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STABILITY OF MIXED EQUILIBRIA

by

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Stability of Mixed Equilibria*

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Abstract

We consider stability properties of equilibria in stochastic evolutionary dynamics. In particular, we study the stability of mixed equilibria in strategic form games. In these games, when the populations are small, all strategies may be stable. We prove that when the populations are large, the unique stable outcome of best-reply dynamics in 2×2 games with a unique Nash equilibrium that is completely mixed is the mixed equilibrium. The proof of this result is based on estimating transition times in Markov chains.

1 Introduction

In *evolutionary game theory*, game-theoretic methods are used to analyze conflict and cooperation between plant or animal populations, and the stability properties of the system. The evolutionary model is a dynamic model, and in a stochastic evolutionary model, the populations are finite and changes are made in discrete time, which yields stochastic processes whose long-run behavior defines the stability of the strategies.

Such stochastic dynamic models have been studied in various classes of games, both in normal form games, starting with Kandori et al. [1993] and Young [1993], and in extensive form games (see for instance Nöldeke and Samuelson [1993] and Hart [2002]). These models usually focus on the

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stability of pure Nash equilibria, like the risk-dominant or the subgame perfect equilibria. We try to extend the question of equilibrium selection to cases where the "reasonable pick" is not a pure equilibrium.

Here we consider 2×2 games with a unique Nash equilibrium that is completely mixed. Each such game is played by two distinct populations of individuals. The individuals change their strategies over time according to a best-reply dynamic, which yields a Markov chain on the space of mixed strategies of the game, whose invariant distributions define the stability of the strategies.

When the population sizes are fixed, all possible outcomes occur with positive probability. However, we show that when the sizes increase, the Nash equilibrium is the unique stable outcome.

Section 2 presents the model both as population dynamics and as dynamics on the space of mixed strategies. Section 3 states the Main Theorem. Section 4 proves the Main Theorem using results on transition times.

2 The Model

2.1 The Populations Dynamics

Let G be a 2×2 two-player strategic game with a unique Nash equilibrium $e = ((p, 1-p), (q, 1-q))$, which is completely mixed. Let $S^1 = \{T, B\}$ and $S^2 = \{R, L\}$ be the pure strategies of players 1 and 2 respectively. For each *population size* m, two disjoint populations of size m are playing G between them. Each individual in each population plays a pure strategy, and at each period one randomly chosen individual of each population may change his strategy according to a best-reply dynamics. The changes are conditionally independent over the two populations.

The dynamics are discrete-time stationary Markov chains on the state space $\Omega_m := \{T, B\}^m \times \{L, R\}^m$, where $\omega = (\omega_i^1, \omega_{i'}^2)_{1 \le i, i' \le m} \in \Omega_m$ is a specification of the pure strategy of each individual. The one-step transition probabilities of the process are given by a transition matrix $Q = Q(m) = (q_{\omega,\tilde{\omega}})_{\omega,\tilde{\omega} \in \Omega_m}$ that satisfies:

- Conditional independence over the populations, i.e.,[1]

$$Q[(\tilde{\omega}^1, \tilde{\omega}^2) \mid \omega] = Q[\tilde{\omega}^1 \mid \omega] \cdot Q[\tilde{\omega}^2 \mid \omega]. \tag{2.1}$$

- For each population $i = 1, 2$, one individual $1 \leq q(i) \leq m$ is chosen, such that

$$Q[q(i) = q \mid \omega] = \frac{1}{m} \text{ for each } 1 \leq q \leq m, \text{ and} \tag{2.2}$$

$$Q[\tilde{\omega}^i_q = \omega^i_q \mid \omega] = 1 \text{ for each } q \neq q(i). \tag{2.3}$$

- For each population $i = 1, 2$ we have

$$Q[\tilde{\omega}^i_{q(i)} = b^i(\omega) \mid \omega] = 1, \tag{2.4}$$

  where b^i is the best-reply strategy of player i when there is a unique such strategy, and $b^i = \omega^i_{q(i)}$ otherwise, when both strategies yield the same payoff.

2.2 Dynamics on the Space of Mixed Strategies

Each state ω can be seen as a pair of mixed strategies of the game — $(x(\omega^1), y(\omega^2)) = ((x, 1-x), (y, 1-y))$ — according to the proportions of the pure strategies played, where x is the proportion of individuals in population 1 that play T, and y is the proportion of individuals in population 2 that play L. Therefore, we have a natural map $(x, y) : \Omega_m \to [0, 1] \times [0, 1]$. As the transition probabilities ((2.1)-(2.4)) depend only on $(x(\omega), y(\omega))$, we can identify Ω_m as a subset of $[0, 1] \times [0, 1]$, i.e., $\Omega_m = \{(k/m, l/m) : 0 \leq k, l \leq m\}$, with the following transition probabilities.

Let $b_x = b_x(y)$ be the "direction" of the best reply of player 1 according to the strategy of player 2 (i.e., according to y), and let $b_y = b_y(x)$ be the best reply direction of player 2. Thus, $b_x = 1$ (or $b_y = 1$) when T (or L) is

[1] For each $\omega \in \Omega_m$, take $Q[\cdot \mid \omega]$ to be a probability distribution over Ω_m, such that $Q[\Omega' \mid \omega] = \sum_{\omega' \in \Omega'} q_{\omega,\omega'}$ for all $\Omega' \subseteq \Omega_m$.

the best reply, $b_x = -1$ ($b_y = -1$) when B (R) is the best-reply, and $b_x = 0$ ($b_y = 0$) when $y = q$ ($x = p$). With these notations we have

$$Q[(\tilde{x}, \tilde{y}) \mid (x, y)] = Q[(\tilde{x}, \cdot) \mid (x, y)] \cdot Q[(\cdot, \tilde{y}) \mid (x, y)],$$

where[2]

$$\begin{aligned} [Q[(x + (1/m)b_x, \cdot) \mid (x, y)] &= (1 + b_x)/2 - xb_x, \text{ and} \\ Q[\cdot, (y + (1/m)b_y) \mid (x, y)] &= (1 + b_y)/2 - yb_y. \end{aligned} \tag{2.5}$$

These transition probabilities yield a Markov chain on $\Omega_m \subset [0, 1] \times [0, 1]$. As e (if at all a state in Ω_m) is a separate ergodic component of the system,[3] we will use Ω_m for $\Omega_m \setminus \{e\}$. Therefore, Ω_m contains one recurrent class, and has a unique invariant distribution — π_m — that describes the long-run behavior of the system.

