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DECLINING VALUATIONS IN SEQUENTIAL AUCTIONS

by

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Declining Valuations in Sequential Auctions

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Abstract

We analyze an independent private values model where a number of objects are sold in sequential first- and second-price auctions. Bidders have unit demand and their valuation for an object is decreasing in the rank number of the auction in which it is sold. We derive efficient equilibria if prices are announced after each auction or if no information is given to bidders. We show that the sequence of prices constitutes a supermartingale. Even if we correct for the decrease in valuations for objects sold in later auctions we find that average prices are declining.

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1 Introduction

If multiple objects are sold in auctions, this is often done sequentially – one object after the other. In this paper we focus on situations where bidders have unit demands and their valuation for an object declines with the rank number of the auction it is sold. This is the case if otherwise identical objects become available at different points in time and bidders prefer to receive an object early. Examples are auctions of goods on a rental basis where one object is available to one buyer at a time (like vacation accommodation, cars, tractors or DVD's) or fish auctions, where the fish is auctioned when it arrives at the port. In addition, our analysis applies to situations where the objects are physically different and the more valuable objects are sold in earlier auctions. Prominent examples are right-to-choose auctions where the winner can choose one object from the pool of objects that are for sale.

We find for sequential first- or second-price (sealed bid) auctions that (expected) discounted prices decline. Furthermore, we show that conditional on the current price, the expected discounted price in any future period is below the current price. In other words: the stochastic process that governs the price development is a supermartingale. The intuition supporting these results is as follows: If discounted equilibrium prices remained constant, a bidder's expected utility from winning would decline (since valuation and price decline at the same rate). Since the probability of winning in a given period is not affected by the devaluation, the (non-conditional) expected utility would decline as well. This means that bidders can do better by increasing their bids in earlier periods, thereby deviating from the putative equilibrium. We provide an example where even a slight decline in valuations across periods results in a substantial price decline. Hence our results can offer an explanation for declining prices even in settings with high discount factors.

Another finding of our analysis concerns the information policy of the seller. We analyze the price dynamics for two different information structures. Under the first no information is revealed between periods. Under the second the seller announces after each period the price paid in that period. We show that both information structures yield efficient equilibria.

By using a revenue equivalence approach we argue that the same price dynamics hold for other sequential bidding mechanisms and information structures. Specifically, we show that any two sequential mechanisms which are efficient and in which information is revealed only about bidders who already quit the auction, yield the same price dynamics. In addition, the sequential formats are revenue equivalent to a Vickrey auction, where all objects are sold at the same time within one auction (and buyers have to wait after the auction until the objects become available). Hence, choosing a sequential format will not reduce the seller's revenue or create inefficiencies. What matters in terms of revenue is not whether the auctions are sequential or simultaneous but whether the goods are available to all buyers instantaneously[1].

We also show how our model translates to situations where the objects for sale are identical but bidders are uncertain about the continuation of the auction process.

Weber (1983) and Milgrom and Weber (2000) assume constant valuations across periods and show that expected prices stay constant. These papers analyze bidding behavior in first- and second-price auctions with and without price announcements but only provide a proof for the first-price auction with price announcements[2]. Beggs and Graddy (1997) analyze a two period second-price auction with declining valuations whereas Jeitschko (1999) and Pezanis-Christou (1997) analyze sequential auctions with uncertain supply[3]. We generalize these papers by allowing for more than two periods, for a more general valuation structure and by looking at different auction mechanisms. Moreover we use a

[1] If the latter is the case, it is obviously not in the interest of the seller to sell the objects sequentially if bidders are impatient.

[2] This proof is for a more general environment than ours since it also covers the case of common values. Milgrom and Weber (2000) point out the problems they face with the proofs for the other settings, e.g. they mention that with price announcements second-price auctions reveal information about remaining bidders and thus break the symmetry. This problem is tackled in our framework of independent private values.

[3] Jeitschko (1999) considers second price auctions with only two stages and Pezanis-Christou (1997) requires the highest bid to be announced between auctions to compute equilibria.

revenue equivalence result to show that our price dynamics holds for a reasonably large class of information structures. Our analysis of the price dynamics establishes results on today's price conditional on yesterday's price and exposes a super-martingale property for special cases. Finally, we provide comparative statics results that shed further light on the structure of price trends.

Several empirical papers report price decline in sequential auctions: Ashenfelter (1989) first observed this trend in wine auctions followed by Jones et al. (1996) for wool auctions and recently van den Berg et al. (2001) for flower auctions. These examples differ from our model in an important aspect: There is only a very short time difference between the availability of different items. Hence it is not clear (and also not analyzed in these papers) to what extend declining valuations matter. Our findings are consistent with Ashenfelter and Genesove (1992), who observe a "declining price anomaly" for real estate right-to-choose auctions, where prices drop faster than associated valuations. They observe that at most 25 % of the price drop can be explained by quality differences. Some theoretical papers give other reasons for declining prices: While McAfee and Vincent (1993) show that risk aversion can cause declining prices, von der Fehr (1994) shows that declining prices can be the result of participation costs. Relaxing the unit demand assumption, Black and de Meza (1992) explain declining prices by a buyer's option to purchase further objects for the same price and Menezes and Monteiro (2003) obtain declining prices if objects are complements.

Our paper is organized as follows: In Subsection 1.1 we present an example and sketch the main ideas. The general model is introduced in Section 2. In Section 3 we derive equilibria and properties of the trend of prices. Section 4 gives comparative statics results on the sequences of expected prices. Section 5 contains the conclusion. Proofs can be found in the appendix.

1.1 An illustrative example: constant proportional devaluation

There are two objects for sale in two subsequent second-price auctions and three bidders with unit demand. The bidders' valuations for the objects are common knowledge. Bidder

i's valuation for the object sold in the first auction is given by θ_i. Her valuation for the object sold in the second auction is $\delta\theta_i$ where $\delta \in (0,1]$. We write $\theta_{(1)}$ for the highest, $\theta_{(2)}$ for the median and $\theta_{(3)}$ for the lowest of the types. The last auction is a normal second-price auction and it is a dominant strategy for the two remaining bidders to bid their valuations, i.e. $\delta\theta_i$. The optimal bid in the first auction is $b(\theta) = \theta - \delta\theta + \delta\theta_{(3)}$. It is optimal to bid one's own valuation minus the utility of winning the second auction[4]. If $\delta = 1$ both objects are sold for the same price $p = \theta_{(3)}$. This is because a bidder can arbitrage away differences in prices: if for example the price in the first auction were higher the winning bidder would do better by losing the first auction and winning the second at a lower price. If valuations for the objects are decreasing, i.e. if $\delta < 1$, then the price paid in the first auction, $(1-\delta)\,\theta_{(2)} + \delta\theta_{(3)}$, is higher than the price in the second auction $\delta\theta_{(3)}$. It is even higher than the discounted price $\theta_{(3)}$ of the second object. The devaluation in prices does not follow the devaluation in valuations of the objects, in fact it is stronger. Intuitively, bidders would prefer winning the first auction for a price of p instead of winning the second auction for a price of δp. The first would give a utility of $\theta - p$ whereas the latter would only give $\delta(\theta - p) < (\theta - p)$. Hence bidding in the first auction is more aggressive and results in higher prices. In the following sections we show that this intuition translates to the incomplete information case.

