We have thus shown that for $k \geq K$ $F^k \subseteq \widehat{F}$. Since for every k $F^k \supseteq FCKRMC$ we have

$\widehat{F} \subseteq FCKRMC \subseteq F^k \subseteq \widehat{F}$ which implies part (a).

[29] More specifically, we can define a function $g \in G$ s.t. $f = \sigma \nabla g$ in the following way: For $s \in \overline{S} \cap S(\sigma)$, let p_s be prices such that $p_s \in P_s / P, p_s \neq f(s')$ for every $s' \in S/S(\sigma)$ and $p_s \neq p_{s'}$ for $s \neq s'$. Define $g : \overline{S} \to R^{K-1}$ as follows $g(s) \equiv \begin{cases} p_s & s \in \overline{S} \cap S(\sigma) \\ f(s) & s \in \overline{S}/S(\sigma) \end{cases}$

It is easy to see that $g \in G$ and that $f = \sigma \nabla g$.

We now turn to the proof of part (b). Before presenting the detailed proof we describe its outline.

Let $\widehat{f} \in \widehat{F}$. There exists $\widehat{\sigma} \in \Sigma^{\infty}$ and $\widehat{g} \in G$ such that $\widehat{f} = \widehat{\sigma}\nabla\widehat{g}$. The proof of proposition 3.2 can be used to establish that there exists a finite $CKRMC$ set of functions $G(\widehat{g})$ in the economy $E_{\overline{S}}$ such that $\widehat{g} \in G(\widehat{g})$ and $G(\widehat{g}) \subseteq G$. For $g \in G(\widehat{g})$ we let $\mu^{g} \equiv \{\mu_i^g\}_{i\in I}$ denote the profile of beliefs that supports g w.r.t $G(\widehat{g})$. Define now,

(6.9) $\widehat{F}(\widehat{f}) \equiv \{f : \exists\sigma \in \Sigma^{\infty} \text{ and } g \in G(\widehat{g}) \text{ s.t } f = \sigma\nabla g\}$.

First, we note that since $\widehat{g} \in G(\widehat{g})$ $\widehat{f} \in \widehat{F}(\widehat{f})$. Also, since Σ^{∞} and $G(\widehat{g})$ are finite so is $\widehat{F}(\widehat{f})$. The main step in the proof is to show that $\widehat{F}(\widehat{f})$ is $CKRMC$. So let $\widetilde{f} \in \widehat{F}(\widehat{f})$ s.t. $\widetilde{f} = \widetilde{\sigma}\nabla\widetilde{g}$ where $\widetilde{\sigma} \in \Sigma^{\infty}$ and $\widetilde{g} \in G(\widehat{g})$. We need to construct a profile of beliefs $\mu^{\widetilde{f}} \equiv \left\{\mu_i^{\widetilde{f}}\right\}_{i\in I}$ that supports $\widetilde{f}$ w.r.t $\widehat{F}(\widehat{f})$. To do that we first need to extend the definition of "support" (page 10) to include subsets of S : Say that a profile of beliefs $\{\mu_i\}_{i\in I}$ supports the price function f on a <u>subset</u> of states S' ($S' \subseteq S$) if the conditions of rationality and market clearing (conditions 1. and 2. in the definition of "support", pages 9 and 10) apply for every $s \in S'$. We now construct the profile $\mu^{\widetilde{f}}$ as follows: First, we will show that there exists a profile of beliefs (on $\widehat{F}(\widehat{f})$), $\mu^{\widetilde{\sigma}} \equiv \left\{\mu_i^{\widetilde{\sigma}}\right\}_{i\in I}$, that supports $\widetilde{f}$ on the set $S(\widetilde{\sigma})$. Then, we will use the profile $\mu^{\widetilde{g}}$ to construct a profile $\mu'^{\widetilde{g}}$ that supports $\widetilde{f}$ on $S\backslash S(\widetilde{\sigma})$. Finally, we will define the belief $\mu_i^{\widetilde{f}}$ by combining the beliefs $\mu_i^{\widetilde{\sigma}}$ and $\mu_i'^{\widetilde{g}}$ so that the profile $\mu^{\widetilde{f}} \equiv \left\{\mu_i^{\widetilde{f}}\right\}_{i\in I}$ supports $\widetilde{f}$ on the whole set of states, S.