3 The Results

We are interested in stable outcomes of the dynamics, i.e., strategies $\omega = (x, y) \in \Omega_m$ such that $\pi_m(\omega) > 0$.

For fixed m, all the recurrent states have positive probability, and hence all such strategies are stable. Therefore, we are looking at the behavior of the process when the populations are large, i.e., in the limit of the invariant distribution π_m as $m \to \infty$. As $m \to \infty$, the state space changes and becomes infinite in the limit, and therefore we consider the probabilities of neighborhoods rather than the probability of a single point. For every $\varepsilon > 0$ and $\omega = (x, y)$, let ω_ε be the ε-neighborhood of ω. Our result is that any neighborhood of e is obtained in the limit with probability one.

Theorem 3.1 (Main Theorem). *For every $\varepsilon > 0$, $\lim_{m\to\infty} \pi_m[e_\varepsilon] = 1$.*

[2]If, for example, the best reply of player 1 is T (and hence $b_x = 1$), then x will increase (by $1/m$) whenever a player not playing T is chosen, and there is a probability of $1 - x$ that this will happen.

[3]I.e., $q_{e,e} = 1$ and $q_{\omega,e} = 0$ for all states $\omega \neq e$.

Our proof is based on estimating transition times. Therefore, we also learn about the number of periods it takes to reach any neighborhood of e.

Proposition 3.2. *For every $\varepsilon > 0$ and for any state ω, the expected number of periods to reach e_ε from ω is*[4] $O(m)$.

4 The Proof

4.1 An Outline

We start with an outline of the proof of the Main Theorem. As mentioned, we use transition times to and from some neighborhood of the equilibrium e. We show that the expected number of periods to reach that neighborhood is much smaller than the expected number of periods to leave it, and we obtain our result as the ratio goes to zero as m goes to infinity.

For simplicity, assume that[5] $p = q = 1/2$ and that m is odd. The system is symmetric and can be divided into four quadrants around e. Therefore, assume that each $(x, y) \in \Omega_m$ satisfies $x > 1/2$ and $y > 1/2$, and without loss of generality, assume that T is the best reply for player 1 (and hence x can only increase, with probability $1 - x$), and R is the best reply for player 2 (and hence y can only decrease, with probability y).

To estimate the expected number of periods required to reach e_ε, assume that at time 0 we are at a state $\omega_0 = (x_0, y_0)$, and let T be the first time we reach the next quadrant (where the "direction" of the dynamics changes), i.e., the minimum t such that $y_t < 1/2$. We show that with high probability, the distance of ω_T from e is less than the distance of ω_0 from e (as long as ω_0 is not too close to e), and the ratio of those distances is less than $1 - \delta$, for some constant δ (see Proposition 4.6). Repeating this for k times, the distance from e is less than $(1-\delta)^k$ times the distance from e of ω_0. Therefore, there exists a constant k_0 such that the distance from e in the k_0-th time is

[4] We use the "big-O" notation: $f(m) = O(g(m))$ if there exists a constant $c = c(\Gamma, \varepsilon)$ that does not depend on m such that $|f(m)| \leq c\,|g(m)|$ for all $m > m_0$ for some m_0.

[5] The same arguments hold for any p and q with an appropriate normalization.

less than ε. As the expectation of T is less than $2m$, the expected number of periods to reach e_ε is less than $k_0 2m = O(m)$.

To show that ω_T is closer to e than ω_0 is, we use two propositions. In the first we show that for small T, with high probability, x cannot increase by much (see Proposition 4.3), and hence the distance from e is small relative to the distance at time 0. In the second proposition, we show that with high probability, T is small (see Proposition 4.4).

After estimating the expected number of periods required to reach e_ε, we estimate the number of periods it takes to leave it. Let ω_0 be a state in e_ε. As the distance of ω_0 from e is less than ε, the distance of ω_T from e must be more than ε in order to have $0 < t \leq T$ such that ω_t is outside e_ε.[6] Using similar arguments to those above, we show that with high probability the distance of ω_T from e is less than[7] ε, and we get that the expected number of periods to leave e_ε is at least m^2.

4.2 Notations

Without loss of generality, assume that for $y > q$, the best reply of player 1 is T. Therefore, for $x > p$ and $y > q$, (2.5) becomes

$$Q[(x + 1/m, \cdot) \,|\, (x, y)] = 1 - x, \qquad Q[(x, \cdot) \,|\, (x, y)] = x,$$
$$Q[(\cdot, y - 1/m) \,|\, (x, y)] = y, \qquad Q[(\cdot, y) \,|\, (x, y)] = 1 - y.$$

We use the following notations:

- We divide Ω_m into four quadrants. Let $\Omega_1 = \{(x, y) \in \Omega : x \geq p, y > q\}$, $\Omega_2 = \{(x, y) \in \Omega : x > p, y \leq q\}$, $\Omega_3 = \{(x, y) \in \Omega : x \leq p, y < q\}$, and $\Omega_4 = \{(x, y) \in \Omega : x < p, y \geq q\}$. Assume that m is large enough such that $\Omega_i \neq \emptyset$ for all i.

