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ENCOURAGING A COALITION FORMATION

by

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ABSTRACT. A 4-person quota game is analyzed and discussed, in which players find it beneficial to pay others, in order to encourage favorable coalition structure.

KEY WORDS: Game theory, cooperative games, power of a coalition, coalition formations, experiments in game theory.

1. INTRODUCTION

In Maschler (1978) I described a set of 123 cooperative games played by high school students. Among the games there were 18 *4-person quota games* with varying quotas $\omega = [\omega_1, \omega_2, \omega_3, \omega_4]$.

Specifically, groups of four students were told who among them, is player 1, ..., 4. If any 2-members i, j decided to form a coalition, they would receive together $\omega_i + \omega_j$ "points", about which they had to decide how to share among themselves and report on a card. The three students who collected the highest number of points by the end of the 123 plays would then get prizes which were coupons for buying books. To boost their ambition I also told them that they should make every effort to gain as many points as possible, because this experiment is scientifically important. Either because of this encouragement, or because of the enjoyment from the competition, they fought fiercely to gain points.

At the end of each play, the participants were asked to report briefly on the back of the card what their reasonings were while playing, and why they agreed to the points they received.

Note that I did not reveal how I determined the characteristic function. I simply provided them with the characteristic function. Nevertheless, the students, who belonged to an elite class, were intelligent enough to realize how the worths of the coalitions were obtained.



In this paper I will use the quota $\omega = [10, 20, 30, 40]$ as a prototype, even though, as said earlier, the experiment used various quotas. For the above quota game, we were interested in finding out if the players would end up anywhere near the nucleolus of the game (10,20,30,40). A deviation of 5 points or less was judged a "success", as this small amount was regarded the least noticeable difference. (10,20,30,40) is the nucleolus of the game, no matter which couple of 2-person coalitions form.¹

Altogether, 18 plays were performed of which one ended up with the students entering a fight.² This game is discarded. Of the 17 games, 9 ended successfully, as predicted. But 8 ended up far from the quota vector.

Nevertheless, even in these games there was always one coalition that ended up within the tolerable distance from the quota. For example, an outcome could be (22, 23, 27, 28; {23, 14}) where players 2 and 3 ended up near their quota but players 1 and 4 were far off.³

This result is interesting as it conforms to a von Neuman Morgenstern solution, discovered by Shapley (1953). Obviously, the students knew nothing about von Neumann Morgenstern solutions, or any other game theoretical solutions. What was it that caused them to act so strangely? The answer became obvious after I read the reports of the students and realized that the coalition that reached the neighborhood of the quota was always the coalition that was first to form. Now, everything became transparent: As long as no coalition is formed, there is a pressure to share in accordance with the quota. When two players are left alone, the quota is meaningless. Why not share the proceeds equally, for example? If in the above example, once coalition {2, 3} was formed there remained two players, 1 and 4 who together would earn 50 points and alone each was worth nothing. Why not share (25,25)? Indeed, in all the 8 cases, the last coalition shared its proceeds somewhere between the quota share and the equal split share. Why not sharing equally when realizing that they were left alone? Is it because of benevolence, to compensate the player with the larger quota who became disappointed, realizing that suddenly he lost his potential wealth? Is it in order to com-

pensate for his regret for not succeeding to be in the first coalition to form? Perhaps psychologists can explain this. I do want to stress, however, that from the few plays that I watched personally and from the reports, it seems to me that the player with the higher quota would have refused an equal share.

In two plays Player 1 was smart. He realized that as long as all players were negotiating he could expect only an amount near 10, but if he was left out, he would have a good chance to improve his lot. In these cases (one of which I happened to watch when it was played), this player made excessive demands so that nobody was willing to join him. The three players competed frantically in order to enter a 2-person coalition, negotiating a quota share. Once a single person was left to negotiate with Player 1, it was easy for Player 1 to explain that now, since only two were left, they should share equally; however, he is benevolent and is willing to give the other player a token more.

