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EVOLUTIONARY DYNAMICS FOR LARGE POPULATIONS IN GAMES WITH MULTIPLE BACKWARD INDUCTION EQUILIBRIA

by

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Evolutionary Dynamics for Large Populations in Games with Multiple Backward Induction Equilibria*

Tomer Wexler†

Abstract

This work follows "Evolutionary dynamics and backward induction" (Hart [2000]) in the study of dynamic models consisting of selection and mutation, when the mutation rate is low and the populations are large. Under the assumption that there is a single backward induction (or subgame perfect) equilibrium of a perfect information game, Hart [2000] proved that this point is the only stable state. In this work, we examine the case where there are multiple backward induction equilibria.

1 Introduction

More than two decades after the first connections between evolutionary biology and game theory were made, a major part of the study in this field still revolves around finding evolutionarily stable states. Evolutionary models replace a player with a population of individuals, and a mixed strategy with the proportions of the various strategies in the population. The evolutionary dynamics consist of selection (toward the better replies) and mutation (which is random and relatively rare). Evolutionarily stable states in these models are states that, in the long run, occur with positive probability (bounded away from zero), no matter how rare the mutations are. It turns out that every evolutionarily stable states is a Nash equilibrium, but the converse does not necessarily hold.

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In finite games of perfect information in extensive form, it seems that the natural candidate to be the equivalent of the evolutionary stable states in the evolutionary models is the backward induction equilibrium (BIE). This equilibrium is obtained by induction starting at the final nodes, with each player choosing the best reply given the choices of the players that play after him. This equilibrium is also known as a subgame perfect equilibrium because the strategies yield an equilibrium in every subgame as well.

Since mutations, like perturbations, make every action possible, it follows that every node is reached, and thus, as the perturbations go to zero, this should yield a subgame perfect equilibrium. However, as the literature shows (see Hart [2000]), equilibria other than the BIE can be evolutionarily stable. In Hart's paper it is assumed that the game is generic and has a unique BIE. The main result there (later expanded by Gorodeisky [2003]) is that the backward induction equilibrium becomes in the limit the only evolutionarily stable outcome as the mutation rate decreases to zero and the populations increase to infinity.

In this paper, we continue Hart's work by examining the case where the BIE is not unique. The assumptions on large populations and rare mutations are maintained. We limit ourselves to games with two players and, by reviewing the different cases, we find the limit distribution of both populations. The result is that the evolutionarily stable states are either one or two out of the infinitely many backward induction equilibria in these games.

Two main tools developed in this paper for the case of two players will be useful in trying to expand the results to multi-player games. The first uses the equations satisfied by the invariant distribution. It is easy to use, but it is quite a rough tool and fits mainly the more obvious cases. The second tool is based on estimating the average time the system spends in each state.

2 The model

We now present the so-called "basic model" as it appears in Hart [2000].

2.1 The game

Let Γ be a finite game in extensive form with perfect information. Thus, we have a rooted tree, where each non-terminal vertex corresponds to a move. Each move of one of the players is called a node and the set of all nodes is denoted N. At each node i there is a different player i. At each node $i \in N$, player i has a choice out of the set A^i, which denotes the set of outgoing branches at i. An action of player i is a^i in A^i, and $A := \prod_{i \in N} A^i$ is the set of $N-$tuples of actions. At each leaf (terminal vertex) there are associated payoffs to all players and $u^i : A \to \mathbf{R}$ is the payoff function of player i. As usual, the payoff functions are extended multilinearly to mixed actions; thus $u^i : A \to \mathbf{R}$, where $X := \prod_{i \in N} X^i$ and $X^i := \Delta(A^i) = \{x^i \in \mathbf{R}_+^{A^i} : \sum_{a^i \in A^i} x^i_{a_i} = 1\}$ is the set of probability distributions over A^i.

We now define a population game called the "gene-normal form." At each node i there is a population of $M(i)$ individuals playing the game in the role of player i. We assume that the populations at different nodes are disjoint. Let $\omega^i_q \in A^i$ denote the pure action of an individual $q \in M(i)$, $\omega^i = (\omega^i_q)_{q \in M(i)}$, and $\omega = (\omega^i)_{i \in N}$. Let, for each $a^i \in A^i$,

$$x^i_{a_i} \equiv x^i_{a_i}(\omega^i) := \frac{|q \in M(i) : \omega^i_q = a^i|}{|M(i)|}$$

be the proportion of population $M(i)$ that plays a^i. We can view $x^i \equiv x^i(\omega^i) := (x^i_{a_i}(\omega_i))_{a^i \in A^i} \in X^i$ as a mixed action of player i. The payoff of an individual $q \in M(i)$ is defined as his average payoff against the other populations, i.e., $u^i(\omega^i_q, x^{-i})$ or $u^i(\omega^i_q, \omega^{-i})$.

2.2 The dynamics

We come now to the dynamic model. A state ω specifies the pure action of each individual in each population, i.e., $\omega = (\omega^i)_{i \in N}$, where $\omega^i = (\omega^i_q)_{q \in M(i)}$. Let $\Omega := \prod_{i \in N} (A^i)^{M(i)}$ be the state space. Our process will be a stationary Markov chain with a one-step transition probability matrix $Q \equiv (Q[\widetilde{\omega}|\omega])_{\widetilde{\omega}, \omega \in \Omega}$, which specifies the transition probabilities. The basic model assumes that all populations are of equal size, i.e., $m = |M(i)|$ for each $i \in N$. Given $\mu, \sigma > 0$, such that $\mu + \sigma \leq 1$, the transition matrix entries are given by performing the following process independently for each $i \in N$:

- Choose a random individual $q(i) \in M(i)$, with probability $1/m$ for each individual to be chosen. All other individuals in $M(i)$, $q'(i) \in M(i) \backslash q(i)$ don't change their action.
- Choose $SE(i)$ ("selection") and $MU(i)$ ("mutation"), with probabilities $(1 - \mu), \mu$, respectively.
- If selection was chosen then define the set of "better actions" $B^i := \{a^i \in A^i : u^i(a^i, \omega^{-i}) > u^i(\omega^i_{q(i)}, \omega^{-i})\}$. If the set is not empty then a random action is chosen from B^i, with probability $1/|B^i|$ each, as the new action of $q(i)$; otherwise, there is no change in $q(i)$'s action.

- If mutation was chosen, then a random action in A^i is chosen, with probability $1/A^i$ for each action, as the new action of $q(i)$.[1]

3 Main results

Consider the two-player game

$$\begin{array}{c} 1 \\ L^1 / \ \backslash R^1 \\ 2 \\ L^2 / \quad \backslash R^2 \end{array}$$

with m individuals in each node. We would like to estimate the joint distribution of both populations when the mutation rate μ' goes to zero (for convenience we define $\mu := \mu'/2$) and the population m goes to infinity in the cases where there are multiple BIE. We define a state of the above system to be $z := (x, y)$ where

- x is the number of individuals in population 1 playing left (L^1).
- y is the number of individuals in population 2 playing left (L^2).

  and we define a state of the above system at time t as $z_t := (x_t, y_t)$. Let H_t be the history until time t (inclusive).