- Define $L_1 = \{(\xi_1, y) \in \Omega_1\}$ where $\xi_1 = \min\{x : x \geq p, xm \in I\!N\}$, $L_2 =$

[6] For all $0 < t \leq T$ and $\omega_t = (x_t, y_t)$ we have $x_t \leq x_T$ and $y_t \leq y_o$, and therefore if the l_∞ distance of ω_0 and ω_T from e is less than ε then so is the distance of ω_t from e.

[7] A distance of more than ε from e at time T may by much larger than the distance at time 0, and the probability of that is exponentially small.

$\{(x, \eta_2) \in \Omega_2\}$ where $\eta_2 = \max\{y : y \leq q, ym \in I\!N\}$, $L_3 = \{(\xi_3, y) \in \Omega_3\}$ where $\xi_3 = \max\{x : x \leq p, xm \in I\!N\}$, and $L_4 = \{(x, \eta_4) \in \Omega_4\}$ where $\eta_4 = \min\{y : y \geq q, ym \in I\!N\}$. L_i is the "first line" of states in Ω_i, according to the dynamics.[8] Let $L = \cup_{i=1}^4 L_i$.

- Define $D : L \to I\!R$ by $D(\omega) = \rho(\omega, \Omega_{i+1}) = \rho(\omega, L_{i+1})$ where $i+1$ is the next quadrant relative to ω, and $\rho(\omega, \Omega') = \min_{\omega' \in \Omega'} \|\omega - \omega'\|_1$. I.e., $D(\omega)$ is the distance between ω and the next quadrant.

- Let $d_1 = 1/(1-p)$, $d_2 = 1/q$, $d_3 = 1/p$, and $d_4 = 1/(1-q)$, and let $d = \max_{i=1,\ldots,4} d_i$. Define[9] $Z_i(\omega) = D(\omega)/d_i$ for each $\omega \in L_i$. Z_i is the normalized distance on L_i. Let $Z = \cup_{i=1}^4 Z_i : L \to I\!R$.

- Extend Z as a distance on all Ω, by using the l_∞ norm according to Z on L. I.e., for $\omega = (x, y) \in \Omega_1$ we have $Z(\omega) = \max\{Z(\xi_1, y), Z(x, \eta_2)\}$.

- For every $\varepsilon > 0$, define $\Omega(\varepsilon) = Z^{-1}([0, \varepsilon])$. $\Omega(\varepsilon)$ is an ε-neighborhood of e. Let $L(\varepsilon) = L \cap \Omega(\varepsilon)$.

- Let $i_0 = i_0(\varepsilon) \in I\!N$ be minimal such that $(1 - \varepsilon/33)^{i_0} \leq \varepsilon$; i.e.,

$$i_0 = \lceil \frac{\ln \varepsilon}{\ln(1 - \varepsilon/33)} \rceil \leq \frac{\ln \varepsilon}{\ln(1 - \varepsilon/33)} + 1 \leq \frac{\ln \varepsilon}{-\varepsilon/33} + 1 = O(1).$$

  For every $i = 0, \ldots, i_0$, define $G_i(\varepsilon) = L((1 - \varepsilon/33)^i) = \{\omega \in L : Z(\omega) \leq (1 - \varepsilon/33)^i\}$; then $G_0(\varepsilon) = L(1) = L$, $G_{i+1}(\varepsilon) \subset G_i(\varepsilon)$, and $G_{i_0}(\varepsilon) \subset L(\varepsilon)$.

- For every $\Omega' \subset \Omega$, define $T(\Omega') = \min\{t \geq 0 : \omega_t \in \Omega'\}$. For every $\Omega', \Omega'' \subset \Omega$, define[10] $\overline{U}(\Omega', \Omega'') = \max_{\omega \in \Omega'} E[T(\Omega'') \,|\omega]$ and $\underline{U}(\Omega', \Omega'') = \min_{\omega \in \Omega'} E[T(\Omega'') \,|\omega]$.

- For an event $\mathbf{A}$ and a state $\omega \in \Omega$ we use $P_\omega[\mathbf{A}]$ for $P[\mathbf{A} \,|\omega_0 = \omega]$.

[8] L_i is the set of states ω in Ω_i such that there exists $\omega' \notin \Omega_i$ that satisfies $q_{\omega',\omega} > 0$.

[9] As will be seen in the proofs, a distance of d_1 in L_1 is "equivalent" by the dynamics to a distance of d_i in L_i.

[10] If $\Omega' = \{\omega'\}$, we will also use $U(\omega', \Omega'') = \overline{U}(\{\omega'\}, \Omega'') = \underline{U}(\{\omega'\}, \Omega'')$.

- Let X and Y be random variables. We use $X \lesssim Y$ if for every α we have $P[X \geq \alpha] \leq P[Y \geq \alpha]$. We use $X \approx Y$ if $X \lesssim Y$ and $Y \lesssim X$.

By the symmetry of the system, we will prove most of the results only for Ω_1. We therefore introduce some more notations on Ω_1, but they can be naturally extended to all Ω_i.

- If $x = p$, there is no unique best reply for population 2. Therefore, define $T_1 = \min\{t \geq 0 : x_t > p\}$. For every $0 \leq t < T_1$ we have $x_t = p$, and therefore $y_{t+1} = y_t = y_0$, and $y_{T_1} = y_0 > q \geq \eta_2$.

- Define $T = T(\Omega_2) = \min\{t \geq 0 : y_t = \eta_2\}$. For $\omega_0 \in \Omega_1$ we have $T > T_1$, and for every $T_1 < t \leq T$ we have $x_t \geq x_{t-1}$, and therefore, $x_{T_1+t} \geq x_{T_1} > p$. For every $T_1 \leq t < T$ we have $y_t > q$.

- Let $T_2 = T - T_1$.

