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BINARY EFFECTIVITY RULES

by

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Binary Effectivity Rules

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Abstract

A social choice rule is a collection of social choice correspondences, one for each agenda. An effectivity rule is a collection of effectivity functions, one for each agenda. We prove that every monotonic and superadditive effectivity rule is the effectivity rule of some social choice rule. A social choice rule is binary if it is rationalized by an acyclic binary relation. The foregoing result motivates our definition of a binary effectivity rule as the effectivity rule of some binary social choice rule. A binary social choice rule is regular if it satisfies unanimity, monotonicity, and independence of infeasible alternatives. A binary effectivity rule is regular if it is the effectivity rule of some regular binary social choice rule. We characterize completely the family of regular binary effectivity rules. Quite surprisingly, intrinsically defined von Neumann-Morgenstern solutions play an important role in this characterization.

Keywords: Social choice correspondences, effectivity functions, Nakamura's number, von Neumann-Morgenstern solutions.

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1 Introduction

The classical problem of social choice is the following: A society has to choose one or more alternatives out of a set of feasible alternatives. Our problem is to find a choice procedure that satisfies certain desirable properties so that it may be used by the society.

It is well known that every choice procedure introduces a power structure among coalitions of members of the society. Perhaps the most prominent example is the

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power structure induced by a social welfare function as a function of the restrictions on the social preference relation (see Section 3 of Sen (1986)). A method of detailed description of the induced power structure was introduced in Moulin and Peleg (1982), who used effectivity functions. Effectivity functions describe fully and precisely the power structure induced by a game form or a social choice correspondence. Abdou and Keiding (1991) deals with the theory and applications of effectivity functions. Many additional applications have been found since 1991. Thus, Peleg (2002) mentions some recent applications of effectivity functions.

Gärdenfors (1981) proposes a new framework for the application of game theory to social choice. His starting point is a new concept of constitution called "rights-system". The axioms imposed by Gärdenfors on a rights-system imply that it is, essentially, a monotonic and superadditive effectivity function. Gärdenfors' ideas have been used recently in Peleg (1998), Peleg, Peters and Storcken (2002), and Keiding and Peleg (2002).

It is well-known that every monotonic and superadditive effectivity function is the effectivity function of some game form (see, e.g., Peleg (1998) for a proof). We prove in Section 3 the following related result: Every monotonic and superadditive effectivity function is the effectivity function of some social choice correspondence. This result motivates our central definition in Section 5.

A social choice rule is a collection of social choice correspondences, one for each agenda (an agenda is a non-empty subset of the set of social alternatives). An effectivity rule is a collection of effectivity functions, one for each agenda. An effectivity rule is monotonic (superadditive) if every effectivity function is monotonic (superadditive). By the result in Section 3, every monotonic and superadditive rule is the effectivity rule of some social choice rule. This is the contents of Section 4.

A social choice rule is binary of it may be rationalized by an acyclic social binary relation (for each profile of preferences of the members of the society). A binary social choice rule is regular if it satisfies independence of infeasible alternatives, and each of its social choice correspondences is monotonic and satisfies unanimity. An effectivity rule is binary and regular if it is the effectivity rule of some regular binary social choice rule. Section 5 contains a complete characterization of regular binary effectivity rules.

2 Definitions and notations

Throughout this paper, A denotes the set of alternatives. A is assumed to be finite and to have at least three members.. A (linear) preference ordering on A is a complete, transitive, and antisymmetric binary relation. We denote by $L = L(A)$ the set of all linear orders on A. If S is a set, then $L^S = \{f \mid f : S \to L\}$ denotes the set of all maps from S to L.

Let B be a set. We denote by $P(B)$ the set of all subsets of B, that is $P(B) = \{B' \mid B' \subseteq B\}$. Also, $P_1(B) = P(B)\backslash\{\emptyset\}$ is the set of all non-empty subsets of B.