Notice that $P\left[(x_t, y_t) \mid H_{t-1}\right] = P\left[(x_t, y_t) \mid (x_{t-1}, y_{t-1})\right]$ because this is a Markovian process.

3.1 The family of games

We consider the case where $U^2(L^1, L^2) = U^2(L^1, R^2)$ (the population in node 2 is indifferent; therefore there are multiple BIE). Without loss of generality assume that $U^2(L^1, L^2) = 0, U^1(L^1, R^2) = 0$; thus the game is (for $\theta, \lambda > 0$)

$$\begin{array}{c} 1 \\ / \ \backslash \\ 2 \ \ (\theta, .) \\ / \quad \backslash \\ (\lambda, 0) \ \ (0, 0) \end{array}$$

3.1.1 The first case: $\lambda/2 = \theta$

Here the set of backward induction equilibria is:

$$BIE = \{(x, y) : x = m, y > m/2\} \cup \{(x, y) : x = 0, y < m/2\} \cup \{(x, y) : y = m/2\}.$$

[1] Examples of this model can be found in Hart [2000].

Let $\pi_{m,\mu}$ denote the unique invariant probability vector[2] of the above system. Denote $\xi := x/m,\ \eta := y/m.$ $\pi_{m,\mu} \in [0,m]^2$ yields a probability measure on $[0,1]^2$ which we denote as $\widehat{\pi}_{m,\mu}(\xi,\eta)$. Our Main Theorem is ($\Phi$ denotes the cumulative normal distribution):

Main Theorem

$$\widehat{\pi}_{m,\mu}(\xi,\eta) \underset{\frac{1}{m},\mu\to 0}{\Longrightarrow} \frac{1}{2}\mathbf{1}_{(1,\frac{1}{2})} + \frac{1}{2}\mathbf{1}_{(0,\frac{1}{2})}$$

(where "$\Longrightarrow$" denotes weak convergence of measures and $\mathbf{1}_\zeta$ is the Dirac measure on ζ). Moreover,

$$\forall\varepsilon > 0, \exists d > 0, \text{s.t.}:\ \forall m, \forall\mu > 0,\quad \forall\alpha,\beta : 0 < \alpha < \beta$$
$$\left|\pi_{m,\mu}\left[\left\{(x,y)\mid x \geq m(1-\varepsilon),\ \alpha \leq \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \leq \beta\right\}\right] - (\Phi(\beta)-\Phi(\alpha))\right| \leq d(\sqrt{\frac{1}{m}} + \sqrt{\mu})$$
and for $\alpha < \beta < 0$
$$\left|\pi_{m,\mu}\left[\left\{(x,y)\mid x \leq m\varepsilon,\ \alpha \leq \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \leq \beta\right\}\right] - (\Phi(\beta)-\Phi(\alpha))\right| \leq d(\sqrt{\frac{1}{m}} + \sqrt{\mu}).$$

In other words, our Main Result is that in the limit when $1/m, \mu \to 0$, the distribution of population 2 is concentrated around two BIE. These are: (i) all the population at node 1 is playing left and half the population at node 2 is playing left; and (ii) all the population at node 1 is playing right and half the population at node 2 is playing left (see Fig. 1 in the appendix).

To prove our result, we start with player 2. Notice that the y_t process is independent of the x_t process, so for every $i < t$ we have $P[y_t \mid (x_i, y_i)] = P[y_t \mid y_i]$ (because the population at node 2 is not affected by the population at node 1).

Lemma 1 *Let $\pi^y_{m,\mu}$ denote the marginal invariant distribution on population 2. For every $\mu > 0, m$, we have*

$$\pi^y_{m,\mu} = Binomial(m, \frac{1}{2}).$$

Proof. We have a one-dimensional random walk, where for every $i > 0, t > 0$

$$P[y_t = i-1 \mid y_{t-1} = i] = \frac{i}{m}\mu$$

[2] The dynamic is irreducible because mutations make every state reachable from any other state, and aperiodic because for each state there is a positive probability of staying.

and for every $i < m$

$$P\left[y_t = i+1 \mid y_{t-1} = i\right] = \frac{m-i}{m}\mu.$$

Therefore, we know for the invariant distribution that

$$\begin{aligned}
&\pi^y_{m,\mu}\left[i\right]\frac{m-i}{m}\mu = \pi^y_{m,\mu}\left[i+1\right]\frac{i+1}{m}\mu \\
\Rightarrow\ &\pi^y_{m,\mu}\left[i+1\right] = \frac{m-i}{i+1}\pi^y_{m,\mu}\left[i\right] \\
\Rightarrow\ &\pi^y_{m,\mu}\left[i\right] = \binom{m}{i}\pi^y_{m,\mu}\left[0\right].
\end{aligned}$$

Now, $\pi^y_{m,\mu}$ is a probability distribution so

$$\begin{aligned}
1 &= \sum_{i=0}^{m}\pi^y_{m,\mu}\left[i\right] = \pi^y_{m,\mu}\left[0\right]\sum_{i=0}^{m}\binom{m}{i} = 2^m\pi^y_{m,\mu}\left[0\right] \\
&\Rightarrow\ \pi^y_{m,\mu}\left[0\right] = \frac{1}{2^m} \Rightarrow \pi^y_{m,\mu}\left[i\right] = \binom{m}{i}\frac{1}{2^m},
\end{aligned}$$

which completes the proof. ■

We define two sets of states ($c > 0$, $\varepsilon > 0$):

- $A_{c,\varepsilon} := \{z : x < (1-\varepsilon)m \text{ and } y \geq m/2 + c\sqrt{m}\}$ (all states in which, given a parameter c and an ε, at least $m/2 + c\sqrt{m}$ individuals in population 2 are playing left and less than $(1-\varepsilon)m$ individuals in population 1 are playing left).

- $B_\varepsilon := \{z : x \geq (1-\varepsilon)m \text{ and } y > m/2\}$ (all states in which, given a parameter ε, more than half the population 2 are playing left and at least $(1-\varepsilon)m$ individuals in population 1 are playing left).

Lemma 2 *For every constants $c_1, c_3 > 0, \varepsilon > 0$ there exists a constant $d > 0$ such that for every $\mu > 0, m$, for any $z_0 \in A_{c_1,\varepsilon}$ we have*

$$P\left[\min_{0\leqslant t\leq \tau m} y_t \leq \frac{m}{2}\,\middle|\, z_0\right] \leq d\sqrt{\mu},$$

where $\tau := \frac{c_3}{\sqrt{\mu}}m$.