- For every $\omega = (x, y) \in \Omega_1$, let $\gamma(\omega) = m(2q + D(\omega)) - 1$. For all ω we have $\gamma(\omega) < 2m$.

- For every $\omega \in \Omega_1$, let[11]

$$\mathbf{T}(\omega) := d_2 m D(\omega)(1 - \frac{mD(\omega) - 2}{2\gamma(\omega)}), \text{ and}$$
$$\mathbf{Z}(\omega) := Z(\omega)(1 - \frac{mD(\omega) - 4}{16\gamma(\omega)}).$$

4.3 Proof of the Main Theorem

Lemma 4.1. *Let $\{X_t\}_{t=1}^{\infty}$ be indicator random variables, and let $S_n = \sum_{t=1}^{n} X_t$ for $n = 1, 2, \ldots$ and $S_0 = 0$. Let $0 \leq p \leq 1$ such that for all t, $P[X_t = 1 \mid S_{t-1}] \leq p$. Then, for all n we have[12] $S_n \lesssim B(n, p)$.*

Proof. We use induction on n. For $n = 1$ we have $P[S_n \geq \alpha] = P[X_1 \geq \alpha] \leq P[B(1, p) \geq \alpha]$.

[11] $\mathbf{T}(\omega)$ is a bound on the "expected" time to reach Ω_2, starting in ω, and $\mathbf{Z}(\omega)$ is a bound on the distance of the expected reaching point in Ω_2.

[12] $B(n, p)$ is the binomial distribution.

Assume by induction for $n-1$.

$$\begin{aligned}
P[S_n \geq \alpha] &= \sum_{\beta} P[S_n \geq \alpha \mid S_{n-1} = \beta] \cdot P[S_{n-1} = \beta] \\
&= \sum_{\beta \geq \alpha} P[S_{n-1} = \beta] + \sum_{\alpha-1 \leq \beta < \alpha} P[X_n = 1 \mid S_{n-1} = \beta] \cdot P[S_{n-1} = \beta] \\
&\leq P[S_{n-1} \geq \alpha] + p \cdot P[\alpha - 1 \leq S_{n-1} < \alpha] \\
&= p \cdot P[S_{n-1} \geq \alpha - 1] + (1-p) \cdot P[S_{n-1} \geq \alpha] \\
&\leq p \cdot P[B(n-1,p) \geq \alpha - 1] + (1-p) \cdot P[B(n-1,p) \geq \alpha] \\
&= P[B(n,p) \geq \alpha].
\end{aligned}$$

□

Lemma 4.2. *Let $\omega = (x, y) \in \Omega_1$.*

1. *$[T_1 \mid \omega] \lesssim G(1/d_1)$, where G is the geometric distribution.*
2. *If $\omega_0 = \omega$ then $T \geq T_2 \geq mD(\omega)$.*
3. *$[T_2 \mid \omega] \approx \sum_{i=1}^{mD(\omega)} G(\eta_2 + i/m)$.*
4. *$E[T_2 \mid \omega] \leq d_2 m D(\omega) \leq d_2 m$.*
5. *Let $\omega' = (x, y') \in \Omega_1$ such that $y' \leq y$; then $[T_2 \mid \omega] \gtrsim [T_2 \mid \omega']$.*
6. *For all α and τ we have $P[x_T - x_{T_1} \geq \alpha \mid \omega, T_2 \leq \tau] \leq P[B(\lfloor \tau \rfloor, 1/d_1) \geq m\alpha]$.*

Proof. 1. If $x > p$, then $T_1 = 0$. Otherwise, $x = p$, and for every t such that $x_t = p$ we have $P[x_{t+1} > p \mid \omega_t] = 1 - p = 1/d_1$.

2. This is clear as $y_t - y_{t-1} \leq 1/m$ for all $t \leq T$ and $y_T - y_{T_1} = D(\omega)$.

3. For every $m\eta_2 \leq i \leq m$, let $T^i = \min\{t \geq T_1 : y_t \leq i/m\}$. Thus, we have $T^i \geq T^{i+1}$ for all i, $T = T^{m\eta_2}$, and $T_1 = T^i$ for all $i \geq my$. For every $m\eta_2 < i \leq my$ and $n \in \mathbb{N}$ we have $P_\omega[T^{i-1} - T^i = n] = P[G(i/m) = n]$. Therefore, for every $m\eta_2 < i \leq my$ we have $[T^{i-1} - T^i \mid \omega] \approx G(i/m)$.

Thus, as $T^{i-1} - T^i$ and $T^{j-1} - T^j$ are independent for $i \neq j$, we have $[T_2 \mid \omega] = [T - T_1 \mid \omega] = [T^{m\eta_2} - T^{my} \mid \omega] = [\sum_{i=m\eta_2}^{my} (T^{i-1} - T^i) \mid \omega] \approx \sum_{i=1}^{mD(\omega)} G(\eta_2 + i/m)$.

4. Follows from (3) and $\eta_2 + i/m \geq q = 1/d_2$.

5. Using (3) we have $[T_2 \mid \omega] \approx \sum_{i=1}^{mD(\omega)} G(\eta_2 + i/m) \gtrsim \sum_{i=1}^{mD(\omega')} G(\eta_2 + i/m) \approx [T_2 \mid \omega']$.

6. Let $\chi := x_T - x_{T_1}$. For every $T_1 < t \leq T$, we have $x_{t-1} > p$ and $y_{n-1} > q$. Therefore, for every $\omega_{t-1} = (x_{t-1}, y_{t-1}) \in \Omega$, we have $P[x_t - x_{t-1} = 1/m \mid \omega_{t-1}] = 1 - x_{t-1} \leq 1 - p$. As $\chi = \sum_{T_1 < t \leq T}(x_t - x_{t-1})$, for all n we have, using Lemma 4.1, $[\chi \mid \omega, T - T_1 = n] \lesssim (1/m)B(n, 1-p)$.