For many years, in private conversations and in class, I explained that this is an example in which high school kids outsmarted game theory. The two students realized that it was smarter to wait and perhaps the others realized that it is in their interest to rush to form a coalition so as not to be stuck with Player 1. There does not exist a solid theory that tells us, in the sense described above, when to rush and form coalitions, when it is advantageous to wait, with whom to join and how much to ask for.⁴ I added that with all due respect to game theory, if I were Player 1, I would have acted similarly.

Nowadays I am somewhat smarter than the two students, as will be described in the next section.

2. SOLVING THE GAME, USING THE POWER OF COALITIONS

We analyze the game from a normative point of view. To do so, we assume that the last coalition that forms shares its worth equally among its members. This is the *standard of fairness* that we assume to be prevailing with the players. Although in reality the players deviated from this standard, as explained above, we

regard these deviations as being an order of second magnitude and choose to ignore them.

As explained in the introduction, it is beneficial for Player 1 to wait until a 2 person coalition forms. However, Player 1 would like that Players 2 and 3 form a coalition, because, being left with Player 4, he would raise his profits from a quota of 10 utils to 25 utils. Any other 2-person coalition that forms among players 2, 3 and 4 will yield him less. *He might be willing to pay some utils to players 2 and 3 in order to "encourage" the formation of this coalition. How much should he offer them?!*

To answer that, we recall Maschler (1963), where it is argued that when a standard of fairness exists among the players then, even if a game is presented to them by a characteristic function, the players perceive the game as having a different worth function, called, *the power function*.⁵ Denoting the power function by w and assuming the above standard of fairness, we proceed as follows:

For 2-person coalitions $\{i, j\}$, $w(ij) = v(ij) = \omega_i + \omega_j$. Indeed, this is all that these coalitions can guarantee. A 3-person coalition $\{i, j, k\}$ can do more than $\max \{\omega_\mu + \omega_\nu : \mu, \nu \in \{i, j, k\}, \mu \neq \nu\}$. Indeed, they can decide which 2-person coalition forms and who will remain to form a coalition with the fourth player. Thus, coalition $\{1, 2, 3\}$ will maximize its proceeds if its members decide to form coalition $\{2, 3\}$ and let Player 1 play with Player 4 and get 25 utils. Thus, $w(123) = 50 + 25 = 75$. Similarly, $w(124) = 80$, $w(134) = 85$, $w(234) = 85$. Single-person coalitions can guarantee only what their complementary coalitions spare them. Thus, $w(1) = 15$, because Player 1 can count on the fact that at least Player 2 will join him⁶ $w(2) = 15$, $w(3) = 20$, $w(4) = 25$. Finally, $w(\phi) = 0$ and $w(1234) = 100$, as the players can safely assume that eventually a couple of 2-person coalitions will form. Note that the power function is a constant sum game, so, trying to form a "new power" of the power function will not yield any different function.

We propose that the players look for the nucleoli of the game $(\{1, 2, 3, 4\}; w)$ for the various coalition structures.⁷

For example, if Player 1 wants that coalition $\{2, 3\}$ form, he should look for the nucleolus of the power game for the coalition structure $\{\{1, 2, 3\}, \{4\}\}$ as he is trying to form the 3-person coalition $\{1, 2, 3\}$ to negotiate a compensation to players 2 and 3. The nucleolus vector for this coalition structure is

$(17.5, 25, 32.5, 25)$.