Remark 1 *Clearly, it is enough to prove this for* $y_0 = \lceil m/2 + c_1\sqrt{m}\rceil$ *because for* $y'_0 \geq y_0$ *we get*

$$\begin{aligned}
& P\left[\min_{0\leq t\leq\tau} y_t \leq \frac{m}{2}\,\middle|\, y'_0\right] \\
= & \sum_{t'=0}^{\tau} P\left[t' = \inf_{0\leq t}\left\{t : y_t = \lceil\frac{m}{2} + c_1\sqrt{m}\rceil\right\}\right] P\left[\min_{t'\leq t\leq\tau} y_t \leq \frac{m}{2}\,\middle|\, y_{t'} = \lceil\frac{m}{2} + c_1\sqrt{m}\rceil\right] \\
\leq & \max_{0\leq t'\leq\tau} P\left[\min_{t'\leq t\leq\tau} y_t \leq \frac{m}{2}\,\middle|\, y_{t'} = \lceil\frac{m}{2} + c_1\sqrt{m}\rceil\right] \\
= & \max_{0\leq t'\leq\tau} P\left[\min_{0\leq t\leq\tau-t'} y_t \leq \frac{m}{2}\,\middle|\, y_0 = \lceil\frac{m}{2} + c_1\sqrt{m}\rceil\right] \\
\leq & P\left[\min_{0\leq t\leq\tau} y_t \leq \frac{m}{2}\,\middle|\, y_0 = \lceil\frac{m}{2} + c_1\sqrt{m}\rceil\right].
\end{aligned}$$

Proof. Define a stopping time $T := \inf\{t : |y_t - y_0| \geq c_1\sqrt{m}\}$. Notice that $P\left[\min_{0\leq t\leq\tau} y_t \leq m/2 \,\middle|\, y_0\right] \leq P\left[T \leq \tau \,|\, y_0\right]$ because the event on the left is contained in the one on the right. Also define $\widehat{y}_t := y_{t\wedge T}$ for every $t \geq 0$ and $w_t := \widehat{y}_t - \widehat{y}_{t-1}$ for every $t \geq 1$, so $\widehat{y}_t = y_0 + \sum_{i=1}^{t} w_t$. Notice that $w_t = 0$ for $t > T$, and for $t \leq T$ we have

$$w_t = \begin{cases} 0 & \text{with probability } 1-\mu \\ +1 & \text{with probability } p(y_{t-1}) = \frac{m-y_{t-1}}{m}\mu \\ -1 & \text{with probability } q(y_{t-1}) = \frac{y_{t-1}}{m}\mu. \end{cases}$$

Define $v_t := w_t - E\left[w_t |\, H_{t-1}\right]$; these are martingale differences; hence

$$\begin{aligned}
&\forall t \quad E\left[v_t\right] = 0. \\
&i \neq j \;\; E\left[v_i v_j\right] = 0. \\
&\forall t \quad Var(v_t) \leq Var(w_t) \leq \left\{\begin{array}{ll} \mu & \text{if } t \leq T \\ 0 & \text{if } t > T \end{array}\right\} \leq \mu. \\
&\forall t \quad Var\Big(\sum_{i=1}^{t} v_i\Big) = \sum_{i=1}^{t} Var(v_i) \leq t\mu.
\end{aligned}$$

Based on Kolmogorov's inequality for martingale differences (see Sheldon [1996]), we get

$$(1)\; P\left[\max_{1\leq k\leq\tau} \Big|\sum_{i=1}^{k} v_i\Big| > \frac{1}{2}c_1\sqrt{m}\right] \leq \frac{\tau\mu}{\frac{1}{4}c_1^2 m} \leq \frac{4c_3}{c_1^2}\sqrt{\mu}.$$

Let $i \le \tau$. If $i \le T$ then $m/2 \le y_i \le m/2 + 2c_1\sqrt{m} + 2$ (recall that $y_0 = \lceil m/2 + c_1\sqrt{m}\rceil$) and we get

$$\begin{aligned} & E\left[w_i \mid H_{i-1}\right] = \; p(y_{i-1}) - q(y_{i-1}) \\ \le \; & \left|\mu\frac{\frac{m}{2} - 2c_1\sqrt{m} - 2}{m} - \mu\frac{\frac{m}{2} + 2c_1\sqrt{m} + 2}{m}\right| \le \frac{4(c_1+1)}{\sqrt{m}}\mu. \end{aligned}$$

This inequality also holds for $i > T$, since then $w_i = 0$. Hence

$$(2)\ \max_{1\le k\le\tau}\sum_{i=1}^{k}\left|E\left[w_i \mid H_{i-1}\right]\right| \le \frac{4(c_1+1)}{\sqrt{m}}\mu\frac{c_3}{\sqrt{\mu}}m \le 4(c_1+1)c_3\sqrt{\mu}\sqrt{m}.$$

Choose $\delta > 0$ such that $4(c_1+1)c_3\sqrt{\delta} < c_1/2$; then for all $\mu < \delta$ the right-hand side in (2) becomes less than $c_1\sqrt{m}/2$. Also choose $d_1 > 0$ such that $d_1 = 4c_3/c_1^2$, so

$$\begin{aligned} & P\left[T \le \tau \mid y_0\right] \le P\left[\max_{1\le k\le\tau}\left|\sum_{i=1}^{k} w_i\right| \ge c_1\sqrt{m} \,\middle|\, y_0\right] \\ \le \; & P\left[\max_{1\le k\le\tau}\left(\left|\sum_{i=1}^{k} v_i\right| + \sum_{i=1}^{k}\left|E\left[(w_i \mid H_{i-1})\right]\right|\right) \ge c_1\sqrt{m} \,\middle|\, y_0\right] \\ & \overset{(2)}{\le} P\left[\max_{1\le k\le\tau}\left|\sum_{i=1}^{k} v_i\right| > \frac{1}{2}c_1\sqrt{m} \,\middle|\, y_0\right] \overset{(1)}{\le} d_1\sqrt{\mu}. \end{aligned}$$

Thus, we have proved that for every $c_1, c_3 > 0, \varepsilon > 0$ there exists $\delta > 0$ and a constant $d_1 > 0$ such that for every m, for any $(x_0, y_0) \in A_{c_1,\varepsilon}$, we have

$$P\left[\min_{0\leqslant t\leqslant\tau} y_t \le \frac{m}{2} \,\middle|\, z_0\right] \le d_1\sqrt{\mu} \quad \text{for all } \mu < \delta.$$