Therefore,

$$\begin{aligned}
P[\chi \geq \alpha \mid \omega, T_2 \leq \tau] =& \sum_{n \leq \tau} P[\chi \geq \alpha \mid \omega, T_2 = n] \cdot P[T_2 = n \mid \omega, T_2 \leq \tau] \\
\leq& \sum_{n \leq \tau} P[B(n, 1-p) \geq m\alpha] \cdot P[T_2 = n \mid \omega, T_2 \leq \tau] \\
\leq& P[B(\lfloor \tau \rfloor, 1-p) \geq m\alpha] \cdot \sum_{n \leq \tau} P[T_2 = n \mid \omega, T_2 \leq \tau] \\
=& P[B(\lfloor \tau \rfloor, 1-p) \geq m\alpha]. \qquad \square
\end{aligned}$$

Proposition 4.3. *Let $\omega = (x, y) \in L_1$ such that $mZ(\omega)^2 \geq 128$; then $P[Z(\omega_T) \geq \mathbf{Z}(\omega) \mid \omega, T_2 \leq \mathbf{T}(\omega)] = O(1/(mZ(\omega)^3))$.*

Proof. As $x_{T_1} \leq \xi_1 + 1/m \leq \xi_3 + 2/m$, we have $x_T - \xi_3 \leq x_T - x_{T_1} + 2/m$, and therefore, for every t we have

$$\begin{aligned}
P[Z(\omega_T) \geq \mathbf{Z}(\omega) \mid \omega, T_2 \leq t] &= P[x_T - \xi_3 \geq d_2 \mathbf{Z}(\omega) \mid \omega, T_2 \leq t] \\
&\leq P[x_T - x_{T_1} \geq d_2 \mathbf{Z}(\omega) - 2d_2/m \mid \omega, T_2 \leq t]. \quad (4.1)
\end{aligned}$$

Let $\mu = E[B(\lfloor \mathbf{T} \rfloor, 1/d_1)] \leq (1/d_1) d_2 m D(\omega)(1 - (mD(\omega) - 2)/(2\gamma(\omega))$. Using Lemma 4.2(6) and (4.1), we have

$$\begin{aligned}
P[Z(\omega_T) \geq \mathbf{Z} \mid \omega, T_2 \leq \mathbf{T}] &\leq P[B(\lfloor \mathbf{T} \rfloor, 1/d_1) \geq d_2 m \mathbf{Z} - 2d_2] \\
&\leq P[B(\lfloor \mathbf{T} \rfloor, 1/d_1) \geq \mu + (d_2 m \mathbf{Z} - d_2 m Z(\omega)(1 - \frac{mD(\omega) - 2}{2\gamma(\omega)}) - 2d_2)] \\
&= P[B(\lfloor \mathbf{T} \rfloor, 1/d_1) \geq \mu + (d_2 m Z(\omega) \frac{7mD(\omega) - 12}{16\gamma(\omega)} - 2d_2)].
\end{aligned}$$

Using $mZ(\omega)^2 \geq 128$ and $\gamma(\omega) < 2m$, we get

$$d_2 mZ(\omega)\frac{7mD(\omega)-12}{16\gamma(\omega)} - 2d_2 \geq \frac{d_1 d_2 mZ(\omega)^2}{64}.$$

As $V[B(\lfloor \mathbf{T} \rfloor, 1/d_1)] \leq 1/d_1\mathbf{T} \leq d_2 mZ(\omega)$, we have, using Chebyshev's inequality,

$$\begin{aligned} P[Z(\omega_T) \geq \mathbf{Z} \,|\omega, T_2 \leq \mathbf{T}] &\leq P[B(\lfloor \mathbf{T} \rfloor, 1/d_1) \geq \mu + \frac{d_1 d_2 mZ(\omega)^2}{64}] \\ &\leq d_2 mZ(\omega)(\frac{64}{d_1 d_2 mZ(\omega)^2})^2 = O(\frac{1}{mZ(\omega)^3}). \quad \square \end{aligned}$$

Proposition 4.4. *Let $\omega = (x, y) \in \Omega_1$, then $P_\omega[T_2 > \mathbf{T}(\omega)] = O(1/(mZ(\omega)^3))$.*

Proof. Let $\mu = E[T_2 \,|\omega]$; then by Lemma 4.2(3) and using the convexity of $1/x$, we get

$$\begin{aligned} \mu = E[\sum_{i=1}^{mD(\omega)} G(\frac{m\eta_2 + i}{m})] &= m\sum_{i=1}^{mD(\omega)} \frac{1}{m\eta_2 + i} \leq \frac{m^2 D(\omega)}{\frac{1}{mD(\omega)}\sum_{i=1}^{mD(\omega)} m\eta_2 + i} \\ &= \frac{2m^2 D(\omega)}{m(2\eta_2 + D(\omega)) + 1} \leq \frac{2m^2 D(\omega)}{m(2q + D(\omega)) - 1} = \frac{2m^2 D(\omega)}{\gamma(\omega)}. \quad (4.2) \end{aligned}$$

Let $\sigma^2 = V[T_2 \,|\omega]$; then

$$\sigma^2 = \sum_{i=1}^{mD(\omega)} V[G(\eta_2 + \frac{i}{m})] \leq \sum_{i=1}^{mD(\omega)} V[G(q)] = \frac{mD(\omega)(1-q)}{q^2}. \quad (4.3)$$