This share is quite reasonable! It can be described by the following proposal of Player 1 to $\{2, 3\}$: "If you form a coalition then my proceeds will rise by 15 (from a quota of 10 to 25 utils). I give you half of it, namely 7.5, by offering 5 utils to Player 2 and 2.5 utils to Player 3. The reason for the unequal division is because it is more valuable for Player 1 that Player 2 enters a 2-person coalition with either Player 3 or 4. This will raise his proceeds by at least 10 (from 10 to 20), whereas if Player 3 forms a coalition, he may be stuck with Player 2, in which case his proceeds will rise only by 5 (from 10 to 15). Thus in effect, Player 1 offers each of the players 2 and 3 half of the minimal excess amount each guarantees him by entering a coalition before he forms a coalition. I cannot imagine a better resolution of the question asked at the beginning of this section. I did not expect such a convincing resolution before looking at the nucleolus and I am grateful that the nucleolus revealed it.

If Player 1 succeeds to "bribe" 2 and 3 to form a coalition, then Player 4 is the big loser. He might try to "bribe" either Player 2 or Player 3 to form a coalition with him. This is going to cost him. How much?

Heuristically, it seems that he should approach Player 3 and offer him 32.5 utils to outbid the offer of Player 1. This is better than trying to outbid by offering 25 utils to Player 2, thereby losing 15 utils from his quota. Indeed, the nucleolus of the power game for the coalition structure $\{\{1, 2\}, \{3, 4\}\}$ is⁸

$(15, 15, 32.5, 37.5)$.

Note that the nucleolus drives Player 1 to his power worth.

The nucleoli of the power game for the various coalition structures in which the total amount of proceeds is 100 utils is

given below. We write the coalitions that form, as much as possible, in the order of formation that, we believe, justifies the outcome:

$$\begin{aligned}
 & \left(17\frac{1}{2}, 25, 32\frac{1}{2}, 25; \{123, 4\} \right), \\
 & \left(20, 22\frac{1}{2}, 20, 37\frac{1}{2}; \{124, 3\} \right), \\
 & \left(22\frac{1}{2}, 15, 27\frac{1}{2}, 35; \{134, 2\} \right), \\
 & \left(15, 23\frac{1}{3}, 28\frac{1}{3}, 33\frac{1}{3}; \{234, 1\} \right), \\
 & \left(15, 15, 32\frac{1}{2}, 37\frac{1}{2}; \{34, 12\} \right), \\
 & \left(16\frac{2}{3}, 21\frac{2}{3}, 28\frac{1}{3}, 33\frac{1}{3}; \{23, 14\} \right), \\
 & (15, 25, 25, 35; \{24, 13\}), \\
 & \left(17\frac{1}{2}, 20\frac{5}{6}, 28\frac{1}{3}, 33\frac{1}{3}; \{1234\} \right).
 \end{aligned}$$

Assuming that the players consider the nucleolus as their solution, it seems that player 1 is in quite a strong position. The only way Player 4 can compete is by either convincing Player 3 or Player 2 to join him in a coalition that forms first. All other coalition structures yield both players less than the offer in the coalition structure $\{123, 4\}$. If Player 4 succeeds to attract either Player 3 or Player 4, Player 1 is driven to his bottom worth of 15.⁹

We are not claiming that either $\{123, 4\}$ or $\{34, 12\}$ will form. An ambitious Player 4, for example, may opt for $\{124, 3\}$ or $\{34, 12\}$ hoping for 37.5 utils, thereby attempting to cause a competition between the losers Player 2 or Player 3 to gain his favor. The formation of the grand coalition is also interesting: It can happen when Player 1 fears that either Player 2 or Player 3 will join Player 4 who has his own fears. Then, players 1 and 4 may agree to cause the formation of the grand coalition by

committing themselves not to listen to offers from Player 2 and/or Player 3. Player 2 can then try to destroy this coalition by forming a coalition with Player 3, but this may not succeed, especially if players 1 and 4 promise an additional token to Player 3. What we want to argue is that when the order of formation of coalitions is considered, and the players regard the nucleolus as their solution, then there are interesting bids and counter bids on various coalition formations.