2 A Model with Private Valuations

A seller offers $k \leq n$ indivisible objects to n risk neutral buyers $i = 1, ..., n$. The seller uses a sequential first- (second-) price auction, i.e. the objects are sold sequentially in periods: Each period consists of a first- (second-)price auction for one of the objects. The seller's valuation for each object is zero. Buyer i's valuation for an object depends on her type θ_i. The types θ_i are independently distributed on $[\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} \geq 0$ and are drawn according

[4]Note that this is not a dominant strategy since it might be profitable to underbid a (significantly) lower type of another bidder who deviates from her equilibrium in the first auction and bids slightly below her own equilibrium bid.

to a common distribution function F with continuous and strictly positive density f. We write $\theta_{(i)}$ for the $i'th$ highest type among $\theta_1, ..., \theta_n$, i.e. $\theta_{(i)}$ denotes the i'th order statistic of $\theta_1, ..., \theta_n$. Each buyer has unit demand and therefore quits after winning. A buyer's valuation for an object is a function of her type and the rank number of the period in which the object is sold, i.e. $D_l(\theta_i)$ denotes bidder $i's$ valuation for the object[5] sold in period l given that her type is θ_i. We make the following assumptions for D_l :

A1 Normalization: $D_1(\theta) = \theta$, $D_l(\theta) \geq 0$ for all l, θ

A2 Time is valuable: $D_l(\theta) \geq D_{l+1}(\theta)$ for all l

A3 Objects are more valuable for higher types: $D_l(\theta)$ is strictly increasing in θ for all l

A4 Continuity: $D_l(\theta)$ is continuous for all l

A5 Increasing loss to delay: $D_l(\theta) - D_{l+1}(\theta)$ is weakly increasing in θ for all l.

Example 1 *Constant proportional devaluation is given by* $D_l(\theta) = \delta^{l-1}\theta$ *for* $\delta \in (0, 1]$.

The last assumption states that higher types face higher devaluation (in absolute terms). Note the similarity of these conditions with those in Rubinstein's bargaining model (Rubinstein (1982)). However our notion of the devaluation function is more general in that it does not assume stationarity (an essential property in Rubinstein's time preferences), i.e. the degree of devaluation may change in time.

3 Equilibria and Price Trends

We allow for two different information policies pursued by the seller: she can either reveal the winning price (i.e. the highest bid in the first-price auction and the second highest bid

[5] The assumption that for all bidders the functions D_l are identical is restrictive. Nevertheless it guarantees that the order of the bidders' valuations is the same in each period. One can interpret the type as a general preference for the kind of objects for sale. Then it seems plausible that a bidder with a higher type should value each object more than a bidder with a lower type.

in the second-price auction) at the end of each period (auction with price announcement) or she can reveal no information at all (auction without price announcement). We restrict our attention to symmetric (Bayes-Nash-) equilibria. The following Theorem characterizes the symmetric equilibrium of the sequential second- and first-price auctions with and without price announcements:

Theorem 1 *The symmetric equilibrium bidding strategy for a type θ-bidder in period l of a sequential* ***first-price auction with or without price announcements*** *is given by b_l defined recursively:*

$$\begin{aligned} b_k(\theta) &= E\left[D_k\left(\theta_{(k+1)}\right) \mid \theta_{(k)} = \theta\right], \\ b_l(\theta) &= E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right) \mid \theta_{(l)} = \theta\right]. \end{aligned}$$

The symmetric equilibrium bidding strategy for a type θ-bidder in period l of a sequential ***second-price auction with or without price announcements*** *is given by b_l defined recursively:*

$$\begin{aligned} b_k(\theta) &= D_k(\theta), \\ b_l(\theta) &= D_l(\theta) - D_{l+1}(\theta) + E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid \theta_{(l+1)} = \theta\right]. \end{aligned}$$

The equilibria exhibit some interesting properties. First, bidding functions are strictly increasing, i.e., bidders of a higher type win earlier. This implies that the sequential auctions are ex-post efficient (and therefore optimal for a social planner). Furthermore, even in the second-price auction we find that bidders shade their bids, i.e. $b_l < D_l$, except for the last period. Note that the bidding functions do not depend on the history of the game. Since types are independent, the only relevant information (used for updating beliefs about remaining bidders' types) in period l of the first-price auction is the type of the bidder who won period $l-1$ since this is the $l-1$'th highest type. Every bidder can deduce this information by inverting the bidding function if prices are announced. The situation in the second-price auction is more complex because the bidder who sets the price in period $l-1$ participates in period l and therefore others might know her type if prices are announced. Theorem 1 shows that this does not lead to inefficiencies due to pooling.

If valuations remain constant over objects the sequence of prices is a martingale (see Weber (1983)), i.e. prices are constant on average over time. In our model with time preferences prices drift down over time. The following Theorem shows the link between (expected) prices in the subsequent period and observed prices in the actual period. Denote by p_l the price in period l in a sequential first- or second-price auction, i.e. $p_l = b_l\left(\theta_{(l)}\right)$ in a sequential first-price auction and $p_l = b_l\left(\theta_{(l+1)}\right)$ in a sequential second-price auction. Denote by $D_{l,l+1} := D_{l+1} \circ D_l^{-1}$ the devaluation function from period l to period $l+1$, i.e. $D_{l,l+1}(v)$ denotes a bidders valuation in period $l+1$ if her valuation is v in period l. Hence $D_{l,l+1}^{-1}(p_{l+1})$ is the price in period $l+1$ corrected for the last period devaluation.

Theorem 2 *In a sequential first-price auction the expected corrected price in period $l+1$ is always lower than the realized price p_l, i.e.*

$$E\left[D_{l,l+1}^{-1}(p_{l+1}) \mid p_l\right] \leq p_l. \tag{1}$$

In a sequential second-price auction given a price p_l in period l the corrected expected price in period $l+1$ is always lower than p_l, i.e.

$$D_{l,l+1}^{-1}\left(E\left[p_{l+1} \mid p_l\right]\right) \leq p_l. \tag{2}$$

Note that on the left hand side of (1) we have the expected corrected price (in period $l+1$) whereas in (2) we have the corrected expected price, both of which are compared with the actual price of period l. The reason for this difference is that in the sequential first-price auction bids are given in terms of expected utility of the second highest bidder, while in the second-price auction bidders bid their own valuation minus the expected outside option. If $D_{l,l+1}$ is convex we get statement (1) also for second-price auctions (using Jensen's inequality) and if $D_{l,l+1}$ is concave we get (2) for first-price auctions. The following Corollary summarizes some implications of Theorem 2.