Therefore, using Chebyshev's inequality together with (4.2) and (4.3),

$$\begin{aligned} P_\omega[T_2 > \mathbf{T}] &\leq P_\omega[T_2 \geq d_2 mD(\omega)(1 - \frac{mD(\omega)-2}{2\gamma(\omega)})] \\ &\leq P_\omega[T_2 \geq \mu + d_2 mD(\omega)(1 - \frac{mD(\omega)-2}{2\gamma(\omega)}) - \frac{4m^2 D(\omega)}{2\gamma(\omega)}] \\ &= P_\omega[T_2 \geq \mu + \frac{(mD(\omega))^2}{2q\gamma(\omega)}] \leq P_\omega[T_2 \geq \mu + \frac{mD(\omega)^2}{4}] = O(\frac{1}{mD(\omega)^3}). \quad \square \end{aligned}$$

Corollary 4.5. *For every $\omega \in L_1$, such that $mZ(\omega)^2 > 128$, we have $P_\omega[Z(\omega_T) \geq \mathbf{Z}(\omega)] = O(1/(mZ(\omega)^3))$.*

Proof. Using Proposition 4.3 and Proposition 4.4, we get

$$\begin{aligned} P_\omega[Z(\omega_T) \geq \mathbf{Z}] &= P[Z(\omega_T) \geq \mathbf{Z} \,|\omega, T_2 \leq \mathbf{T}] \cdot P_\omega[T_2 \leq \mathbf{T}] \\ &\quad + P[D(\omega_T) \geq \mathbf{Z} \,|\omega, T_2 > \mathbf{T}] \cdot P_\omega[T_2 > \mathbf{T}] \\ &\leq P[D(\omega_T) \geq \mathbf{Z} \,|\omega, T_2 \leq \mathbf{T}] + P_\omega[T_2 > \mathbf{T}] = O(\frac{1}{mZ(\omega)^3}). \quad \square \end{aligned}$$

Using the same methods, we can show the same result for all quadrants, and obtain the following proposition:

Proposition 4.6. *Let $\varepsilon > 33/\sqrt{m}$. For all $\omega \in L$ such that $Z(\omega) > \varepsilon$, we have $P_\omega[Z(\omega_T) \geq Z(\omega_0)(1 - \varepsilon/33)] \leq C/m$, for some constant $C = C(\varepsilon)$.*

Proof. For $Z(\omega) > \varepsilon > 33/\sqrt{m}$ we have $(mD(\omega)-4)/(16\gamma(\omega)) \geq Z(\omega)/33 \geq \varepsilon/33$. Therefore, using Corollary 4.5, we get

$$\begin{aligned} P_\omega[Z(\omega_T) \geq Z(\omega_0)(1 - \frac{\varepsilon}{33})] &\leq P_\omega[Z(\omega_T) \geq Z(\omega_0)(1 - \frac{mD(\omega) - 4}{16\gamma(\omega)})] \\ &= P_\omega[Z(\omega_T) \geq \mathbf{Z}(\omega)] = O(\frac{1}{mZ(\omega)^3}) = O(\frac{1}{m}). \quad \square \end{aligned}$$

Let $\varepsilon > 0$ be fixed, and let $C = \max\{C(\varepsilon), C(\varepsilon/2)\}$ for the constants from Proposition 4.6. Let m be large enough such that

$$\frac{\varepsilon}{2} > \frac{33}{\sqrt{m}}, \text{ and } \frac{C}{m} \leq \frac{1}{2i_0}. \tag{4.4}$$

Proposition 4.7. *For every*[13] *$i = 0, \ldots, i_0 - 1$ we have $\overline{U}(G_i, L(\varepsilon)) \leq 2dm + \overline{U}(G_{i+1}, L(\varepsilon)) + \overline{U}(G_0, L(\varepsilon))/(2i_0)$.*

Proof. Let $0 \leq i \leq i_0 - 1$, and let $\omega = (x, y) \in G_i$ such that $U(\omega, L(\varepsilon)) = \overline{U}(G_i, L(\varepsilon))$. If $Z(\omega) \leq \varepsilon$ then $\omega \in L(\varepsilon)$ and therefore $\overline{U}(G_i, L(\varepsilon)) = 0$. Otherwise $\varepsilon < Z(\omega) \leq (1 - \varepsilon/33)^i$, and therefore, $(1 - \varepsilon/33)^{i+1} \geq Z(\omega)(1 -$

[13] As ε is fixed, we use i_0 and G_i for $i_0(\varepsilon)$ and $G_i(\varepsilon)$.

$\varepsilon/33$). Therefore, using Proposition 4.6 and (4.4), we have

$$P_\omega[\omega_T \notin G_{i+1}] = P_\omega[Z(\omega_T) > (1-\frac{\varepsilon}{33})^{i+1}]$$
$$\leq P_\omega[Z(\omega_T) \geq Z(\omega)(1-\frac{\varepsilon}{33})] \leq \frac{C}{m} \leq \frac{1}{2i_0},$$

and therefore,

$$\begin{aligned} E[U(\omega_T, L(\varepsilon)) \mid \omega] &= E[U(\omega_T, L(\varepsilon)) \mid \omega, \omega_T \in G_{i+1}] \cdot P_\omega[\omega_T \in G_{i+1}] \\ &+ E[U(\omega_T, L(\varepsilon)) \mid \omega, \omega_T \notin G_{i+1}] \cdot P_\omega[\omega_T \notin G_{i+1}] \\ &\leq \overline{U}(G_{i+1}, L(\varepsilon)) + \overline{U}(G, L(\varepsilon))/(2i_0). \end{aligned}$$

Using the generalization of Lemma 4.2 (parts 1 and 4), we have

$$\begin{aligned} U(\omega, L(\varepsilon)) &= E[T(L(\varepsilon)) \mid \omega] \leq E[T + U(\omega_T, L(\varepsilon)) \mid \omega] \\ &= E[T_1 \mid \omega] + E[T_2 \mid \omega] + E[U(\omega_T, L(\varepsilon)) \mid \omega] \\ &\leq d + dm + \overline{U}(G_{i+1}, L(\varepsilon)) + \overline{U}(G, L(\varepsilon))/(2i_0). \quad \square \end{aligned}$$

We now use Proposition 4.7 to prove Proposition 3.2.