Now, take $d = \max(d_1, \frac{1}{\sqrt{\delta}})$ and the result follows. ■

Lemma 3 *For every constants $c_1 > 0, \varepsilon > 0$ there exist constants $c_2, d > 0$ such that for every $\mu > 0, m$, for any $z_0 \in A_{c_1,\varepsilon}$, we have*

$$P\left[\max_{0\le t\le c_2 m} x_t > (1 - \frac{\varepsilon}{2})m \,\middle|\, z_0\right] \ge 1 - d(\frac{1}{m} + \sqrt{\mu}).$$

Proof. Define a stopping time $T := \inf\{t : y_t \le m/2\}$; also define

$$w_t := \begin{cases} x_t - x_{t-1} & \text{if } x_{t-1} \le (1 - \frac{\varepsilon}{2})m \text{ and } t \le T \\ \frac{\varepsilon}{6} & \text{if } x_{t-1} > (1 - \frac{\varepsilon}{2})m \text{ or } t > T \end{cases}$$

and we know that for $t \leq T$

$$x_t - x_{t-1} = \begin{cases} 0 & \text{with probability } 1-p-q \\ +1 & \text{with probability } p(x_{t-1}, y_{t-1}) = \frac{m - x_{t-1}}{m}(1-\mu) \\ -1 & \text{with probability } q(x_{t-1}, y_{t-1}) = \frac{x_{t-1}}{m}\mu. \end{cases}$$

Notice that when $x_i \leq (1-\varepsilon/2)m$ and $i \leq T$ we get $p(x_{i-1}) \geq (1-\mu)\varepsilon/2$ and $q(x_{i-1}) \leq (1-\varepsilon/2)\mu$. So we can choose $\delta > 0$ such that for every $\mu \leq \delta$ we get $p(x_{i-1}) \geq \varepsilon/3$ and $q(x_{i-1}) \leq \varepsilon/6$ and it follows that $E[w_i \mid H_{i-1}] \geq \varepsilon/3 - \varepsilon/6 = \varepsilon/6$. This inequality also holds in the other case, where $w_t = \varepsilon/6$. Define $\widehat{x}_t := \sum_{i=1}^{t} w_i + x_0$ and notice that when $t \leq T$ and $\max_{0 \leq i \leq t} x_i \leq (1-\varepsilon/2)m$ we have $\widehat{x}_t = x_t$.

Define $v_t := w_t - E[w_t \mid H_{t-1}]$; these are martingale differences so it follows that

$$\forall t \quad Var(v_t) \leq Var(w_t) \leq Var(x_t - x_{t-1}) \leq 1$$
$$\forall t \quad Var(\sum_{i=1}^{t} v_i) = \sum_{i=1}^{t} Var(v_i) \leq t.$$

Choose $c_2 = \lceil 6/\varepsilon \rceil$ and $d_1 = 4c_2/\varepsilon^2$; from Chebyshev's inequality we get

$$P\left[|\sum_{i=1}^{c_2 m} v_i| \geq \frac{\varepsilon}{2} m \right] \leq \frac{c_2 m}{(\frac{\varepsilon}{2} m)^2} = (\frac{4c_2}{\varepsilon^2})\frac{1}{m} = d_1 \frac{1}{m}.$$

Therefore

$$\begin{aligned} & P\left[\max_{0 \leq t \leq c_2 m} \widehat{x}_t > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0 \right] \\ \geq\ & P\left[\widehat{x}_{c_2 m} > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0 \right] \\ \geq\ & P\left[\sum_{i=1}^{c_2 m} w_i > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0 \right] \\ =\ & P\left[\sum_{i=1}^{c_2 m} E[w_i \mid H_{i-1}] + \sum_{i=1}^{c_2 m} v_i > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0 \right] \\ \geq\ & P\left[\lceil \frac{6}{\varepsilon} \rceil m \frac{\varepsilon}{6} + \sum_{i=1}^{c_2 m} v_i > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0 \right] \\ \geq\ & P\left[|\sum_{i=1}^{c_2 m} v_i| < \frac{\varepsilon}{2} m \,\middle|\, z_0 \right] \geq 1 - d_1 \frac{1}{m}. \end{aligned}$$

By Lemma 2 (with $c_1 = c_1, c_3 = c_2, \varepsilon = \varepsilon$), there exists $d_2 \geq d_1$, which guarantees $P\left[T \leq c_2 m \mid z_0\right] \leq P\left[T \leq \frac{c_2 m}{\sqrt{\mu}} \,\middle|\, z_0\right] \leq d_2\sqrt{\mu}$, and so

$$P\left[\max_{0\leq t\leq c_2 m} \widehat{x}_t > (1-\frac{\varepsilon}{2})m \textit{ and } T > c_2 m \,\middle|\, z_0\right] \geq 1 - d_2(\frac{1}{m} + \sqrt{\mu}).$$

Let $t_0 := \min\{t : \widehat{x}_t > (1-\varepsilon/2)m\}$. Notice that if $\max\limits_{0\leq t\leq c_2 m} \widehat{x}_t > (1-\varepsilon/2)m$ and $T > c_2 m$ then $\widehat{x}_{t_0} = x_{t_0}$, and so $x_{t_0} > (1-\varepsilon/2)m$. Therefore

$$\begin{aligned} & P\left[\max_{0\leq t\leq c_2 m} x_t > (1-\frac{\varepsilon}{2})m \,\middle|\, z_0\right] \\ \geq\ & P\left[\max_{0\leq t\leq c_2 m} \widehat{x}_t > (1-\frac{\varepsilon}{2})m \textit{ and } T > c_2 m \,\middle|\, z_0\right] \\ \geq\ & 1 - d_2(\frac{1}{m} + \sqrt{\mu}). \end{aligned}$$

This holds for all $\mu < \delta$. Take $d = \max(d_2, 1/\sqrt{\delta})$; the result follows. ■

Proposition 4 *For every constants $c_1 > 0, \varepsilon > 0$ there exists a constant $c_2 > 0$ such that for any $c_3 \geq c_2$ there exists $d > 0$ such that for every $\mu > 0, m$, for any $z_0 \in A_{c_1,\varepsilon}$, if we define $g := c_3/\max(\sqrt{\mu}, \sqrt{1/m})$, we have*

$$P\left[\bigcap_{c_2 m\leq t\leq gm} \{z_t \in B_\varepsilon\} \,\middle|\, z_0\right] \geq 1 - d(\sqrt{\frac{1}{m}} + \sqrt{\mu}). \quad (*)$$

Proof. Define stopping times $T_1 := \inf\{t : x_t > (1-\varepsilon/2)m\}$ and $T_2 := \inf\{t : y_t \leq m/2\}$. Define

$$w_t := \begin{cases} 0 & \text{with probability } 1-\mu \\ +1 & \text{with probability } \frac{m-x_{t-1}}{m}(1-\mu) \\ -1 & \text{with probability } \frac{x_{t-1}}{m}\mu. \end{cases}$$