Corollary 3

1. *The sequence of prices* $(p_l)_{l\leq k}$ *is a supermartingale.*

2. *The sequence of expected prices is the same for a sequential first- and second-price auction and we have*

$$E[D_{l,l+1}^{-1}(p_{l+1})] \leq E[p_l] \quad \text{and} \quad E[p_{l+1}] \leq E[D_{l,l+1}(p_l)].$$

If $D_{l,l+1}$ *is concave we have*

$$D_l^{-1}(E[p_l]) \geq D_{l+1}^{-1}(E[p_{l+1}]).$$

3. *For proportional devaluation, i.e.* $D_l(\theta) = \prod_{i=1}^{l-1} \delta_i \theta,\ \delta_i \in (0,1)$, *the sequence* $\left(D_l^{-1}(p_l)\right)_{l\leq k}$ *is a supermartingale. Moreover (1) and (2) hold.*

Due to the Revenue-Equivalence-Theorem the seller's expected revenue is the same for all mechanisms that implement the efficient allocation and result in the same utility level for a type-$\underline{\theta}$-bidder. Corollary 3 shows that even the per period revenue for sequential first- and second-price auctions is the same[6]. The following Theorem shows revenue equivalence in each period for a large class of efficient sequential auction mechanisms. A sequential auction mechanism is a mechanism[7] in which bidders submit bids in each period and the object (sold in that period) is given to the bidder with the highest bid. Payments to the

[6]By applying techniques known from the case of one object (see Myerson (1981)), we can also characterize the revenue-maximizing sequential auction. If we assume that virtual valuations $D_l(\theta) - D_l'(\theta)\frac{1-F(\theta)}{f(\theta)}$ fulfill the properties A2, A3 and A5, then the sequential first- or second-price auction with a reserve price of $r_l = D_l(\theta_{r,l})$ in period l, where $\theta_{r,l}$ is defined by

$$D_l(\theta_{r,l}) - D_l'(\theta_{r,l})\frac{1-F(\theta_{r,l})}{f(\theta_{r,l})} = 0,$$

is optimal. For the special case of constant proportional devaluation the optimal reserve prices are given by $r_l = \delta r_{l-1}$, $r_1 - \frac{1-F(r_1)}{f(r_1)}$, if the "standard" virtual valuation $\theta - \frac{1-F(\theta_i)}{f(\theta_i)}$ is strictly increasing in θ.

[7]For a more formal definition, see the proof of Theorem 4.

seller depend on the submitted bids[8]. We consider four properties of sequential auction mechanisms:

P1 The mechanism results in an efficient allocation of the objects, i.e., the l'th object is sold to the bidder with the l'th highest type.

P2 A bidder of type $\underline{\theta}$ has zero expected payments in each period.

P3 After each period the type of the winning bidder in that period is announced.

P4 Each bidder's (expected) payment (in a period) only depends on her own bid (of that period).

Theorem 4 *For any two sequential auction mechanisms both satisfying either P1, P2, P3 or P1, P2, P4 the expected sum of payments in period l (i.e. for the l'th object) is the same.*

Our previous analysis translates to a model with discounting or equivalently to a model with an uncertain number of objects. Assume that bidders in period l expect a continuation of the auction process with (commonly known) probability δ_l, i.e. with probability $1-\delta_l$ period l was the last period. Since agents are risk-neutral, this model is equivalent to a model in which discounting applies to both payments and valuations with the same discount factor δ_l (between period l and $l+1$). Formally, if $\delta_1, \ldots, \delta_{k-1}$ are the inter-period discount rates then winning an object in period $l+1$ for the (nominal) price of p_{l+1} yields a utility level of $\delta_l\left(\theta - p_{l+1}\right)$ for a type-θ-agent finding herself in period l.

[8] An example is a sequential version of the Clarke-Groves mechanism: In period l bidders submit types $\widehat{\theta}_i$, the bidder with the highest announcement wins the l'th object and pays $D_l\left(\widehat{\theta}^l_{(2)}\right) + \sum_{j=2}^{k-l+1}\left(D_j\left(\widehat{\theta}^l_{(j+1)}\right) - D_j\left(\widehat{\theta}^l_{(j)}\right)\right)$, where $\widehat{\theta}^l_{(j)}$ is the j'th highest announced type in period l. After each period the highest announced type is made public and the winning bidder of that period quits the mechanism (i.e. does not participate in subsequent periods). Since truthtelling is a dominant strategy in the Clarke-Groves mechanism, this is also true for its sequential version (which results in the same allocation and payments and only differs in the bidders' knowledge on others' types).

Corollary 5 *The equilibrium bidding function for the sequential first-price auction (with or without price announcement) with uncertain continuation is given by*

$$\begin{aligned} b_l(\theta) &= E\left[\theta_{(l+1)} - \delta_l\left(\theta_{(l+1)} - b_l\left(\theta_{(l+1)}\right)\right) \mid \theta_{(l)} = \theta\right], \\ b_k(\theta) &= E\left[\theta_{(k+1)} \middle| \theta_{(k)} = \theta\right]. \end{aligned}$$

For the sequential second-price auction with uncertain continuation the bidding function is given by

$$\begin{aligned} b_l(\theta) &= \theta - \delta_l\left(\theta - E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid \theta_{(l+1)} = \theta\right]\right) \\ b_k(\theta) &= \theta. \end{aligned}$$

The sequence of actual prices $(p_l)_{l\leq k}$ with $p_l = b_l\left(\theta_{(l)}\right)$ for the first-price auction and $p_l = b_l\left(\theta_{(l+1)}\right)$ for the second-price auction is a supermartingale, i.e. given any realization p_l expected prices decline on average.

4 Comparative Statics

In this section we study how changes in various parameters of the model (e.g., the number of bidders, the devaluation factor and the distribution of valuations) affect the price dynamics.

If we denote the expected price in period l by $\overline{p}_l$, we get as a direct consequence of Theorem 1

$$\begin{aligned} \overline{p}_l &= E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right)\right] + \overline{p}_{l+1} \text{ for } l < k \text{ and} \\ \overline{p}_k &= E\left[D_k\left(\theta_{(k+1)}\right)\right]. \end{aligned} \tag{3}$$

From this it is easy to see that for a given k, prices increase in the number of bidders n and converge to the highest type's valuation of the good.