Proposition 3.2. *There exists a constant C_1 such that $\overline{U}(\Omega, L(\varepsilon)) \leq C_1 m$.*

Proof. Using Proposition 4.7 and $\overline{U}(G_{i_0}, L(\varepsilon)) = 0$, we get $\overline{U}(G_i, L(\varepsilon)) \leq (2dm + \overline{U}(G_0, L(\varepsilon))/(2i_0))(i_0 - i)$ for all $i = 0, \ldots, i_0$, and in particular, for $i = 0$, we get $\overline{U}(G_0, L(\varepsilon)) \leq 2dmi_0 + \overline{U}(G_0, L(\varepsilon))/2$, or $\overline{U}(G_0, L(\varepsilon)) \leq 4dmi_0$.

Using Lemma 4.2 (parts 1 and 4) and Lemma A.1 of Gorodeisky [2003], for every $\omega \in \Omega$, we have $U(\omega, L(\varepsilon)) \leq U(\omega, L) + \overline{U}(L, L(\varepsilon)) = E[T \mid \omega] + \overline{U}(G_0, L(\varepsilon)) \leq 2dm + \overline{U}(G_0, L(\varepsilon))$. Therefore, $\overline{U}(\Omega, L(\varepsilon)) \leq 2dm + 4dmi_0$. $\square$

We now estimate the expected time required to leave e_ε. To do so, we bound the probability that $Z(\omega_T) \geq \varepsilon$, both when $Z(\omega_0) > \varepsilon m/2$ (using Proposition 4.6) and when $Z(\omega_0) \leq \varepsilon m/2$ (Corollary 4.10). Again, for simplicity, we prove those results for L_1, but they can be extended to all L.

Let $\mathbf{S} = (3/4)d_2\varepsilon m$.

Proposition 4.8. *For all $\omega \in L_1$ we have $P[Z(\omega_T) \geq \varepsilon \mid \omega, T_2 \leq \mathbf{S}] \leq e^{-\varepsilon m/25}$.*

Proof. Let $\mu = E[B(\lfloor \mathbf{S} \rfloor, 1/d_1)] \leq \mathbf{S}/d_1$. Using Lemma 4.2(6) together with $x_T - \xi_3 \leq x_T - x_{T_1} + 2/m$, and Theorem 1 of[14] Hoeffding [1963], we have[15]

$$\begin{aligned} P[Z(\omega_T) \geq \varepsilon \mid \omega, T_2 \leq \mathbf{S}] &\leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_1) \geq d_2 \varepsilon m - 2d_2] \\ &\leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_1) \geq \mu + (\frac{d_2 \varepsilon m}{4} - 2d_2)] \leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_1) \geq \mu + \frac{d_2 \varepsilon m}{5}] \\ &\leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_1) \geq \mu + \lfloor \mathbf{S} \rfloor/5] \leq e^{-2\lfloor \mathbf{S} \rfloor/25} \leq e^{-\varepsilon m/25}. \quad \square \end{aligned}$$

Proposition 4.9. *For all $\omega \in L_1$ such that $Z(\omega) \leq \varepsilon/2$, we have $P_\omega[T_2 > \mathbf{S}] \leq e^{-\varepsilon m/25}$.*

Proof. Let $y = \eta_2 + \lfloor \varepsilon m/2 \rfloor/m$. By Lemma 4.2(3) we have[16] $[T_2 \mid \omega_0 = (\xi_1, y)] \approx \sum_{i=1}^{\lfloor \varepsilon m/2 \rfloor} G(\eta_2 + i/m) \lesssim \sum_{i=1}^{\lfloor \varepsilon m/2 \rfloor} G(q) = NB(\lfloor \varepsilon m/2 \rfloor, 1/d_2)$.

Let $\mu = E[B(\lfloor \mathbf{S} \rfloor, 1/d_2)] \geq (\mathbf{S} - 1)/d_2$; then

$$\begin{aligned} P[T_2 > \mathbf{S} \mid \omega_0 = (\xi_1, y)] &\leq P[NB(\lfloor \varepsilon m/2 \rfloor, 1/d_2) > \mathbf{S}] \\ &= P[B(\lfloor \mathbf{S} \rfloor, 1/d_2) < \lfloor \frac{\varepsilon m}{2} \rfloor] \leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_2) < \mu - (\frac{\varepsilon m}{4} - \frac{1}{d_2})] \\ &\leq P[B(\lfloor \mathbf{S} \rfloor, 1/d_2) \geq \mu - \lfloor \mathbf{S} \rfloor/5] \leq e^{-2\lfloor \mathbf{S} \rfloor/25} \leq e^{-\varepsilon m/25}. \quad \square \end{aligned}$$

Using Lemma 4.2(5), for all $\omega = (\xi_1, y')$ such that $y' \leq y$, we have $P[T_2 > \mathbf{S} \mid \omega_0 = (\xi_1, y')] \leq P[T_2 > \mathbf{S} \mid \omega_0 = (\xi_1, y)] \leq e^{-\varepsilon m/25}$.

Corollary 4.10. *For all $\omega \in L_1$ such that $Z(\omega) \leq \varepsilon/2$, we have $P_\omega[Z(\omega_T) \geq \varepsilon] \leq 2e^{-\varepsilon m/25}$.*

Proof. The proof is the same as the proof of Corollary 4.5, using Propositions 4.8 and 4.9. $\square$

Let $\mathbf{W} = \Omega \setminus \Omega(\varepsilon)$, and $\mathbf{L} = L(\varepsilon) \setminus L(\varepsilon/2)$.