3. CONCLUSION

We started with a game with a solution that looked trivial: A couple of 2-person coalitions will form and no-matter what, the payoff will be in accordance with the quota. This argument does not take into account that coalitions do not form simultaneously. It also does not take into account that a standard of fairness exists, according to which (at least normatively), *if all things are equal let us share equally*. Our analysis revealed that coalition formation is relevant and drastically affects the payoff outcome. One still cannot predict which coalitions will form, but whatever structure forms determines the payoff vector. The coalitions that form and the order of formation are strategic variables, and in order to achieve a desirable formation one may have to pay the participants. The present analysis suggests that the nucleolus of the power game can be employed to decide how much each player should get theoretically. For example, one cannot predict in our example if the coalition structure will be e.g.¹⁰, $\{\{1, 2, 3\}, \{4\}\}$ or $\{\{3, 4\}, \{1, 2\}\}$, if Player 1 takes the initiative and tries to convince Players 2 and 3 to form a coalition. Presumably, the final actual outcome will be determined by who will be willing to sacrifice somewhat more from the theoretical nucleolus prediction as in the spirit of the bargaining set theory. The theoretical nucleolus of the power game for a coalition structure predicts only a "center" around which real outcomes in games played by knowledgeable players should come about.¹¹

We have based our analysis on a single principle; namely, that a standard of fairness exists in the population of the

players. Can one generalize this analysis to a general solution theory for TU cooperative games? I feel that such task is not easy to achieve. Issues that should be addressed are: what types of standards of fairness can one assume when a subgame based on a certain coalition is not a unanimity game? How then should a power function be defined? More importantly, issues concerning coalition formation and the order of the formation can be more involved: a player may want that coalition S form provided that coalition T *does not form*; otherwise he would prefer the formation of another coalition, etc. It is therefore quite a challenge to generalize these ideas to a general theory.

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NOTES

1. Regardless of whether we attribute 0 to the worth of the 3 and 4-person coalitions or take the super additive cover of this game.
2. A student reported that he caused a rift between players 1 and 4.
3. Whenever we find it convenient, we shall refrain from using commas and curly brackets when specifying a coalition. Thus, we often write 12, instead of $\{1, 2\}$.
4. Of course there are several papers concerning coalition formation that look at the issue in other senses.
5. Rapoport and Kahan investigated experimentally the existence of standard of fairness and power perceptions among players playing characteristic function games. See Rapoport and Kahan (1979, 1982) and Kahan and Rapoport (1984), which summarizes their findings.
6. We take into consideration that Player 1 can wait until a 2-person coalition forms. We could be less biased and more pessimistic, allowing for the possibility that he may be forced to be in the coalition that forms first. We will then have to define $w(1) = \min\{10, 15\} = 10$. It turns out that the insight gained in this paper does not change much.
7. For the definition and computation of the nucleolus for a game when a coalition structure forms, see Schmeidler (1969), Owen (1977), Wall-

meier (1980), Potters and Tijs (1992), Maschler et al. (1992). In our game the easiest way to compute the nucleolus when a three person coalition forms, is to compute its kernel which consists of a single imputation and therefore equals the nucleolus (see Maschler and Peleg, 1966). When a couple of 2-person coalitions form, perhaps brute force is the easiest way to compute the nucleolus.

8. It is (11.25, 18.75, 32.5, 37.5) if we decide that $w(1) = 10$.
9. If we define $w(1) = 10$, three imputations change as follows: $(10, 23\frac{1}{3}, 28\frac{1}{3}, 33\frac{1}{3}; \{234, 1\})$, $(11\frac{1}{4}, 18\frac{3}{4}, 32\frac{1}{2}, 37\frac{1}{2}; \{34, 12\})$, $(12\frac{1}{2}, 25, 27\frac{1}{2}, 35; \{24, 13\})$.
10. Note that this notation does not represent the actual coalitions that form. For example, the first coalition structure means that $\{2, 3\}$ forms first, after being promised some compensation from Player 1.
11. The reader is referred to Maschler (1978, 1992) for further discussions on experimental findings.

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