Let $N = \{1, \ldots, n\}$ be the set of players, $n \geq 2$. A *social choice correspondence* (SCC) is a function $H : L^N \to P_1(A)$. Let H be an SCC. H satisfies *non-imposition* (NI) if for every $x \in A$ there exists $R^N \in L^N$ such that $H(R^N) = \{x\}$. H satisfies *unanimity* if

$$[x \in A \text{ and } x\ R^i\ y \text{ for all } y \in A \text{ and } i \in N] \Rightarrow H(R^N) = \{x\}.$$

Let $R^N \in L^N$ and let $x \in A$. $R_1^N \in L^N$ is obtained from R^N by an improvement of the position of x if

(i) for all $a, b \in A \backslash \{x\}$ and all $i \in N$, $a\ R^i\ b \Leftrightarrow a\ R_1^i\ b$, and

(ii) for all $a \in A \backslash \{x\}$ and all $i \in N$, $x\ R^i\ a \Rightarrow x\ R_1^i\ a$.

An SCC $H : L^N \to P_1(A)$ is *monotonic* if it satisfies the following: If $R^N \in L^N$, $x \in H(R^N)$, and R_1^N is obtained from R^N by an improvement of the position of x, then $x \in H(R_1^N)$ and $H(R_1^N) \subseteq H(R^N)$.

Let $R \in L$ and $a \in A$. We denote $L(a, R) = \{b \in A \mid a\ R\ b\}$. An SCC H is *Maskin-monotonic* if

$$[a \in H(R^N),\ Q^N \in L^N, \text{ and } L(a, R^i) \subseteq L(a, Q^i) \text{ for all } i \in N] \Rightarrow a \in H(Q^N).$$

We shall now define a few basic properties of effectivity functions. Effectivity functions were introduced in Moulin and Peleg (1982). An *effectivity function* (EF) is a function $E : P(N) \to P(P_1(A))$ that satisfies the following conditions: (i) $E(N) = P_1(A)$, (ii) $E(\emptyset) = \emptyset$, and (iii) $A \in E(S)$ for all $S \in P_1(N)$. Let E be an EF. E is *superadditive* if it satisfies the following condition: If $S_i \in P_1(N)$, $B_i \in E(S_i)$, $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$ (in particular, $B_1 \cap B_2 \neq \emptyset$). E is *monotonic* if

$$[B \subseteq B^*,\ S \subseteq S^*, \text{ and } B \in E(S)] \Rightarrow B^* \in E(S^*).$$

As a general interpretation, $B \in E(S)$ means that the coalition S can enforce the social outcome to be in B. Thus, superadditivity and monotonicity are natural properties.

We shall now show how EFs are linked to SCCs (see, again Moulin and Peleg (1982)). Let H be an SCC that satisfies (NI), $S \in P_1(N)$, and $B \in P_1(A)$. S is *effective* for B if there exists $R^S \in L^S$ such that $H(R^S, Q^{N\backslash S}) \subseteq B$ for all $Q^{N\backslash S} \in L^{N\backslash S}$. The EF E^H which is associated with H is given by $E^H(\emptyset) = \emptyset$ and

$$E^H(S) = \{B \in P_1(A) \mid S \text{ is effective for } B\}$$

for $S \in P_1(N)$. As the reader may easily verify, E^H is superadditive and monotonic.

We recall now some properties of simple games. A *simple game* is a pair $G = (N, W)$, where N is a set of players and $W \subseteq P_1(N)$, $W \neq \emptyset$, is the set of winning coalitions. G is *monotonic* if $S \in W$ and $S \subseteq T \subseteq N$ imply $T \in W$. A simple game

G is *proper* if $S \in W \Rightarrow N \backslash S \notin W$, for all $S \in P_1(N)$. In the sequel, we deal only with monotonic and proper simple games. G is *weak* if

$$V = \bigcap\{S \mid S \in W\} \neq \emptyset.$$

V is the set of *vetoers* of G. If G is not weak, then the Nakamura number of G, $\nu(G)$ (see Nakamura (1979)), is given by

$$\nu(G) = \min\{|U| \mid U \subseteq W \text{ and } \cap \{S \mid S \in U\} = \emptyset\}.$$

(Here and in the sequel, if B is a finite set, then $|B|$ denotes the number of members of B.) If G is weak, then we define $\nu(G) = \infty$.