For further analysis we confine our attention to the case where $D_l(\theta) := \delta^{l-1}\theta$. Fix the number of objects k and denote by $\overline{p}_l(n)$ the expected price in the l'th auction if the number of bidders in the first auction was n. As the number of bidders becomes large (for a fixed number of goods), we find that expected prices decrease approximately with δ, i.e. $\lim_{n\to\infty} \frac{\overline{p}_l(n)}{\overline{p}_{l-1}(n)} = \delta$ for all $l \leq k$. For a fixed number of bidders and objects the difference in prices $\overline{p}_l - \overline{p}_{l+1} = \delta^{l-1}(1-\delta) E\left[\theta_{(l+1)}\right]$ is decreasing in l. The ratio of prices, i.e. the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$, depends on the distribution of types, or more precisely on the expected values of the l'th order statistics. For high δ, we will see that the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$ is increasing.

Theorem 6 *Fix a number of bidders n and a number of objects $2 < k < n$.*

1. *For every distribution F there exists a threshold $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$ is increasing.*
2. *If the sequence $\left(\frac{E\left[\theta_{(l)}\right]}{E\left[\theta_{(l-1)}\right]}\right)_{l\leq k}$ is increasing (decreasing) and if for $\delta \in (0,1)$ we find that*
$$\frac{E\left[\theta_{(k+1)}\right]}{(1-\delta) E\left[\theta_{(k)}\right] + \delta E\left[\theta_{(k+1)}\right]} \underset{(<)}{>} \frac{E\left[\theta_{(k)}\right]}{E\left[\theta_{(k-1)}\right]}$$
then the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$ is increasing (decreasing).

To analyze the proportion of the decline in prices that can be directly attributed to the factor δ we look at the ratio $\frac{(1-\delta)\overline{p}_l}{\overline{p}_l - \overline{p}_{l+1}}$. Because of (3) we have

$$\frac{(1-\delta)\overline{p}_l}{\overline{p}_l - \overline{p}_{l+1}} = \frac{\sum_{m=l}^{k-1}\left(\delta^{m-l} - \delta^{m-l+1}\right) E\left[\theta_{(m+1)}\right] + \delta^{k-l} E\left(\theta_{(k+1)}\right)}{E[\theta_{(l+1)}]}.$$

Therefore the portion of the total price difference that cannot be directly explained by the devaluation is greater if δ is higher. The share that can be explained by the direct effect (of the devaluation) is always higher than $\frac{E[\theta_{(k+1)}]}{E[\theta_{(l+1)}]}$, but can be arbitrarily close to this value (if δ is close to 1). In the next example we show that this share can be arbitrarily small[9].

[9] In general, this can be achieved by a distribution that has its mass concentrated around 0 and 1. If the mass concentrated in a small environment of 0 becomes large, $\frac{E[\theta_{(k+1)}]}{E[\theta_{(l+1)}]}$ is close to zero.

Example 2 *Consider the case of a distribution where the type is 0 with probability very close to q and 1 with probability very close to $1-q$. If we have two objects and three bidders (3) yields that expected prices are close*[10] *to:*

$$\begin{aligned}\overline{p}_1 &= (1-q)^3 + 3(1-\delta)(1-q)^2 q, \\ \overline{p}_2 &= \delta (1-q)^3 .\end{aligned}$$

If we have $q = \frac{3}{4}$ and $\delta = 0.99$ we get that $\frac{\overline{p}_2}{\overline{p}_1} \approx 0.91$ and $\frac{1-\delta}{1-\frac{\overline{p}_2}{\overline{p}_1}} \approx \frac{1}{9}$, indicating that $\frac{8}{9}$ of the total price drop cannot be explained directly by the factor δ. This shows that the indirect effect can be substantial and that a significant decline in prices may occur even if devaluation at first sight seems to be too small to have a significant impact on the trend of prices.

5 Conclusion

We show that in sequential first- and second-price auctions with or without price announcements we have declining prices if valuations decrease for objects sold in later auctions. Even if we account for the general decrease in valuations, which is given by a common general devaluation function D_l, $l = 1, \ldots, k$, expected prices decline in later auctions. Even if the decline in valuations is relatively small it can have a substantial effect on the development of prices, hence our model might explain declining prices for environments where devaluation seems to be negligible at first sight. A revenue equivalence result shows that our findings translate to a large class of sequential selling mechanisms[11] and therefore are applicable to other market mechanisms as well.

[10] Note that a discrete distribution which only puts mass on 0 and 1 does not fulfill the assumptions made in this paper. In particular there do not exist pure strategy equilibria if the distribution of types has mass points. The prices $\overline{p}_1$ and $\overline{p}_2$ (which are calculated from (3) with a discrete distribution that puts mass q on 0 and $1-q$ on 1) can be approximated arbitrarily close by choosing continuous distributions that have mass concentrated around 0 and 1.

[11] Our findings also translate to non-sequential selling mechanisms as the comparison to the Clarke-Groves mechanism shows.

A Appendix

Proof of Theorem 1

We prove the statement for the sequential first-price auction without and sequential second-price auction with price announcements. The other proofs follow similar arguments and are omitted here.

Consider first the sequential first-price auction without price announcements. We prove that it is never beneficial for a bidder to imitate a type different to her own type θ in some of the periods of the auction. The expected utility of a type-θ-bidder who always bids according to $b_l(\theta)$ in period l (if she is still in the auction) and who faces bidders following the same strategies b_l is denoted by $U(\theta)$. A bidder who deviates from the strategy b_l by bidding as if she were of type θ^l in period l (if she did not win in period $m < l$), expects a utility of $U(\theta, \theta^1, \ldots, \theta^k)$. To improve the exposition we set $D_l = b_l = 0$ for $l > k$ and we use the following abbreviations:

$$\widehat{E}(x_l) := \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta^i)\right) \frac{(n-l-1)!}{(n-i)!} \int_{\theta^{l+1}}^{\max\left(x_l, \theta^{l+1}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1},$$

$$E(x_l) := \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!} \int_{\theta}^{x_l} \cdots \int_{\theta}^{x_{i-2}} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}.$$

We show that $U(\theta) \geq U(\theta, \theta^1, \ldots, \theta^k)$ for all $(\theta, \theta^1, \ldots, \theta^k) \in [\underline{\theta}, \overline{\theta}]^{k+1}$ by using an induction argument, i.e. we show that for $l=1$ and $x_{l-1}=x_0=\overline{\theta}$ we have that

$$\begin{aligned}
& U\left(\theta,\theta^1,\ldots,\theta^k\right)=\left(D_l\left(\theta\right)-b_l\left(\theta^l\right)\right)F^{n-l}\left(\theta^l\right)+\left(n-1\right)\int_{\theta^l}^{x_{l-1}}\widehat{E}\left(x_l\right)f\left(x_l\right)dx_l \qquad (4)\\
& +\left(n-l\right)\left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta^{l+1}\right)\right)\int_{\theta^l}^{x_{l-1}}F^{n-l-1}\left(\min\left(x_l,\theta^{l+1}\right)\right)f\left(x_l\right)dx_l\\
\leq\ & \left(D_l\left(\theta\right)-b_l\left(\theta\right)\right)F^{n-l}\left(\theta\right)+\left(n-l\right)\left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta\right)\right)\int_{\theta}^{x_{l-1}}F^{n-l-1}\left(\theta\right)f\left(x_l\right)dx_l\\
& +\left(n-1\right)\int_{\theta}^{x_{l-1}}E\left(x_l\right)f\left(x_l\right)dx_l=U\left(\theta\right).
\end{aligned}$$