Proposition 4.11. $\underline{U}(L(\varepsilon/2), \mathbf{W}) \geq (1 + \underline{U}(L(\varepsilon), \mathbf{W})(1 - 2e^{-\varepsilon m/25})$.

[14] By Theorem 1 of Hoeffding [1963] we have $P[B(t, 1-p) \geq (1-p)t + \delta t] \leq e^{-2t\delta^2}$.
[15] (4.4) implies that $d_2 \varepsilon m/4 - 2d_2 \geq d_2 \varepsilon m/5$.
[16] NB is the Negative Binomial distribution.

Proof. Let $\omega \in L(\varepsilon/2)$ such that $U(\omega, \mathbf{W}) = \underline{U}(L(\varepsilon/2), \mathbf{W})$. As $1 \leq mZ(\omega) \leq \varepsilon m/2$, we have, using Lemma 4.2(2), $T \geq T_2 \geq 1$. Using Corollary 4.10, we also have $P_\omega[\omega_T \notin L(\varepsilon)] = P_\omega[Z(\omega_T) > \varepsilon] \leq 2e^{-\varepsilon m/25}$.

For $\omega_0 = (x_0, y_o) \in L$, and for every $0 \leq t \leq T$ and $\omega_t = (x_t, y_t)$, we have $x_t \leq \max\{x_0, x_T\}$ and $y_t \leq \max\{y_0, y_T\}$, and therefore $Z(\omega_t) \leq \max\{Z(\omega_0), Z(\omega_T)\}$. If $\omega_T \in L(\varepsilon)$ (and $\omega_0 \in L(\varepsilon)$) , then for all $0 \leq t \leq T$ we have $Z(\omega_t) \leq \varepsilon$, or $\omega_t \in \Omega(\varepsilon)$. Therefore,

$$\begin{aligned}\underline{U}(L(\varepsilon/2), \mathbf{W}) &= E[T(\mathbf{W}) \,|\omega] \geq E[T(\mathbf{W}) \,|\omega, \omega_T \in L(\varepsilon)] \cdot P_\omega[\omega_T \in L(\varepsilon)] \\ &\geq E[T + U(\omega_T, \mathbf{W}) \,|\omega, \omega_T \in L(\varepsilon)] \cdot (1 - 2e^{-\varepsilon m/25}) \\ &\geq (1 + \underline{U}(L(\varepsilon), \mathbf{W})(1 - 2e^{-\varepsilon m/25}). \qquad \square\end{aligned}$$

Proposition 4.12. $\underline{U}(\mathbf{L}, \mathbf{W}) \geq (\varepsilon m/2 + \underline{U}(L(\varepsilon), \mathbf{W})(1 - C/m))$.

Proof. Let $\omega \in \mathbf{L}$ such that $U(\omega, \mathbf{W}) = \underline{U}(\mathbf{L}, \mathbf{W})$. As $Z(\omega) > \varepsilon/2$, we have, using Lemma 4.2(2), $T \geq T_2 > \varepsilon m/2$, and, using Proposition 4.6 together with $Z(\omega)(1 - \varepsilon/33) \leq Z(\omega) \leq \varepsilon$ and (4.4), we have $P_\omega[\omega_T \notin L(\varepsilon)] = P_\omega[Z(\omega_T) > \varepsilon] \leq P_\omega[Z(\omega_T) \geq Z(\omega)(1 - \varepsilon/33)] \leq C/m$.

As in the proof of Proposition 4.11, we have

$$\begin{aligned}\underline{U}(\mathbf{L}, \mathbf{W}) &\geq E[T + U(\omega_T, \mathbf{W}))\,|\omega, \omega_T \in L(\varepsilon)] \cdot P_\omega[\omega_T \in L(\varepsilon)] \\ &\geq (\varepsilon m/2 + \underline{U}(L(\varepsilon), \mathbf{W}))(1 - C/m). \quad \square\end{aligned}$$

Corollary 4.13. *There exists a constant C_2 such that $\underline{U}(L(\varepsilon), \mathbf{W}) \geq C_2 m^2$.*

Proof. As $\underline{U}(L(\varepsilon), \mathbf{W}) = \min\{\underline{U}(L(\varepsilon/2), \mathbf{W}), \underline{U}(\mathbf{L}, \mathbf{W}\}$, then we either have $\underline{U}(L(\varepsilon), \mathbf{W}) = \underline{U}(\mathbf{L}, \mathbf{W})$, or $\underline{U}(L(\varepsilon), \mathbf{W}) = \underline{U}(L(\varepsilon/2), \mathbf{W})$.

In the first case, using Proposition 4.12, we get $\underline{U}(L(\varepsilon), \mathbf{W}) \geq (\varepsilon m/2 + \underline{U}(L(\varepsilon), \mathbf{W}))(1 - C/m)$, and therefore, $\underline{U}(L(\varepsilon), \mathbf{W}) \geq \varepsilon m^2/(2C)$. In the second case, using Proposition 4.11, we get $\underline{U}(L(\varepsilon), \mathbf{W}) \geq ((1+\underline{U}(L(\varepsilon), \mathbf{W}))(1-2e^{-\varepsilon m/25})$, and therefore, we get $\underline{U}(L(\varepsilon), \mathbf{W}) \geq (1 - 2e^{-\varepsilon m/25})2e^{\varepsilon m/25}$. $\square$

We can now prove the Main Theorem:

Proof of Theorem 3.1. Using Proposition A.5 of Gorodeisky [2003] with Proposition 3.2 and Corollary 4.13, there are constants C_1 and C_2 such that

$$\pi_m[e_\varepsilon] = \pi_m[\Omega(\varepsilon)] \geq \frac{C_2 m^2}{C_1 m + C_2 m^2} \xrightarrow[m\to\infty]{} 1. \qquad \square$$

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