It is always true that $E\left[w_t \mid H_{t-1}\right] \geq -\mu$. Define $\widehat{x}_t := \sum\limits_{i=1}^{t} w_i + x_0$. Notice that for $t \leq T_2$ we have $w_t = x_t - x_{t-1}$, so $\widehat{x}_t = x_t$. From Lemma 4 there exists $d_1, c_2 > 0$ such that we have

$$(1)\ P\left[T_1 \leq c_2 m \mid z_0\right] \geq 1 - d_1(\frac{1}{m} + \sqrt{\mu}).$$

Define $v_t := w_t - E\left[w_t \mid H_{t-1}\right]$; these are martingale differences so it follows that

$$\forall t \quad Var(v_t) \leq Var(w_t) \leq 1 \Rightarrow \forall t \quad Var(\sum_{i=1}^{t} v_i) = \sum_{i=1}^{t} Var(v_i) \leq t.$$

Given $c_3 > c_2$ choose $d_2 = \max((16c_3/\varepsilon^2), d_1)$, from Kolmogorov's inequality for martingale differences we get for each $t_1 \leq c_2 m$

$$P\left[\max_{t_1 \leq t \leq gm} |\sum_{i=t_1}^{t} v_i| > \frac{\varepsilon}{4}m\right] \leq \frac{gm}{(\frac{\varepsilon}{4}m)^2} \leq (\frac{16c_3}{\varepsilon^2})\frac{1}{\sqrt{m}} \leq d_2\sqrt{\frac{1}{m}}.$$

Now, choose $\delta > 0$ such that $c_3\sqrt{\delta} < \varepsilon/4$ so that we have

$$(2)\ \sum_{i=0}^{gm} E\left[w_i \mid H_{i-1}\right] \geq gm(-\mu) \geq -c_3 m\sqrt{\delta} > -\frac{\varepsilon}{4}m.$$

Therefore for each $t_1 \leq c_2 m$ (note that in time T_1, $\widehat{x}_t = (1-\varepsilon/2)m$)

$$\begin{aligned}
& P\left[\exists t,\ t_1 \leq t \leq gm : \widehat{x}_t < (1-\varepsilon)m \,|\, T_1 = t_1, z_0\right] \\
\leq\ & P\left[\min_{t_1 \leq t \leq gm} \sum_{i=t_1}^{t} w_i < -\frac{\varepsilon}{2}m \,\middle|\, T_1 = t_1, z_0\right] \\
=\ & P\left[\min_{t_1 \leq t \leq gm} (\sum_{i=t_1}^{t} E\left[w_i \mid H_{i-1}\right] + \sum_{i=t_1}^{t} v_i) < -\frac{\varepsilon}{2}m \,\middle|\, T_1 = t_1, z_0\right] \\
& \overset{(2)}{\leq} P\left[\max_{t_1 \leq t \leq gm} |\sum_{i=t_1}^{t} v_i| > \frac{\varepsilon}{4}m \,\middle|\, T_1 = t_1, z_0\right] \leq d_2\sqrt{\frac{1}{m}}\,.
\end{aligned}$$

Hence

$$(3)\ P\left[\bigcap_{T_1 \leq t \leq gm} \{\widehat{x}_t \geq (1-\varepsilon)m\} \,\middle|\, T_1 \leq c_2 m, z_0\right] \geq 1 - d_2\sqrt{\frac{1}{m}}$$

so

$$\begin{aligned}
& P\left[\bigcap_{T_1 \leq t \leq gm} \{\widehat{x}_t \geq (1-\varepsilon)m\}\ \mathit{and}\ T_1 \leq c_2 m \,\middle|\, z_0\right] \\
=\ & P\left[\bigcap_{T_1 \leq t \leq gm} \{\widehat{x}_t \geq (1-\varepsilon)m\} \,\middle|\, T_1 \leq c_2 m, z_0\right] P\left[T_1 \leq c_2 m \mid z_0\right] \\
& \overset{(1)+(3)}{\geq} (1 - d_2\sqrt{\frac{1}{m}})(1 - d_2(\frac{1}{m} + \sqrt{\mu})) \geq 1 - 2d_2(\sqrt{\frac{1}{m}} + \sqrt{\mu}).
\end{aligned}$$

Using Lemma 2 with $\varepsilon = \varepsilon, c_1 = c_1, c_3 = c_3$ we can choose $d_3 \geq d_2$ and we get $P\left[T_2 > gm |\, z_0\right] \geq (1 - d_3\sqrt{\mu})$. Therefore

$$\begin{aligned}
& P\left[\left(\bigcap_{T_1 \leq t \leq gm} \{\widehat{x}_t \mid \widehat{x}_t \geq (1-\varepsilon)\}\, m\ \mathit{and}\ T_1 \leq c_2 m\right)\ \mathit{and}\ T_2 > gm \,\middle|\, z_0\right] \\
\geq\ & 1 - d_3\sqrt{\mu} - 2d_3(\sqrt{\frac{1}{m}} + \sqrt{\mu}) = 1 - 3d_3(\sqrt{\frac{1}{m}} + \sqrt{\mu}).
\end{aligned}$$

Define $d := 3d_3$. As we have already seen, for $t \leq T_2$ we have $\widehat{x}_t = x_t$. The above event includes

$$\left\{ \exists t_1 \leq c_2 m : x_{t_1} > (1 - \frac{\varepsilon}{2})m \; and \; \forall t, t_1 \leq t \leq gm : \; z_t \in B_\varepsilon \right\},$$

which in turn includes

$$\bigcap_{c_2 m \leq t \leq gm} \{z_t \in B_\varepsilon\},$$

and the proof is completed. ■

Proposition 5 *For every constants $c_1 > 0, \varepsilon > 0$ there exists a constant $d > 0$ such that for every $\mu > 0, m$, we have*

$$\pi_{m,\mu}\left[A_{c_1,\varepsilon}\right] \leq d(\sqrt{\frac{1}{m}} + \sqrt{\mu}).$$

Proof. Since $\pi_{m,\mu}$ is the unique invariant vector of our stationary Markov process, we have

$$\begin{aligned} & \forall z' \forall z_0, P\left[z_t = z' \mid z_0\right] \underset{t\to\infty}{\rightarrow} \pi_{m,\mu}\left[z'\right] \\ \Rightarrow \quad & P\left[z_t \in A_{c_1,\varepsilon} \mid z_0\right] \underset{t\to\infty}{\rightarrow} \pi_{m,\mu}\left[A_{c_1,\varepsilon}\right]. \end{aligned}$$

Using Proposition 5 and $A_{c_1,\varepsilon} \cap B_\varepsilon = \emptyset$ we know that there exists a constant $c_2 > 0$ such that for every $c_3 \geq c_2$ we get a constant d_1 for which the claim $(*)$ of Proposition 4 holds.