Since we have[12]

$$\left(D_l\left(\theta\right)-b_l\left(\theta^l\right)\right)F^{n-l}\left(\theta^l\right)=\int_{\underline{\theta}}^{\theta^l}\left(D_l\left(\theta\right)-D_l\left(x\right)+D_{l+1}\left(x\right)-b_{l+1}\left(x\right)\right)dF^{n-l}\left(x\right) \qquad (5)$$

it suffices to show that for all $\left(\theta,\theta^{l+1},\ldots,\theta^k\right)\in[\underline{\theta},\overline{\theta}]^{k-l+1}$ the following three statements hold:

1. for $x_l\geq\theta$

$$\begin{aligned}
& \left(D_l\left(\theta\right)-D_l\left(x_l\right)+D_{l+1}\left(x_l\right)-b_{l+1}\left(x_l\right)\right)F^{n-l-1}\left(x_l\right) \qquad (6)\\
\leq\ & \left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta\right)\right)F^{n-l-1}\left(\theta\right)+E\left(x_l\right)
\end{aligned}$$

2. for $x_l\leq\theta$

$$\begin{aligned}
& \left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta^{l+1}\right)\right)F^{n-l-1}\left(\min\left(x_l,\theta^{l+1}\right)\right)+\widehat{E}\left(x_l\right) \qquad (7)\\
\leq\ & \left(D_l\left(\theta\right)-D_l\left(x_l\right)+D_{l+1}\left(x_l\right)-b_{l+1}\left(x_l\right)\right)F^{n-l-1}\left(x_l\right)
\end{aligned}$$

3. for $x_l\geq\theta$

$$\begin{aligned}
& \left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta^{l+1}\right)\right)F^{n-l-1}\left(\min\left(x_l,\theta^{l+1}\right)\right)+\widehat{E}\left(x_l\right) \qquad (8)\\
\leq\ & \left(D_{l+1}\left(\theta\right)-b_{l+1}\left(\theta\right)\right)F^{n-l-1}\left(\theta\right)+E\left(x_l\right).
\end{aligned}$$

[12] This can easily be seen by using the following representation of b_i:

$$b_l\left(\theta\right)=\frac{1}{F^{n-l}\left(\theta\right)}\int_{\underline{\theta}}^{\theta}\left(D_l\left(x\right)-D_{l+1}\left(x\right)+b_{l+1}\left(x\right)\right)dF^{n-l}\left(x\right).$$

This is done by three induction arguments[13].

1. Subtracting $\left(D_l(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta)$ on both sides of (6) gives

$$\begin{aligned} & \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta)\right) F^{n-l-1}(x_l) \\ & + \int_\theta^{x_l} \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) dF^{n-l-1}(x_{l+1}) \\ \leq\ & (n-l-1) \int_\theta^{x_l} \Big(\left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) F(\theta)^{n-l-2} + E(x_{l+1}) \Big) dF(x_{l+1}). \end{aligned}$$

   Since $D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta) \leq 0$ this is true if for all $x_{l+1} \geq \theta$ we have

$$\begin{aligned} & \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) F^{n-l-2}(x_{l+1}) \\ \leq\ & \left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) F^{n-l-2}(\theta) + E(x_{l+1}). \end{aligned}$$

   For $l = k$ and $x_k \geq \theta$ this is true since $D_k(\theta) - D_k(x_k) \leq 0$.

2. For $x_l \leq \theta^{l+1}$ this is true since b_{l+1} is increasing and we have

$$D_l(\theta) - D_{l+1}(\theta) - \left(D_l(x_l) - D_{l+1}(x_l)\right) \geq 0.$$

   Assume now that $x_l > \theta^{l+1}$. Then (7) is equivalent to

$$\begin{aligned} & (n-l-1) \int_{\theta^{l+1}}^{x_l} \left(D_{l+2}(\theta) - b_{l+2}\left(\theta^{l+2}\right)\right) F^{n-l-2}\left(\min\left(x_{l+1}, \theta^{l+2}\right)\right) dF(x_{l+1}) \\ & + (n-l-1) \int_{\theta^{l+1}}^{x_l} \widehat{E}(x_{l+1})\, dF(x_{l+1}) \\ \leq\ & \underbrace{\left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta)\right)}_{\geq 0} F^{n-l-1}(x_l) \\ & + \int_{\theta^{l+1}}^{x_l} \left(D_{l+1}(\theta) - D_{l+1}(y) + D_{l+2}(y) - b_{l+2}(y)\right) dF^{n-l-1}(y). \end{aligned}$$

   This is true if for $x_{l+1} \leq \theta$ we have that (7) is correct for "$l = l + 1$". For $l = k$ the statement is true since $0 \leq \left(D_k(\theta) - D_k(x_k)\right) F^{n-(k+1)}(x_k)$ for $x_k \leq \theta$.

[13] The induction is over l starting from $l = k$ going backwards to $l = 1$.

3. To show that (8) holds we again consider two cases: If $x_l \leq \theta^{l+1}$, the argument is similar to the second case of 2 and therefore omitted here.

   If we have $x_l \geq \theta^{l+1}$, we have to show that

$$\begin{aligned}
& \left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) F^{n-(l+1)}\left(\theta^{l+1}\right) + (n-l-1) \int_{\theta^{l+1}}^{x_l} \widehat{E}\left(x_{l+1}\right) dF\left(x_{l+1}\right) \\
& + (n-l-1)\left(D_{l+2}(\theta) - b_{l+2}\left(\theta^{l+2}\right)\right) \int_{\theta^{l+1}}^{x_l} F^{n-l-2}\left(\min\left(x_{l+1}, \theta^{l+2}\right)\right) dF\left(x_{l+1}\right) \\
\leq\ & \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) + (n-l-1) \int_{\theta^{l}}^{x_l} E\left(x_{l+1}\right) dF\left(x_{l+1}\right) \\
& + (n-l-1)\left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) \int_{\theta}^{x_l} F^{n-l-2}(\theta) f\left(x_{l+1}\right) dx_{l+1}.
\end{aligned}$$

   This is statement (4) formulated for $l+1$. Therefore the Theorem holds for general l if (4) holds for $l = k$ which is the case since

$$\left(D_k(\theta) - b_k\left(\theta^k\right)\right) F^{n-k}\left(\theta^k\right) \leq \left(D_k(\theta) - b_k(\theta)\right) F^{n-k}(\theta).$$