Let G be a simple game and let $B \subseteq A$, $|B| \geq 2$. Further, let $x, y \in B$, $x \neq y$, and let $R^N \in L^N$. x *dominates* y at R^N if there exists $S \in W$ such that $x\ R^i\ y$ for all $i \in S$. The *core* of G with respect to R^N, $C(G, B, R^N)$, is the set of undominated alternatives in B at R^N. If G is not weak, then $C(G, B, R^N) \neq \emptyset$ for all $R^N \in L^N$ if and only if $\nu(G) > |B|$ (see, e.g., Theorem 2.6.14 of Peleg (1984) for a proof). Obviously, if G is weak, then $C(G, B, R^N) \neq \emptyset$ for all $R^N \in L^N$ and $B \subseteq A$.

3 Representation of effectivity functions by social choice correspondences

In Section 2 we have associated with every SCC $H : L^N \to P_1(A)$ a superadditive and monotonic EF $E^H : P(N) \to P(P_1(A))$. In this section we shall be interested in the converse problem: Let $E : P(N) \to P(P_1(A))$ be a monotonic and superadditive EF; is it possible to find an SCC $H : L^N \to P_1(A)$ such that $E^H = E$? Quite surprisingly, the answer is positive.

Let E be an EF. An SCC H that satisfies $E^H = E$ is called a *representation* of E. It is well known that a superadditive and monotonic EF can be represented by a game form (see Peleg (1998) for the relevant definitions and the result). However, an SCC $H : L^N \to P_1(A)$ is a very special (multi-valued) game form: The set of strategies of each player is L, the set of linear preferences on A.

We are ready now to formulate the above-mentioned result.

THEOREM 3.1. *Let $E : P(N) \to P(P_1(A))$ be a monotonic and superadditive EF. Then there exists an SCC $H : L^N \to P_1(A)$ such that $E^H = E$. Moreover, we may assume that H is monotonic and satisfies unanimity.*

Our first step towards a proof of this result is to recall some results of Abdou and Keiding (1991). Let $E : P(N) \to P(P_1(A))$ be an EF and let $R^N \in L^N$. For $S \in P_1(N)$ and $B \subseteq A$ we say that $x \in A \backslash B$ is *uniformly dominated* by B via S at R^N if $B \in E(S)$ and $B\ R^S\ A \backslash B$ (that is $b\ R^i\ a$ for all $b \in B$, $a \in A \backslash B$, and $i \in S$). The *u-core* of E at R^N, written u-C(E, R^N), is the set of all alternatives that are not uniformly dominated at R^N (by some $B \in P_1(A)$ via some $S \in P_1(N)$). If E

is superadditive and monotonic, then u-C$(E, R^N) \neq \emptyset$ for all $R^N \in L^N$ (see Abdou and Keiding (1991), p.145). Also, as the reader may easily verify, u-C(E, R^N) is monotonic and satisfies unanimity.

We shall also need the notion of u-effectiveness: Let $S \in P_1(N)$ and $B \in P_1(A)$; then S is u-*effective* for B at R^N if there exists $T^S \in L^S$ such that for each $x \in A \backslash B$, there is $S' \in P_1(S)$ satisfying the conditions

(a) x is uniformly dominated via S' at $(T^S, R^{N \backslash S})$,

(b) $B \ R^{S'} \ x$.

The following result, which we shall use in the proof of Theorem 3.1, is proved in Abdou and Keiding (1991), p.148.

Lemma 3.2. *Let $E : P(N) \to P(P_1(A))$ be a monotonic and superadditive EF, and let $R^N \in L^N$. If $S \in P_1(N)$ is u-effective for $B \in P_1(A)$ at R^N, then $B \in E(S)$.*

We are now ready for the proof of Theorem 3.1.

Proof: Define an SCC $H : L^N \to P_1(A)$ by

$$H(R^N) = \text{u-C}(E, R^N),$$

for all $R^N \in L^N$. By the foregoing remarks, $H(R^N) \neq \emptyset$ for all $R^N \in L^N$ and H is monotonic and satsifies unanimily. It remains to prove that $E^H = E$.