Notice that since this is a Markovian process, the claim will still be valid starting at a certain time t', so

$$\forall z_{t'} \in A_{c_1,\varepsilon}, \; P\left[\bigcap_{c_2 m + t' \leq t \leq gm + t'} \{z_t \in B_\varepsilon\} \;\middle|\; z_{t'}\right] \geq 1 - d(\sqrt{\frac{1}{m}} + \sqrt{\mu}).$$

Now, for each $\mu > 0, m$ we know that

$$\forall \varepsilon > 0, \exists t_0^{m,\mu} \equiv: \forall t \geq t_0^{m,\mu}, \left|P\left[\{z_t \in A_{c_1,\varepsilon}\} \mid z_0\right] - \pi_{m,\mu}\left[A_{c_1,\varepsilon}\right]\right| < \sqrt{\mu}.$$

Define $T_1 := \inf\{t' : t' \geq t_0, z_{t'} \in A_{c_1,\varepsilon}\} \wedge (t_0 + gm)$. For every $T_1 = t_1$ we get from Proposition 5 (choose $c_3 = c_2$) that

$$\begin{aligned} & P\left[\#\{t_1 \leq t \leq t_0 + gm : z_t \in A_{c_1,\varepsilon}\} \leq c_2 m \mid T_1 = t_1, z_0\right] \geq 1 - d_1(\sqrt{\frac{1}{m}} + \sqrt{\mu}) \\ \Rightarrow \quad & P\left[\#\{T_1 \leq t \leq t_0 + gm : z_t \in A_{c_1,\varepsilon}\} \leq c_2 m \mid z_0\right] \geq 1 - d_1(\sqrt{\frac{1}{m}} + \sqrt{\mu}) \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow\ E\left[\#\left\{t_0 \le t \le t_0+gm : z_t \in A_{c_1,\varepsilon}\right\} \mid z_0\right] \\
&\le\ (1-d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu}))c_2 m + d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu})gm \le c_2 m + d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu})gm \\
&\Rightarrow\ E\left[\sum_{t=t_0}^{t_0+gm} 1_{z_t \in A_{c_1,\varepsilon}} \,\middle|\, z_0\right] \le c_2 m + d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu})gm \\
&\Rightarrow\ \frac{1}{gm}\sum_{t=t_0}^{t_0+gm} P\left[\{z_t \in A_{c_1,\varepsilon}\} \mid z_0\right] \le \frac{c_2 m + d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu})gm}{gm} \\
&\le\ (\sqrt{\mu}+\sqrt{\frac{1}{m}}) + d_1(\sqrt{\frac{1}{m}}+\sqrt{\mu}) \le (d_1+1)(\sqrt{\frac{1}{m}}+\sqrt{\mu}) \\
&\Rightarrow\ \exists t \ge t_0 : P\left[\{z_t \mid z_t \in A_{c_1,\varepsilon}\} \mid z_0\right] \le (d_1+1)(\sqrt{\frac{1}{m}}+\sqrt{\mu}).
\end{aligned}
$$

Now, since we chose t_0 such that for any $t \ge t_0, |P[\{z_t \mid z_t \in A_{c_1,\varepsilon}\} \mid z_0] - \pi_{m,\mu}[\{z \mid z \in A_{c_1,\varepsilon}\}]| \le \sqrt{\mu}$ we get that $\pi_{m,\mu}[A_{c_1,\varepsilon}] \le (d_1+2)(\sqrt{1/m}+\sqrt{\mu})$. We define $d = d_1 + 2$ and the result follows. ■

Theorem 6 *For every $\varepsilon > 0$ there exists $d > 0$ such that for every $\mu > 0, m$ for every $\alpha, \beta : 0 < \alpha \le \beta$, we have*

$$
\left|\pi_{m,\mu}\left[\left\{(x,y) \mid x \ge m(1-\varepsilon),\ \alpha \le \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right\}\right] - (\Phi(\beta)-\Phi(\alpha))\right| \le d(\sqrt{\frac{1}{m}}+\sqrt{\mu})
$$

and

$$
\left|\pi_{m,\mu}\left[\left\{(x,y) \mid x \le m\varepsilon,\ -\beta \le \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le -\alpha\right\}\right] - (\Phi(-\alpha)-\Phi(-\beta))\right| \le d(\sqrt{\frac{1}{m}}+\sqrt{\mu}).
$$

Proof. Given that our system is symmetric (a state with x individuals playing left at node 1 and y individuals playing left at node 2 is symmetric to a state with $m-x$ individuals playing left at node 1 and $m-y$ individuals playing left at node 2), we get that $\pi_{m,\mu}[(x,y)]$ is the same as $\pi_{m,\mu}[(m-x,m-y)]$. Therefore, it is enough to estimate the probability $\pi_{m,\mu}[x,y]$ for every $0 \le x \le m, m/2 \le y \le m$ and thus we prove only the first statement of the Theorem.

We know from Lemma 1 that the marginal invariant distribution of population 2 is $\pi^y_{m,\mu} := Binomial(m, 1/2)$. Choose $c_1 > 0$ such that $c_1 < \alpha$; using the Berry-Esséen Theorem (see Alan [1993]) we have a constant d_1 such that

$$
\left|\pi_{m,\mu}\left[\alpha \le \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right] - (\Phi(\beta)-\Phi(\alpha))\right| \le d_1\frac{1}{\sqrt{m}}.
$$

Now, using Proposition 5 we have a $d_2 > 0$, such that

$$
\begin{aligned}
& \pi_{m,\mu}\left[\left\{(x,y)\,|\,x < m(1-\varepsilon),\ \alpha \le \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right\}\right] \le \pi_{m,\mu}\left[A_{c_1,\varepsilon}\right] \le d_2(\sqrt{\frac{1}{m}} + \sqrt{\mu}) \\
\Rightarrow\ & \left|\pi_{m,\mu}\left[\left\{(x,y)\,|\,x \ge m(1-\varepsilon),\ \alpha \le \frac{y-\frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right\}\right] - (\Phi(\beta)-\Phi(\alpha))\right| \\
\le\ & d_1\frac{1}{\sqrt{m}} + d_2(\sqrt{\frac{1}{m}} + \sqrt{\mu}) \le (d_1+d_2)(\sqrt{\frac{1}{m}} + \sqrt{\mu}).
\end{aligned}
$$