We turn now to the details of the proof. Lemma 2.4 below is the main step in the definition of the beliefs $\mu^{\widetilde{\sigma}} \equiv \left\{\mu_i^{\widetilde{\sigma}}\right\}_{i\in I}$ which will be given later on.

<u>Lemma 2.4:</u> Let $\widetilde{\sigma} \in \Sigma^{\infty}$. There exists a profile of beliefs on Σ^{∞}, $\widetilde{\beta} \equiv \left\{\widetilde{\beta}_i\right\}_{i\in I}$, such that for <u>every</u> collection of price functions $C = \{f_\sigma\}_{\sigma\in\Sigma^{\infty}}$, $f_\sigma \in F_\sigma$, the profile of beliefs on C $\mu^{C} \equiv \{\mu_i^C\}_{i\in I}$,defined by, $\mu_i^C(f_\sigma) \equiv \widetilde{\beta}_i(\sigma)$[30], supports $f_{\widetilde{\sigma}}$ on $S(\widetilde{\sigma})$.

<u>Proof:</u> Let $\widetilde{\sigma} \in \Sigma^{\infty}$ and let k be some index such that $k \geq K+1$. The definition of Σ^{∞} implies that there exists a function $f_{\widetilde{\sigma}} \in F^k$ such that $f_{\widetilde{\sigma}} \in F_{\widetilde{\sigma}}$. Let $\mu^{f_{\widetilde{\sigma}}}$ be the profile of beliefs on F^{k-1} that supports $f_{\widetilde{\sigma}}$. Since for every $s \notin S(\widetilde{\sigma})$ $f_{\widetilde{\sigma}}(s) \notin P$ and since for every $s \in S(\widetilde{\sigma})$ $\widetilde{\sigma}_s \in P$ it follows that for $s \in S(\widetilde{\sigma})$ and $i \in I$ $\mu_i^{f_{\widetilde{\sigma}}}(|\Pi_i(s), \widetilde{\sigma}_s)$ depends only on the beliefs $\mu_i^{f_{\widetilde{\sigma}}}(F_\sigma), \sigma \in \Sigma, i \in I$. Specifically, for $s \in S(\widetilde{\sigma})$

$\mu_i^{f_{\widetilde{\sigma}}}(s'\,|\Pi_i(s), \widetilde{\sigma}_s) = \frac{\alpha(s')\cdot\mu_i^{\widetilde{\sigma}}(\{F_\sigma:\sigma_{s'}=\sigma_{\widetilde{s}}\})}{\sum_{\widehat{s}\in\Pi_i(s)}\alpha(\widehat{s})\cdot\mu_i^{\widetilde{\sigma}}(\{F_\sigma:\sigma_{\widehat{s}}=\sigma_{\widetilde{s}}\})}$.

Define now beliefs $\widetilde{\beta}_i, i \in I$, on Σ as follows,

$\widetilde{\beta}_i(\sigma) \equiv \mu_i^{f_{\widetilde{\sigma}}}(F_\sigma)$.

The lemma is established by observing that since $k-1 \geq K$ $\mu_i^{f_{\widetilde{\sigma}}}(F_\sigma) = 0$ for every $\sigma \notin \Sigma^{\infty}$.

[30] An equality between two beliefs means of course an equality between the probability distributions which constitute the beliefs. So if $\widetilde{\beta}_i = (\widetilde{\beta}_i^1, .., \widetilde{\beta}_i^k)$, $\mu_i^C(f_\sigma) = \widetilde{\beta}_i(\sigma)$ means that $\mu_i^C = (\mu_i^{C,1}, .., \mu_i^{C,k})$ and that for every $1 \leq j \leq k$ and $\sigma \in \Sigma^{\infty}$ $\mu_i^{C,j}(f_\sigma) = \widetilde{\beta}_i^j(\sigma)$.