Let $B \in P_1(A)$ and $S \in P_1(N)$. If $B \in E(S)$, then for any profile $T^S \in L^S$ such that $B \ T^S \ A \backslash B$, we obtain

$$H(T^S, R^{N \backslash S}) = \text{u-C}(E, (T^S, R^{N \backslash S})) \subseteq B$$

for every $R^{N \backslash S} \in L^{N \backslash S}$. Thus, $B \in E^H(S)$.

Conversely, if $B \in E^H(S)$, then there exists $T^S \in L^S$ such that u-C$(E, (T^S, R^{N \backslash S}))$ $\subseteq B$ for all $R^{N \backslash S} \in L^{N \backslash S}$. Choose $R_0^N \in L^N$ such that

$$B \ R_0^S \ A \backslash B \text{ and } A \backslash B \ R_0^{N \backslash S} \ B.$$

We obtain that each $x \in A \backslash B$ is uniformly dominated at $(T^S, R_0^{N \backslash S})$ via some coalition $S' \subseteq S$, and $B \ R_0^{S'} \ x$. Thus, S is u-effective for B at R_0^N. By Lemma 3.2, $B \in E(S)$. □

An EF may be considered as a constitution for the society N (see Gärdenfors (1981) and Peleg (1998)). If $S \in P_1(N)$ and $B \in P_1(A)$, then $B \in E(S)$ implies that the group S has the right to enforce the social state to be in B. A representation of E by an SCC H may help the members of N to exercise their rights simultaneously in the following sense. Let $S_i \in P_1(N)$, $B_i \in E(S_i)$, $i = 1, 2$, and let $S_1 \cap S_2 = \emptyset$. Then there exist profiles $R^{S_i} \in L^{S_i}$, $i = 1, 2$, such that

$$H(R^{S_i}, Q^{N \backslash S_i}) \subseteq B_i, \text{ for all } Q^{N \backslash S_i} \in L^{N \backslash S_i}, \ i = 1, 2.$$

Hence

$$H(R^{S_1}, R^{S_2}, Q^{N\setminus(S_1\cup S_2)}) \subseteq B_1 \cap B_2, \text{ for all } Q^{N\setminus(S_1\cup S_2)}.$$

Thus, S_1 and S_2 may exercise B_1 and B_2 respectively at the same time. Notice that the problem of congestion at public facilities does not arise in our framework. For example, the Center for the Study of Rationality of the Hebrew University of Jerusalem has only one copying machine. Let N be the set of the menbers of the Center and let A be the set of all *possible* social states. If $a \in A$, then a specifies at most one user of the copying machine at any point of the time grid.

Clearly, representations of an EF E by game forms also help in simultaneous exercising of rights, and they need not be supplemented by a tie-breaking rule. Also, they may be used to analyze the strategic behavior of the players in N, subject to the legal constraints of E, when their preferences are known (see, e.g., Peleg, Peters and Storcken (2002)). Multi-valued game forms, like SCCs, are inconvenient for analyzing strategic behavior. Thus, we still need to justify the relevance of Theorem 3.1. At this stage we only point out the following corollary.

Corollary 3.3. *An EF $E : P(N) \to P(P_1(A))$ is monotonic and superadditive if and only if it is the EF E^H of some SCC $H : L^N \to P_1(A)$ that is monotonic and satisfies unanimity.*

Corollary 3.3 will be generalized in the next section, and it will motivate the central definition of this paper in Section 5.

4 Effectivity rules

We now turn to the investigation of dynamic choice procedures, or social choice rules. A social choice rule defines a choice set for each agenda or subset of alternatives, given a profile of preferences on the set of all alternatives. We remark that social choice rules appear already in Arrow's basic model of social choice (see Sen (1986), p.1077).

Definition 4.1. *A social choice rule (SCR) is a function $K : P_1(A) \times L^N \to P_1(A)$ that satisfies $K(B, R^N) \subseteq B$ for all $B \in P_1(A)$ and $R^N \in L^N$.*

Thus, if $K : P_1(A) \times L^N \to P_1(A)$ is an SCR, then for every $B \in P_1(A)$, $K(B, \cdot) : L^N \to P_1(B)$ is an SCC. An SCR $K : P_1(A) \times L^N \to P_1(A)$ satisfies (NI) (respectively unanimity, monotonicity, Maskin monotonicity) of each SCC $K(B, \cdot)$, $B \in P_1(A)$, satisfies (NI) (respectively unanimity, monotonicity, Maskin monotonicity).