Take $d := d_1 + d_2$ and the result follows. ■

Proof of the Main Theorem. Let $\nu := \frac{1}{2}\mathbf{1}_{(1,\frac{1}{2})} + \frac{1}{2}\mathbf{1}_{(0,\frac{1}{2})}$; we will show that $\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \ge \nu(G)$ for any open set $G \subset [0,1]^2$ (see Billingsley [1968]). Given such a set G with $(1,\frac{1}{2}) \in G$ we can find $\delta_1, \delta_2 > 0$ and a rectangle $F := \{(\xi,\eta)\,|0 \le 1-\xi \le \delta_1, |1/2-\eta| \le \delta_2\}$ such that $F \subset G$. We know that if we choose $\varepsilon = \delta_1$ in Theorem 3.6, we get, for every $\alpha, \beta : 0 < \alpha < \beta$,

$$
\begin{aligned}
& \forall\alpha,\beta : \widehat{\pi}_{m,\mu}\left[F\right] \ge \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \ge 1-\delta_1,\ \frac{1}{2}+\delta_2 \ge \eta \ge \frac{1}{2}\right\}\right] \\
\ge\ & \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \ge 1-\delta_1,\ \frac{1}{2}m + \frac{1}{4}\beta\sqrt{m} \ge \eta m \ge \frac{1}{2}m + \frac{1}{4}\alpha\sqrt{m}\right\}\right] \\
\ge\ & \Phi(\beta) - \Phi(\alpha).
\end{aligned}
$$

Now, this holds for all $0 < \alpha < \beta$; taking $\alpha \to 0$ and $\beta \to \infty$ therefore yields

$$
\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \ge \liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(F) \ge \Phi(\infty) - \Phi(0) = \frac{1}{2}.
$$

Similarly, given an open set G' with $(0,\frac{1}{2}) \in G'$ we have

$$
\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G') \ge \frac{1}{2}.
$$

For any other open set G'' if it does not contain $(1,\frac{1}{2})$ and $(0,\frac{1}{2})$ then $\nu(G'') = 0$ and the claim is trivial. If it contains both points, then it contains two open sets G, G' such that $(1,\frac{1}{2}) \in G$, $(0,\frac{1}{2}) \in G'$ and $G \cap G' = \emptyset$ so

$$
\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G'') \ge \liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) + \liminf_{\frac{1}{m},\mu\to 0} \pi_{m,\mu}(G') \ge 1.
$$

With this we have finished showing that $\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \ge \nu(G)$ for any open set G, which, together with Theorem 3.6, yields the Theorem. ■

3.1.2 The second case: $\lambda/2 > \theta$

Here the set of backward induction equilibria is:

$BIE = \{(x,y) : x = m, y > m\theta/\lambda\} \cup \{(x,y) : x = 0, y < m\theta/\lambda\} \cup \{(x,y) : y = m\theta/\lambda\}$.

Theorem 7

$$(1)\ \widehat{\pi}_{m,\mu}(\xi,\eta) \underset{\frac{1}{m},\mu\to 0}{\Longrightarrow} \mathbf{1}_{(1,\frac{1}{2})}$$

and for every $\varepsilon > 0$, there exist constants $c, d > 0$ such that for every $\mu > 0, m$ for every $\alpha, \beta : -\infty \leq \alpha \leq \beta \leq \infty$, we have

$$(2)\ \left|\pi_{m,\mu}\left[\left\{(\xi,\psi) \mid \xi \geq (1-\varepsilon),\ \alpha \leq \frac{\eta m - \frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \leq \beta\right\}\right] - (\Phi(\beta) - \Phi(\alpha))\right| \leq de^{-cm}.$$

Thus, evolutionary stability here yields a unique BIE (see Fig. 2 in the appendix).

Proof. We start with the proof of inequality (2).

Define $U_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda > \theta\right\}\right]$, $V_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda < \theta\right\}\right]$ and $W_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda = \theta\right\}\right]$. Now we have an inequality for binomial distribution (see Hoeffding [1963]), namely,

$$P\left[B(t,p) \leq pt - \delta t\right] \leq e^{-2t\delta^2}$$

for all $\delta > 0$. Therefore, knowing that $\pi^y_{m,\mu} = Binomial(m, 1/2)$ and $\theta/\lambda < 1/2$, we have

$$\exists c' : \sum_{i=0}^{m} (V_i + W_i) \leq e^{-c'm}.$$

Since we have a random walk on the population at node 1, for every $i,\ 0 \leq i < m$, we get

$$\begin{aligned} & U_i(1-\mu)\frac{m-i}{m} + V_i\mu\frac{m-i}{m} + W_i\mu\frac{m-i}{m} \\ = \ & U_{i+1}\mu\frac{i+1}{m} + V_{i+1}(1-\mu)\frac{i+1}{m} + W_i\mu\frac{i+1}{m} \\ \Rightarrow \ & U_i(1-\mu)\frac{m-i}{m} \leq U_{i+1}\mu\frac{i+1}{m} + e^{-c'm}. \end{aligned}$$

Using it for $i + 1 < (1 - \varepsilon/2)m$ we get

$$\begin{aligned} & U_i(1-\mu)\frac{m-i}{m} - e^{-c'm} \leq U_{i+1}\mu\frac{i+1}{m} \\ \Rightarrow \ & U_i\frac{(1-\mu)}{\mu}\frac{m-i}{i+1} - \frac{m}{\mu}e^{-c'm} \leq U_{i+1} \\ \Rightarrow \ & \frac{(1-\mu)}{\mu}\left(U_i\frac{\frac{\varepsilon}{2}}{(1-\frac{\varepsilon}{2})} - me^{-c'm}\right) \leq U_{i+1}. \end{aligned}$$

Now, assume by contradiction that there exists i', $i' < (1-\varepsilon)m$ such that $U_{i'} \geq e^{-c'm/2}$; then

$$\begin{aligned}
\exists m' \quad &: \quad \forall m > m',\ \frac{1}{2}e^{-\frac{c'm}{2}}\frac{\frac{\varepsilon}{2}}{(1-\frac{\varepsilon}{2})} > me^{-c'm} \\
&\Rightarrow \quad \forall m > m',\ \frac{(1-\mu)}{\mu}\left(\frac{1}{2}U_{i'}\frac{\frac{\varepsilon}{2}}{(1-\frac{\varepsilon}{2})}\right) \leq U_{i'+1} \\
&\Rightarrow \quad \forall m > m',\ \frac{(1-\mu)\frac{\varepsilon}{2}}{2\mu(1-\frac{\varepsilon}{2})}U_{i'} \leq U_{i'+1}.
\end{aligned}$$