The next step is to use proposition 3.2 to define the set $G(\widehat{g})$[31]. Define $m = \left|\overline{S}\right|$. Define $\overline{P}_s, s \in \overline{S}$, to be m finite sets of prices such that each set $\overline{P}_s$ contains at least $2m$ prices in $P_s \backslash P$ and for $s \in \overline{S}$ $\widehat{g}(s) \in \overline{P}_s$. In addition, if $p \in \overline{P}_s$ then $p \in \overline{P}_{s'}$ for every $s' \in S(p)$[32]. Define now,

$G(\widehat{g}) \equiv \left\{g : g(s) \in \overline{P}_s \text{ for } s \in \overline{S} \text{ and } g \text{ satisfies conditions (6.3)-(6.6).}\right\}$

The set $G(\widehat{g})$ is finite (because the sets $\overline{P}_s, s \in \overline{S}$ are finite) and $\widehat{g} \in G(\widehat{g})$. A proof which is identical to the proof of proposition 3.2 establishes that $G(\widehat{g})$ is a $CKRMC$ set of functions in the economy $E_{\overline{S}}$.

To show that $\widehat{F}(\widehat{f})$ is $CKRMC$ we need one additional (small) intermediate step. Define, $\overline{\Sigma^\infty} \equiv \left\{\overline{\sigma} : \exists \sigma^\infty \in \Sigma^\infty \text{ s.t. } \overline{\sigma} = \{\sigma_s^\infty\}_{s \in \widehat{S}}\right\}$. We will later show that $\overline{\Sigma^\infty} \subseteq \Sigma^\infty$ but since this has not been established yet we define $\widetilde{F}(\widehat{f})$ as follows,

$\widetilde{F}(\widehat{f}) \equiv \left\{f : \exists \sigma \in \overline{\Sigma^\infty} \cup \Sigma^\infty \text{ and } g \in G(\widehat{g}) \text{ s.t. } f = \sigma \nabla g\right\}$.

We now show that $\widetilde{F}(\widehat{f})$ is $CKRMC$. Of course, once we establish that $\overline{\Sigma^\infty} \subseteq \Sigma^\infty$ it will follow that $\widetilde{F}(\widehat{f}) = \widehat{F}(\widehat{f})$ and therefore that $\widehat{F}(\widehat{f})$ is $CKRMC$. So let $\widetilde{f} \in \widetilde{F}(\widehat{f})$ s.t. $\widetilde{f} = \widetilde{\sigma} \nabla \widetilde{g}$ where $\widetilde{\sigma} \in \overline{\Sigma^\infty} \cup \Sigma^\infty$ and $\widetilde{g} \in G(\widehat{g})$. We now define the beliefs $\left\{\mu_i^{\widetilde{\sigma}}\right\}_{i \in I}$ and $\left\{\mu'^{\widetilde{g}_i}\right\}_{i \in I}$ which support the prices specified by $\widetilde{f}$ on the sets of states $S(\widetilde{\sigma})$ and $S \backslash S(\widetilde{\sigma})$ respectively. We will then define $\mu_i^{f_{\widetilde{\sigma}}}$ by combining $\mu_i^{\widetilde{\sigma}}$ and $\mu'^{\widetilde{g}_i}$, and obtain a profile of beliefs $\mu^{\widetilde{f}} \equiv \left\{\mu_i^{\widetilde{f}}\right\}_{i \in I}$ that supports the prices specified by $\widetilde{f}$ on <u>every</u> $s \in S$.

For every $s \in \overline{S}$ select a price p_s such that (a) $p_s \in \overline{P}_s$. (b) For $s \neq s'$ $p_s \neq p_{s'}$. (c) $p_s \neq f(s')$ for every $s' \in S \backslash S(\widetilde{\sigma})$. We note that a selection that satisfies properties (a)-(c) is possible because $S \backslash S(\widetilde{\sigma}) \subseteq \overline{S}$ and because for every $s \in \overline{S}$ $\overline{P}_s$ contains at least $2m$ prices. For every $\sigma \in \Sigma^\infty$ define a price function f_σ as follows,

$$f_\sigma(s) \equiv \begin{cases} \sigma_s & s \in S(\sigma) \\ p_s & s \in S \backslash S(\sigma) \end{cases}$$