We shall give two examples:

Example 4.2. Let $G = (N, W)$ be a proper and monotonic simple game such that $\nu(G) > |A|$. Define an SCR K by

$$K(B, R^N) = \mathrm{C}(G, B, R^N), \text{ for all } B \in P_1(A) \text{ and } R^N \in L^N. \tag{4.1}$$

Then K satisfies unanimity and Maskin monotonicity.

Example 4.2 exhibits a well-behaved SCR for any number of alternatives. Notice, however, that if $|A| \geq n$, then G is weak.

Example 4.3. (The Wicksell SCR.) For each $B \in P_1(A)$ choose $s(B) \in B$, interpreted as the status quo when the agenda B is the set of available alternatives for N. Define

$$K(B, R^N) = \begin{cases} x & \text{if } x \in B \text{ and } x \; R^i \; b \text{ for all } b \in B \text{ and } i \in N, \\ s(B) & \text{otherwise.} \end{cases} \tag{4.2}$$

Then K satisfies unanimity and monotonicity.

Parallel to the introduction of a social choice rule as a family of social choice functions, indexed by subsets of A, we introduce indexed families of effectivity functions in order to get an effectiveness notion which corresponds to SCRs.

Definition 4.4. *An effectivity rule (ER) is a collection $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ of EFs, where $E(B, \cdot) : P(N) \to P(P_1(B))$.*

An ER $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ is monotonic (superadditive) if every EF $E(B, \cdot)$, $B \in P_1(A)$, is monotonic (superadditive). Let $K : P_1(A) \times L^N \to P_1(A)$ be an SCR. The corresponding *ER* D^K is given by $D^K = \{E^{K(B,\cdot)} \mid B \in P_1(A)\}$. For every K, D^K is superadditive and monotonic.

Example 4.2, continued. For $B \in P_1(A)$ define an EF $E(B, \cdot) : P(N) \to P(P_1(B))$ in the following way: $E(B, S) = P_1(B)$ if $S \in W$; $E(B, S) = \{B\}$ if $S \notin W$, $S \neq \emptyset$; and $E(B, \emptyset) = \emptyset$. Then $D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$, where K is given by (4.1).

We prodeed now to our second example.

Example 4.3, continued. For $B \in P_1(A)$ we compute $E(B, \cdot) = E^{K(B,\cdot)}$ (where K is given by (4.2)): $E(B, N) = P_1(B)$; $E(B, S) = \{B' \mid B' \subseteq B \text{ and } s(B) \in B'\}$ if $1 \leq |S| \leq n - 1$; and $E(B, \emptyset) = \emptyset$.

We conclude this section with the following remark.

Remark 4.5 An ER $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ is monotonic and superadditive if and only if it is the ER D^K of some SCR $K : P_1(A) \times L^N \to P_1(A)$ that is monotonic and satisfies unanimity.

5 Binary effectivity rules

In this section, we introduce the key notions of this paper, which are those of binariness, for social choice rules as well as for effectivity rules.

The basic idea behind the notion of binariness for a social choice rule is that (i) the alternatives chosen given any issue B and any profile R^N are those that are maximal for a suitable binary relation R (so that the social choice rule is composed of two operations, namely first a social welfare function assigning a binary relation and secondly the operation of finding maximal elements for this relation), and (ii) the binary relation is defined by the outcome of the social choice rule on subsets containing only two alternatives. This is made precise in the following subsection, where we introduce binary effectivity rules as the ERs associated with binary SCRs. In Subsection 5.2 we show how to move backwards from effectivity rules satisfying a suitable condition to binary SCRs. The correspondence between binary SCRs and binary ERs is then exploited in the final Subsection 5.3 which contains a characterization of binary effectivity rules.

5.1 Binary SCRs

Let $K : P_1(A) \times L^N \to P_1(A)$ be an SCR and let $R^N \in L^N$. We define a binary relation on A by

$$x \ R \ y \Leftrightarrow x \in K(\{x, y\}, R^N);$$

also, $x \ R \ x$ for all $x \in A$. The SCR is binary if it can be rationalized by R, such as described by the following definition.