So choosing $\delta > 0$ such that

$$\frac{(1-\delta)\frac{\varepsilon}{2}}{2\delta(1-\frac{\varepsilon}{2})} > 2$$

we get for every $\mu < \delta,\ m > m'$ and i, $i' \leq i < (1-\varepsilon/2)m-1$ that $U_{i+1} \geq 2U_i$; hence (denote $k := \lfloor (1-\varepsilon/2)m - 1 \rfloor$)

$$\begin{aligned}
&U_k \geq 2^{\left(\frac{\varepsilon}{2}m-3\right)}U_{i'} \geq 2^{\left(\frac{\varepsilon}{2}m-3\right)}\frac{1}{m^2} \\
\Rightarrow\ &\exists m'' > m' : \forall m \geq m'',\ U_k > 1,
\end{aligned}$$

which is a contradiction. Therefore, it is always true for $i \leq (1-\varepsilon)m,\ m > m'$ that $U_i < e^{-c'm/2}$, so

$$\begin{aligned}
\exists m'' \quad &> \quad m', c > 0 : \forall m \geq m'',\ \sum_{i=0}^{(1-\varepsilon)m}(U_i + V_i) \leq \sum_{i=0}^{(1-\varepsilon)m} U_i + \sum_{i=0}^{(1-\varepsilon)m} V_i \\
&\leq \quad (1-\varepsilon)me^{-\frac{c'm}{2}} + e^{-c'm} \leq e^{-cm}
\end{aligned}$$

for all $c > 0$. Choosing $d = \max(m'', \frac{1}{\delta})$ we get

$$\pi_{m,\mu}\left[\{(x,y) \mid x \geq (1-\varepsilon)m\}\right] \geq 1 - de^{-cm}.$$

Using the marginal distribution on population 2 (the same argument as in Theorem 6) proves inequality (2).

We now move to the proof of statement (1). Let $\nu := \mathbf{1}_{(1,\frac{1}{2})}$. Again, we will show that $\liminf_{\frac{1}{m},\mu \to 0} \widehat{\pi}_{m,\mu}(G) \geq \nu(G)$ for any open set G. Given an open set G with $(1,\frac{1}{2}) \in G$ we can find $\delta_1, \delta_2 > 0$ and a square $F := \{(\xi,\eta)\,|0 \leq 1-\xi \leq \delta_1, |1/2-\eta| \leq \delta_2\}$ such that $F \subset G$. We know that if we choose $\varepsilon = \delta_1$ as in the previous lemma, we get in the limit, for every $\alpha, \beta : -\infty \le \alpha \le \beta \le \infty$,

$$\begin{aligned} & \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \ge 1-\delta_1,\ \alpha \le \frac{\eta m - \frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right\}\right] = \Phi(\beta) - \Phi(\alpha) \\ \Rightarrow \quad & \forall\alpha,\beta : \widehat{\pi}_{m,\mu}\left[F\right] \ge \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \ge 1-\delta_1,\ \frac{1}{2}m + \delta_2 m \ge \eta m \ge \frac{1}{2}m\right\}\right] \\ \ge \quad & \Phi(\beta) - \Phi(\alpha). \end{aligned}$$

Therefore

$$\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \ge \liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(F) \ge 1,$$

which concludes the proof. ■

3.1.3 The third case: $\lambda/2 < \theta$

Here the set of backward induction equilibria is:

$$BIE = \{(x,y) : x = m, y > m\theta/\lambda\} \cup \{(x,y) : x = 0, y < m\theta/\lambda\} \cup \{(x,y) : y = m\theta/\lambda\}.$$

Theorem 8

$$(1)\ \widehat{\pi}_{m,\mu}(\xi,\eta) \underset{\frac{1}{m},\mu\to 0}{\Longrightarrow} \mathbf{1}_{(0,\frac{1}{2})}$$

and for every $\varepsilon > 0$, there exists constants $c, d > 0$ such that for every $\mu > 0, m$ for every $\alpha, \beta : -\infty \le \alpha \le \beta \le \infty$, we have

$$(2)\ \left|\pi_{m,\mu}\left[\left\{(\xi,\eta) \mid \xi \le \varepsilon,\ \alpha \le \frac{\eta m - \frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \le \beta\right\}\right] - (\Phi(\beta) - \Phi(\alpha))\right| \le de^{-cm}.$$

Thus, also here evolutionary stability yields a unique BIE (see Fig. 3 in the appendix).

Proof. We start with the proof of inequality (2).

Define $U_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda < \theta\right\}\right]$, $V_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda > \theta\right\}\right]$ and $W_i := \pi_{m,\mu}\left[\left\{(x,y) : x = i, \frac{y}{m}\lambda = \theta\right\}\right]$. Since $\pi^y_{m,\mu} = Binomial(m, 1/2)$, there exists $c' > 0$ such that

$$\exists c' : \sum_{i=0}^{m} (V_i + W_i) \le e^{-c'm}.$$

Again, since we have a random walk on the population at node 1, for every $i,\ 0 \leq i < m$, we have

$$\begin{aligned} & V_i(1-\mu)\frac{m-i}{m} + U_i\mu\frac{m-i}{m} + W_i\mu\frac{m-i}{m} \\ = \quad & V_{i+1}\mu\frac{i+1}{m} + U_{i+1}(1-\mu)\frac{i+1}{m} + W_i\mu\frac{i+1}{m} \\ \Rightarrow \quad & U_i\mu\frac{m-i}{m} + e^{-c'm} \geq U_{i+1}(1-\mu)\frac{i+1}{m}. \end{aligned}$$

From here, steps similar to those in the previous theorem yield inequality (2).

We now move to the proof of statement (1). Let $\nu := \mathbf{1}_{(0,\frac{1}{2})}$; we will show that $\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \geq \nu(G)$ for any open set G. Given an open set G with $(0,\frac{1}{2}) \in G$ we can find $\delta_1, \delta_2 > 0$ and a square $F := \{(\xi,\eta)\,|0 \leq \xi \leq \delta_1, |1/2-\eta| \leq \delta_2\}$ such that $F \subset G$. We know that if we choose $\varepsilon = \delta_1$ as in the previous lemma, we get in the limit, for every $\alpha, \beta : -\infty \leq \alpha \leq \beta \leq \infty$,

$$\begin{aligned} & \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \leq \delta_1,\ \alpha \leq \frac{\eta m - \frac{1}{2}m}{\frac{1}{4}\sqrt{m}} \leq \beta\right\}\right] = \Phi(\beta) - \Phi(\alpha) \\ \Rightarrow \quad & \forall\alpha,\beta : \widehat{\pi}_{m,\mu}\left[F\right] \geq \widehat{\pi}_{m,\mu}\left[\left\{(\xi,\eta)\middle|\xi \geq 1-\delta_1,\ \frac{1}{2}m + \delta_2 m \geq \eta m \geq \frac{1}{2}m\right\}\right] \\ \geq \quad & \Phi(\beta) - \Phi(\alpha). \end{aligned}$$

Therefore

$$\liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(G) \geq \liminf_{\frac{1}{m},\mu\to 0} \widehat{\pi}_{m,\mu}(F) \geq 1,$$

which concludes the proof. ■

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