We now give the proof for the sequential second-price auction with price announcements. We write $v_l(\theta; x_1 ... x_{n-l})$ for the utility of a bidder with type θ, who finds herself in period l given her remaining opponents have types $x_1, ..., x_{n-l}$ and everyone announces her type truthfully. If $x_i < \theta$ for all $i = 1..n-l$ the θ-type buyer wins the l'th auction and we have

$$v_l(\theta; x_1, ..., x_{n-l})) = D_l(\theta) - b_l(\max\{x_1, ..., x_{n-l}\}). \tag{9}$$

We show by induction that it is optimal to bid according to $b_l(\theta_i)$ in period l if it is optimal to bid according to b_m in period m for $m > l$ and if all other bidders (always) bid according to b_l. In period $l = k$ bidding $D_k(\theta_i)$ is a dominant strategy. To show that it is optimal to bid according to b_l in period l consider first the bidder who submitted the highest bid in period $l-1$.

The expected utility of a bidder in period l who sets the price in period $l-1$ does only depend on her type θ, her bid in period l given by[14] $b_l\left(\theta^l\right)$ and in period $l-1$ given by $b_{l-1}\left(\theta^{l-1}\right)$. Her bid in period $l-1$ influences her expected utility since she updates her

[14]Note that bidding outside the range of b_l has the same effect as bidding $b_l(\theta) = \underline{\theta}$ or $b_l(\theta) = \overline{\theta}$.

beliefs about other agents' types distributions by inferring that these are given by $F[\theta| \theta \leq \theta^{l-1}] = \frac{F(\theta)}{F(\theta^{l-1})}$ for $\theta \leq \theta^{l-1}$. Bids in periods 1 to $l-2$ have no influence since all relevant information about other agents' types is given by the fact that these are smaller than[15] θ^{l-1}. A bidder's expected utility in period l if she is type θ, bids as if she were of type θ^l (in period l), submitted $b_{l-1}(\theta^{l-1})$ in period $l-1$ and $b_m(\theta)$ in periods $m > l$ is given by:

$$
\begin{aligned}
& U_l\left(\theta, \theta^l, \theta^{l-1}\right) \qquad (10) \\
= \; & \frac{n-l}{F^{n-l}\left(\theta^{l-1}\right)} \int_{\underline{\theta}}^{\theta^l} \left[D_l(\theta) - b_l(x_1)\right] F^{n-l-1}(x_1) f(x_1)\, dx_1 \\
& + \frac{n-l}{F^{n-l}\left(\theta^{l-1}\right)} \int_{\theta^l}^{\theta^{l-1}} \int_{\underline{\theta}}^{\min\{x_1,\theta\}} \cdots \int_{\underline{\theta}}^{\min\{x_1,\theta\}} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_1)\, dx_1 \\
& + \frac{n-l}{F^{n-l}\left(\theta^{l-1}\right)} (n-l-1) \\
& \int_{\theta^l}^{\theta^{l-1}} \int_{\min\{x_1,\theta\}}^{x_1} \int_{\underline{\theta}}^{x_2} \cdots \int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_1)\, dx_1 .
\end{aligned}
$$

The first addend describes the case where the bidder wins in period l. The second addend describes the case where she does not win period l but wins period $l+1$. The last addend describes the case where she neither wins period l nor period $l+1$.

We show that $\frac{\partial}{\partial \theta^l} U_l\left(\theta, \theta^l, \theta^{l-1}\right) \geq 0$ for $\theta^l < \theta$ and $\frac{\partial}{\partial \theta^l} U_l\left(\theta, \theta^l, \theta^{l-1}\right) \leq 0$ for $\theta^l > \theta$ if the same is true for period $l+1$. Since

$$
b_l\left(\theta^l\right) = D_l\left(\theta^l\right) - \frac{1}{F^{n-l-1}\left(\theta^l\right)} \int_{\underline{\theta}}^{\theta^l} \cdots \int_{\underline{\theta}}^{\theta^l} v_{l+1}\left(\theta^l; x_2 \ldots x_{n-l}\right) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2
$$

we have to determine the sign of

$$
\begin{aligned}
& \frac{F^{n-l}\left(\theta^{l-1}\right)}{(n-l) f\left(\theta^l\right)} \frac{d}{d\theta^l} U_l\left(\theta, \theta^l, \theta^{l-1}\right) \qquad (11) \\
= \; & \int_{\underline{\theta}}^{\theta^l} \cdots \int_{\underline{\theta}}^{\theta^l} \left[D_l(\theta) - D_l\left(\theta^l\right) + v_{l+1}\left(\theta^l; x_2 \ldots x_{n-l}\right)\right] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\
& - \int_{\underline{\theta}}^{\min\left(\theta^l,\theta\right)} \cdots \int_{\underline{\theta}}^{\min\left(\theta^l,\theta\right)} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\
& - (n-l-1) \int_{\min\left(\theta^l,\theta\right)}^{\theta^l} \int_{\underline{\theta}}^{x_2} \cdots \int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2
\end{aligned}
$$

[15] This is due to independence of types.

First assume[16] $\theta^l > \theta$: For $l = k$ we obtain

$$(11) = \int_{\underline{\theta}}^{\theta^l} ... \int_{\underline{\theta}}^{\theta^l} \left[D_k(\theta) - D_k\left(\theta^l\right)\right] f(x_{n-k})\, dx_{n-k} \cdots f(x_2)\, dx_2,$$

which is smaller than zero, since $\theta^l > \theta$ and D_k is strictly increasing.

Using (9) we have that (11) equals:

$$\underbrace{\int_{\underline{\theta}}^{\theta} ... \int_{\underline{\theta}}^{\theta} (D_l(\theta) - D_{l+1}(\theta)) - \left(D_l\left(\theta^l\right) - D_{l+1}\left(\theta^l\right)\right) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2}_{\leq 0}$$
$$+ (n-l-1) \int_{\theta}^{\theta^l} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} \left[D_l(\theta) - D_l\left(\theta^l\right) + v_{l+1}\left(\theta^l; x_2 ... x_{n-l}\right)\right.$$
$$\left. - v_{l+1}(\theta; x_2 ... x_{n-l})\right] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2.$$

Since $\theta^l > x_i$ (a.e.) for $i = 2, \ldots, n-l$ and since x_2 denotes the highest of the other bidders' types (remaining in the auction) we have

$$\begin{aligned} v_{l+1}\left(\theta^l; x_2 ... x_{n-l}\right) &= D_{l+1}\left(\theta^l\right) - D_{l+1}(x_2) \\ &\quad + \frac{1}{F^{n-l-2}(x_2)} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} v_{l+2}(x_2; \widetilde{x}_3, ..., \widetilde{x}_{n-l}) f(\widetilde{x}_{n-l})\, d\widetilde{x}_{n-l} \cdots f(\widetilde{x}_3)\, d\widetilde{x}_3. \end{aligned}$$