Lemma 2.4 implies that there exists a profile of beliefs $\left\{\widetilde{\beta}_i\right\}_{i \in I}$ on $\{f_\sigma\}_{\sigma \in \Sigma^\infty}$ that supports $f_{\widetilde{\sigma}}$ on $S(\widetilde{\sigma})$. To see why this is true we note for $\widetilde{\sigma} \in \Sigma^\infty$ this is a straightforward application of the lemma while for $\widetilde{\sigma} \in \overline{\Sigma^\infty}$ we only need to observe that $\widetilde{\sigma} = (\sigma'_s)_{s \in S(\widetilde{\sigma})}$ for some $\sigma' \in \Sigma^\infty$ and that the profile of beliefs $\{\beta'_i\}_{i \in I}$ that supports $f_{\sigma'}$ on $S(\sigma')$ (obviously) supports $f_{\widetilde{\sigma}}$ on $S(\widetilde{\sigma})$. We now define $\mu_i^{\widetilde{\sigma}} \equiv \widetilde{\beta}_i$.

The next step is to define the profile $\left\{\mu_i'^{\widetilde{g}}\right\}_{i \in I}$. Pick an arbitrary $\overline{\sigma}$ in $\overline{\Sigma^\infty}$ and associate with each $\widetilde{g} \in G(\widehat{g})$ the price function $\overline{\sigma}(\widetilde{g}) \equiv \overline{\sigma} \nabla \widetilde{g}$. Let $\left\{\mu_i^{\widetilde{g}}\right\}_{i \in I}$ be the profile of beliefs that supports $\widetilde{g}$ w.r.t $G(\widehat{g})$ in the economy $E_{\overline{S}}$. Define the belief $\mu_i'^{\widetilde{g}}$ on $\widetilde{F}(\widehat{f})$ as follows,

$$\mu_i'^{\widetilde{g}}(f) \equiv \begin{cases} \mu_i^{\widetilde{g}}(g) & \text{if } f = \overline{\sigma}(g) \\ 0 & \text{otherwise} \end{cases}$$

Since $\overline{\sigma}$ is one-to-one $\mu_i'^{\widetilde{g}}$ is well-defined. It is straightforward to show that the profile

[31] We remind that $G(\widehat{g})$ is used in the definition of $\widehat{F}(\widehat{f})$ in 6.9.

[32] We remind that $S(p)$ is the set of states on which p is $EXPR$ and since $p \notin P$ $S(p) \subseteq \overline{S}$.

$\left\{\mu_i^{\prime\widetilde{g}}\right\}_{i\in I}$ supports $\widetilde{g}$ on $\overline{S}$. In particular, $\left\{\mu_i^{\prime\widetilde{g}}\right\}_{i\in I}$ supports $\widetilde{g}$ on $S\backslash S(\widetilde{\sigma})$.

Let $\mu_i^{\widetilde{\sigma}} = (\mu_i^{\widetilde{\sigma},1}, ..., \mu_i^{\widetilde{\sigma},k})$[33]. Similarly, let $\mu_i^{\prime\widetilde{g}} = (\mu_i^{\prime\widetilde{g},1}, .., \mu_i^{\prime\widetilde{g},l})$. Define $\mu_i^{\widetilde{f}}$ as follows:

$\mu_i^{\widetilde{f}} \equiv (\mu_i^{\widetilde{f},1}, ..., \mu_i^{\widetilde{f},k+l})$ where for $1 \leq j \leq k$ $\mu_i^{\widetilde{f},j} \equiv \mu_i^{\widetilde{\sigma},j}$ and for $k+1 \leq j \leq k+l$ $\mu_i^{\widetilde{f},j} \equiv \mu_i^{\prime\widetilde{g},j-k}$. It is now simple to verify that the profile $\left\{\mu_i^{\widetilde{f}}\right\}_{i\in I}$ supports $\widetilde{f}$ on the whole set of states, S. So let $s \in S$. The price that each player i observes is $\widetilde{f}(s)$. Now if $s \in S(\widetilde{\sigma})$ then $\widetilde{f}(s) = \widetilde{\sigma}_s$ and player i applies the belief $\mu_i^{\widetilde{\sigma}}$. By definition the profile $\left\{\mu_i^{\widetilde{\sigma}}\right\}_{i\in I}$ supports $f_{\widetilde{\sigma}}$ on $S(\widetilde{\sigma})$. If $s \in S\backslash S(\widetilde{\sigma})$ then player i observes the price $\widetilde{f}(s) = \widetilde{g}(s)$. Since $P(\{f_\sigma : \sigma \in \Sigma^\infty\})$ is disjoint to $P(G(\widehat{g}))$[34] $\mu_i^{\widetilde{\sigma}}$ assigns the price $\widetilde{g}(s)$ probability zero. Thus, when player i observes $\widetilde{g}(s)$ he applies the belief $\mu_i^{\prime\widetilde{g}}$. By definition the profile $\left\{\mu^{\prime\widetilde{g}}\right\}_{i\in I}$ supports $\widetilde{g}$ on $S\backslash S(\widetilde{\sigma})$.

We have shown that $\widetilde{F(\widehat{f})}$ is $CKRMC$. To complete the proof all that is left to do is show that $\overline{\Sigma^\infty} \subseteq \Sigma^\infty$ because (as we have already pointed out) that would imply that $\widetilde{F}(\widehat{f}) = \widehat{F}(\widehat{f})$ and therefore that $\widehat{F}(\widehat{f})$ is $CKRMC$. The inclusion $\overline{\Sigma^\infty} \subseteq \Sigma^\infty$ is actually, now, immediate: Since $\widetilde{F(\widehat{f})}$ is $CKRMC$ $\widetilde{F(\widehat{f})} \subseteq F^k$ for every k. Since for every $\overline{\sigma} \in \overline{\Sigma^\infty}$ $F_{\overline{\sigma}} \cap \widetilde{F(\widehat{f})} \neq \varnothing$ it follows that for every k $F_{\overline{\sigma}} \cap F^k \neq \varnothing$ which by definition implies that $\overline{\sigma} \in \Sigma^\infty$.

The proof of lemma 2.3 (b) is now complete and therefore theorem 2 for case 3 has been established.

Proof of Theorem 3:

One direction is immediate: Let $\widehat{f}$ be a price function such that there exists a model (Ω, β) that is $CKRMC$ and a state $\widehat{\omega}$ such that $T_f(\widehat{\omega}) = \widehat{f}$. We have to show that $\widehat{f}$ is $CKRMC$. We will do that by showing that the set $F = F(\Omega)$ is $CKRMC$. So let $f \in F$ and let $\omega \in \Omega$ be a state such that $T_f(\omega) = f$. Let $\{\mu_i^\omega\}_{i\in I} \equiv (\ \mu_i^{\omega,1}, .., \mu_i^{\omega,m})_{i\in I}$ and $\{z_i^\omega\}_{i\in I}$ be the profiles of beliefs and demands in ω. Define a profile of beliefs $\left\{\mu_i^f\right\}_{i\in I}, \mu_i^f = (\mu_i^{f,1}, ..., \mu_i^{f,m})$, on F as follows: For a Borel set of functions $\overline{F} \subseteq F$ define $\mu_i^{f,k}(\overline{F}) \equiv \mu_i^{\omega,k}(T_f^{-1}(\overline{F}))$ for $1 \leq k \leq m$. Since T_f is a measurable transformation $\mu_i^{f,k}$ is well defined. It is easy to see that the profile of beliefs $\left\{\mu_i^f\right\}_{i\in I}$ and the profile of demand strategies $\{z_i^\omega\}_{i\in I}$ support the function f w.r.t to the set F.

We turn now to the second direction. Let $\widehat{f}$ be a function that is $CKRMC$ and let $\widehat{F}$ be a Borel set of functions such that $\widehat{f} \in \widehat{F}$ and $\widehat{F}$ is $CKRMC$. It follows from part

[33]We remind that for $1 \leq j \leq k$ $\mu_i^{\widetilde{\sigma},j}$ is the j'th probability distribution in the lexicographic sequence of probabilities that constitutes $\mu_i^{\widetilde{\sigma}}$.