In addition we have $\theta < x_2$ and consequently $v_{l+1}(\theta; x_2 ... x_{n-l}) = v_{l+2}(\theta; x_3 ... x_{n-l})$. Since $D_l - D_{l+1}$ is increasing we have

$$\begin{aligned} & (n-l-1) \int_{\theta}^{\theta^l} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} \left[D_l(\theta) - D_l\left(\theta^l\right) + D_{l+1}\left(\theta^l\right) - D_{l+1}(x_2)\right. \\ & \left. + v_{l+2}(x_2; x_3 ... x_{n-l}) - v_{l+2}(\theta; x_3 ... x_{n-l})\right] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\ \leq\ & (n-l-1) \int_{\theta}^{\theta^l} \left[\int_{\underline{\theta}}^{\theta} ... \int_{\underline{\theta}}^{\theta} [D_{l+1}(\theta) - D_{l+1}(x_2)\right. \\ & + v_{l+2}(x_2; x_3 ... x_{n-l}) - v_{l+2}(\theta; x_3 ... x_{n-l})] f(x_{n-l})\, dx_{n-l} \cdots f(x_3)\, dx_3 \\ & + (n-l-2) \int_{\theta}^{x_2} \int_{\theta}^{x_3} ... \int_{\theta}^{x_3} [D_{l+1}(\theta) - D_{l+1}(x_2) \\ & \left. + v_{l+2}(x_2; x_3 ... x_{n-l}) - v_{l+2}(\theta; x_3 ... x_{n-l})] f(x_{n-l})\, dx_{n-l} \cdots f(x_3)\, dx_3\right] f(x_2)\, dx_2. \end{aligned}$$

[16] This is the more complicated case since "overbidding", i.e. overstating her own type might lead to winning in period l instead of winning in some later period.

The integrand of the outer integral is smaller than zero by induction[17] since $x_2 > \theta$. Assume now that we have $\theta > \theta^l$. We get $\frac{d}{d\theta^l} U_l\left(\theta, \theta^l, \theta^{l-1}\right) \geq 0$ since $D_l - D_{l+1}$ is increasing and

$$(11) = \int_{\underline{\theta}}^{\theta^l} ... \int_{\underline{\theta}}^{\theta^l} \left[D_l(\theta) - D_{l+1}(\theta) - \left(D_l\left(\theta^l\right) - D_{l+1}\left(\theta^l\right)\right)\right] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2.$$

Consider now a bidder who did not submit the highest bid in period $l-1$. Assume y to be the highest of the other bidders' types which is known since it can be inferred from the announced price of the previous period. We show that for $\theta > y$ it is optimal for bidder i to win period l, which implies that bidding according to b_l is optimal (given it is optimal to bid according to b_m in forthcoming periods $m > l$). If $\theta < y$, bidder i finds it optimal not to win period l which is achieved by bidding according to b_l as well.

The difference in utility between winning period l and not winning for a type θ agents is

$$\begin{aligned} & D_l(\theta) - b_l(y) && (12) \\ & -\frac{1}{F^{n-l-1}(y)} \int_{\underline{\theta}}^{\min\{y,\theta\}} ... \int_{\underline{\theta}}^{\min\{y,\theta\}} v_{l+1}(\theta; x_2 ... x_{n-l})\, f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\ & -\frac{1}{F^{n-l-1}(y)} (n-l-1) \\ & \int_{\min\{y,\theta\}}^{y} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 ... x_{n-l})\, f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2. \end{aligned}$$

This equation has the same sign as (11) if $\theta^l = y$. Therefore we already proved that (12) is negative if $\theta < y$ and positive if $\theta > y$ and the induction argument given above is also valid here.

Proof of Theorem 2:

In the sequential first-price auction denote the type who sets the price p_l, by θ i.e.

$$b_l(\theta) = E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right) \mid \theta_{(l)} = \theta\right] = p_l.$$

Since $D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right) \leq \theta_{(l+1)}$ and $D_l - D_{l+1}$ is increasing, we have

$$D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + D_{l+1}\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right) \geq D_l\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right)$$

[17] Obviously it is smaller than zero if $l = k-1$.

which yields the statement.

For the sequential second-price auction we have

$$b_l(\theta) = D_l(\theta) - D_{l+1}(\theta) + E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid \theta_{(l+1)} = \theta\right] .$$

We define p_0 by $p_l = D_l(p_0)$ and get

$$\begin{aligned} & E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid b_l\left(\theta_{(l+1)}\right) = p_l\right] \\ = \; & D_l(p_0) - D_l\left(b_l^{-1}\left(D_l(p_0)\right)\right) + D_{l+1}\left(b_l^{-1}\left(D_l(p_0)\right)\right) \\ \leq \; & D_{l+1}(p_0) . \end{aligned}$$

Where the inequality holds because $D_l - D_{l+1}$ is increasing and because $D_l > b_l$.

Proof of Corollary 3:

1. This follows directly from Theorem 2 and the fact that $D_{l,l+1}^{-1}(x) \geq x$.
2. This follows directly from Theorem 2, the fact that $D_l - D_{l+1}$ is increasing and $b_{l+1} \leq D_{l+1}$.
3. This results from the linearity of $D_{l,l+1}$and Theorem 2.

Proof of Theorem 4:

A sequential $k-$period auction is given by the strategy set $\mathbf{R}^+$, the sets of participating bidders of period l, $H_l \subseteq \{1, \ldots, n\}$, the sellers information policy, allocation functions $s = (s_1, \ldots, s_k)$ and payment functions $t = (t_1, \ldots, t_k)$ specified as follows: In period l all bidders submit bids $b_{l,i} \in \mathbf{R}^+$. The allocation function $s_l : \mathbf{R}^n \mapsto \{1, \ldots, n\}$ allocates the l'th object to the highest participating bidder of that period[18], i.e. $s_l(b_{l,1}, \ldots, b_{l,n}) = \arg\max_{i \in H_l} b_{l,i}$. Bidder i has to make a payment to the seller which is given by $-t_{l,i}(b_{l,1}, \ldots, b_{l,n})$ whereas this is zero for non-participating bidders (i.e. $t_{l,i} = 0$ if $i \notin H_l$) and does not depend on bids of non-participating bidders (i.e. for all $j \notin H_l$ we have $t_{l,i}(b_{l,j}, b_{l,-j}) = t_{l,i}\left(\widetilde{b}_{l,j}, b_{l,-j}\right)$ for all $i \in H_l$ and $b_{l,j}$, $\widetilde{b}_{l,j} \in \mathbf{R}^+$). In period $l+1$

[18] If there is more than a single highest bidder any tie-breaking rule can be applied.

the set of participating bidders is given by $H_{l+1} = H_l \backslash \{s_l\}$. Before period l information concerning the winning type of the previous period might be revealed to all agents (e.g. if we have efficient equilibria the seller can do so by announcing the highest bid of the previous period). The information policy is common knowledge.

Consider period l of a $k-$period sequential auction where everyone bids according to an efficient equilibrium in previous periods. The belief about other types' distribution of an agent who participates in period l depends on the previously observed history and on her own type. We will denote this distribution (for an agent i) by $F_{l,i}(\theta_i, \theta_{-i})$. If the winner's type of period $l-1$ is known $F_{l,i}$ does only depend on this type since types are distributed independently. If no announcements are made, $F_{l,i}$ only depends on the θ_i (since we assumed truthful bidding in previous periods). In period l no agent should have an incentive to bid as if she were of a different type given all other agents stick to the equilibrium. We denote by $U_{l,i}\left(\theta_i, \widehat{\theta}_i\right)$ the expected utility of an agent i of type θ_i of reaching period l who behaves subsequently as if she were of type $\widehat{\theta}_i$ and who faces agents that are bidding according to the equilibrium (and do not imitate other types). In addition we denote by $t_{l,i}\left(\widehat{\theta}_i, \theta_{-i}\right)$ the payment of a bidder who bids as if she were of type $\widehat{\theta}_i$ in period l (given the other agents bid according to their equilibrium strategy). We have that

$$\begin{aligned}
& U_{l,i}\left(\theta_i, \widehat{\theta}_i\right) \qquad (13)\\
= \; & D_l(\theta_i)\, E_{\theta_{-i}}\left[\mathbf{1}\left(\widehat{\theta}_i > \theta_{(l)}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right] - E_{\theta_{-i}}\left[t_{l,i}\left(\widehat{\theta}_i, \theta_{-i}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right] \\
& + \sum_{j=l+1}^{k}\left(D_j(\theta_i)\, E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > \widehat{\theta}_i > \theta_{(j)}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right]\right. \\
& \left. - E_{\theta_{-i}}\left[t_{j,i}\left(\widehat{\theta}_i, \theta_{-i}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right]\right).
\end{aligned}$$

Consider the case where the winning type of the previous period is known, i.e. $F_{l,i}(\theta_i, \theta_{-i}) = F_{l,i}(\theta_{-i})$. Since imitating another type cannot be profitable we have that $U_{l,i}(\theta_i) := U_{l,i}(\theta_i, \theta_i) = \max_{\widehat{\theta}_i} U_{l,i}\left(\theta_i, \widehat{\theta}_i\right)$ and therefore the Envelope-Theorem yields

$$\frac{dU_{l,i}(\theta_i)}{d\theta_i} = \sum_{j=l}^{k} \frac{dD_j(\theta_i)}{d\theta_i} E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > \theta_i > \theta_{(j)}\right) \mid F_{l,i}(\theta_{-i})\right]. \qquad (14)$$

Combining (13) and (14) shows that the ex-ante expected payment an agent has to make in period l, i.e. $E_\theta\left[t_{l,i}(\theta)\right] = E_\theta\left[E_{\theta_{-i}}\left[t_{l,i}(\theta_i,\theta_{-i}) \mid F_{l,i}(\theta_{-i})\right]\right]$, is the same for all sequential auctions (with announcement of winning types) if this is true for $E_\theta\left[t_{j,i}(\theta)\right]$, $j = l+1,\ldots,n$. By induction we can conclude that this indeed must be the case.

If no announcements are made, beliefs are updated by using the information that all winning types of previous periods are higher than the own type (i.e. $F_{l,i}$ only depends on her own type θ_i). In this case we have

$$E_{\theta_{-i}}\left[t_{l,i}\left(\widehat{\theta}_i,\theta_{-i}\right) \mid F_{l,i}(\theta_i,\theta_{-i})\right] = t_{l,i}\left(\widehat{\theta}_i\right)$$

and the Envelope-Theorem and the argumentation apply to this case as well.

Proof of Corollary 5:

The Corollary is an immediate consequence of the proof of Theorem 1 and Corollary 3. We have (expected) payoffs which are identical to a sequential auction (as analyzed in the proof of Theorem 1) in a period l up to a factor of $\prod_{i=1}^{l-1}\delta_i$ if we set $D_1(\theta) = \theta$ and $D_{l+1}(\theta) = \delta_l D_l(\theta)$. Therefore the analysis is the same as in Theorem 1.

Proof of Theorem 6:

From formula (3) we get $\overline{p}_l = \left(\delta^{l-1} - \delta^l\right) E[\theta_{(l+1)}] + \overline{p}_{l+1}$. Hence

$$\frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\overline{p}_l}{\overline{p}_{l-1}} \Leftrightarrow \frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\left(\delta^{l-1}-\delta^l\right)E[\theta_{(l+1)}]}{\left(\delta^{l-2}-\delta^{l-1}\right)E[\theta_{(l)}]} = \delta\frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}. \tag{15}$$

1. Because of (15) we have to show that for every distribution there exists $\delta \in (0,1)$ with $\frac{\overline{p}_{l+1}}{\overline{p}_l} > \delta\frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}$ for all $l \leq k$. Let m be defined by $m = \arg\max_{l\leq k}\frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}$. We know that $\frac{E[\theta_{(m+1)}]}{E[\theta_{(m)}]} < 1$. Since for $\delta \to 1$ we have $\frac{\overline{p}_{l+1}}{\overline{p}_l} \to 1$ for all l there exists a δ sufficiently close to 1 such that $\frac{\overline{p}_{l+1}}{\overline{p}_l} > \frac{E[\theta_{(m+1)}]}{E[\theta_{(m)}]}$ for all $l \leq k$.

2. We have $\frac{\overline{p}_k}{\overline{p}_{k-1}} \underset{(<)}{>} \frac{\overline{p}_{k-1}}{\overline{p}_{k-2}}$ because of $\frac{\overline{p}_k}{\overline{p}_{k-1}} = \delta\frac{E\left[\theta_{(k+1)}\right]}{(1-\delta)E\left[\theta_{(k)}\right]+\delta E\left[\theta_{(k+1)}\right]}$ the assumption and (15). Similarly to (15) we have that

$$\frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\overline{p}_l}{\overline{p}_{l-1}} \Leftrightarrow \frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \delta\frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}.$$

Therefore if $\frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\overline{p}_l}{\overline{p}_{l-1}}$ we have (from the assumptions) that $\frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \delta \frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]} \underset{(<)}{>} \delta \frac{E[\theta_{(l)}]}{E[\theta_{(l-1)}]}$ which implies that $\frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \frac{\overline{p}_{l-1}}{\overline{p}_{l-2}}$. The statement therefore follows by induction.

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