[34]We remind that for a set of price functions F $P(F)$ is the set of prices in the range of the functions in F, that is, $P(F) \equiv \{p : \exists f \in F \text{ and } s \in S \text{ s.t. } p = f(s)\}$

(3) of Theorem 2 that we can assume w.l.o.g that $\widehat{F}$ is finite. We will construct a model (Ω, β) that is $CKRMC$ such that $F(\Omega) = \widehat{F}$. Define now (Ω, β) as follows:

(A.5.1) $\quad \Omega \equiv \widehat{F} \times \widehat{F} \times I$

and β is the product of the Borel sets in $\widehat{F} \times \widehat{F}$ and I.

To understand the idea behind this definition of Ω it would be useful to point out why a simpler definition would <u>not</u> work. So suppose we would have defined Ω to be $\widehat{F}$ where we associate with the state f the price function f and the profiles of demands and beliefs $\left\{z_i^f\right\}_{i \in I}$ and $\left\{\mu_i^f\right\}_{i \in I}$ that support f w.r.t $\widehat{F}$ (the beliefs are now interpreted as beliefs on states.) Let i be a specific player and suppose that $\mu_i^{f,1}(\overline{f}) > 0$ for some $\overline{f} \in F$. This means that player i in the state f assigns a positive probability to the state $\overline{f}$ which implies that he is assigning a positive probability to the event where his beliefs are $\mu_i^{\overline{f}}$, but these beliefs are different than his beliefs in the state f, (μ_i^f), so this construction contradicts the assumption that a player knows his own beliefs.

To avoid this contradiction we need a richer set of states, in particular we need a state where the function that is materialized is $\overline{f}$ but player i has the belief μ_i^f and the demand z_i^f. The definition of Ω in (A.5.1) implements this requirement in the following way: The state $\widehat{\omega} = (\overline{f}, f, i)$ is a state in which $\overline{f}$ is materialized and player i has the belief μ_i^f and the demand z_i^f. The complete and formal definition of $\widehat{\omega}$ is as follows: Let $z_j^{\widehat{\omega}}$ and $\mu_j^{\widehat{\omega}}$ denote respectively the demand and belief on Ω of player j in $\widehat{\omega}$. Define $z_j^{\widehat{\omega}} \equiv z_j^{\overline{f}}$ for every $j \neq i$ and $z_i^{\widehat{\omega}} \equiv z_i^f$. So all the players different from i have demand strategies that support $\overline{f}$ while player i has a demand strategy that supports the function f. (We note that since there is a continuum of players the fact that a single player i has a demand that is different than $z_i^{\overline{f}}$ does not change the fact that $\overline{f}$ specifies prices that clear the market.) The beliefs are defined according to the correspondence between states and functions that are materialized in them. Start with player i :

$$\mu_i^{\widehat{\omega}}(\omega) \equiv \begin{cases} \mu_i^f(g) & \omega = (g, f, i) \\ 0 & \text{otherwise} \end{cases}$$ [35]

and for a player $j \neq i$ define:

$$\mu_j^{\widehat{\omega}}(\omega) \equiv \begin{cases} \mu_j^{\overline{f}}(g) & \omega = (g, \overline{f}, j) \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to check that these definitions satisfy requirements 1. and 2. in the definition of a $CKRMC$ model.

[35] The belief $\mu_i^{\widehat{\omega}}$ is a finite sequence of probabilities-$\mu_i^{\widehat{\omega}} = (\mu_i^{\widehat{\omega},1}, ...\mu_i^{\widehat{\omega},m})$. So the definition $\mu_i^{\widehat{\omega}}(\omega) \equiv \mu_i^f(g)$ should be read as follows: Let $\mu_i^f = (\mu_i^{f,1}, .., \mu_i^{f,m})$ for any $1 \leq k \leq m$ $\mu_i^{\widehat{\omega},k}(\omega) = \mu_i^{f,k}(g)$.

The definition of the belief of a player j different from i should be read in a similar way.