Definition 5.1. *An SCR* $K : P_1(A) \times L^N \to P_1(A)$ *is* binary *if for all* $B \in P_1(A)$ *and* $R^N \in L^N$,

$$K(B, R^N) = \{x \in B \mid x \ R \ y \textit{ for all } y \in B\}.$$

Example 4.2, continued. If $G = (N, W)$ is a monotonic and proper simple game, and $\nu(G) > |A|$, then $\mathrm{C}(G, B, R^N)$ is a binary SCR.

Let again $K : P_1(A) \times L^N \to P_1(A)$ be an SCR and let $R^N \in L^N$. The asymmetric part of R, P, is given by

$$x \ P \ y \Leftrightarrow \{x\} = K(\{x, y\}, R^N).$$

If K is binary, then it satisfies *pairwise rejection* (PRJ), that is for every $B \in P_1(A)$,

$$[x \ P \ y \text{ and } x \in B] \Rightarrow y \notin K(B, R^N).$$

Thus, P must be acyclic. K satisfies also *reward for pairwise optimality* (RPO),

$$[x \ R \ y \text{ for all } y \in B \text{ and } x \in B] \Rightarrow x \in K(B, R^N).$$

Clearly, binariness is equivalent to (PRJ) and (RPO).

Motivated by Remark 4.5, we define now binary EFs.

Definition 5.2. *An ER* $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ *is* binary *if there exists a binary SCR* K *that satisfies (NI), such that* $D^K = D$.

Clearly, the ER of Example 4.2 is binary. Our task now is to find "nice" properties of binary ERs that characterize them. Unfortunately, Definition 5.2 is too broad and does not allow us to do this. Therefore we consider a subset of the set of all binary EFs.

Let $K : P_1(A) \times L^N \to P_1(A)$ be an SCR. K satisfies *independence of infeasible alternatives* (IIA) if for all $B \in P_1(A)$ and all $Q^N, R^N \in L^N$,

$$[x\ R^i\ y \Leftrightarrow x\ Q^i\ y,\ \text{for all } x, y \in B \text{ and } i \in N] \Rightarrow K(B, R^N) = K(B, Q^N).$$

Definition 5.3 *A binary SCR is* regular *if it satisfies unanimity, monotonicity, and (IIA). A binary ER is* regular *if it is the ER of some regular SCR.*

We make now the following observation.

Claim 5.4. *Let $K : P_1(A) \times L^N \to P_1(A)$ be a regular binary SCR, let $D = D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$, and let $R^N \in L^N$. If $B = \{x, y\} \subseteq A$, $\{x\} \in E(B, S)$, and $x\ R^i\ y$ for all $i \in S$, then $K(B, R^N) = \{x\}$.*

The proof of Claim 5.4 is straightforward. However, it enables us to prove our first necessary condition.

Theorem 5.5. *Let $K : P_1(A) \times L^N \to P_1(A)$ be a regular binary SCR, let $D = D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$, and let $\{x_1, \ldots, x_q\} \subseteq A$. If $\{x_i\} \in E(\{x_i, x_{i+1}\}, S_i)$, $i = 1, \ldots, q-1$, and $\{x_q\} \in E(\{x_1, x_q\}, S_q)$, then $\cap_{i=1}^q S_i \neq \emptyset$.*

The proof of Theorem 5.5 is similar to the proof of the second half of Theorem 2.6.14 in Peleg (1984). For the sake of completeness it is included here.

Proof of Theorem 5.5. Assume, on the contrary, that $\cap_{i=1}^q S_i = \emptyset$. We define now $R^N \in L^N$ in the following way. Let $i \in N$. Then there exists $k = k(i)$ such that $i \notin S_k$. We define $R^i \in L$ such that

$$x_{k+1}\ R^i\ x_{k+2}\ R^i\ \ldots\ R^i\ x_q\ R^i\ x_1\ R^i\ \ldots\ R^i\ x_k.$$

Now let $i \in S_j$, $1 \leq j \leq q$. If $j < q$, then $x_j\ R^i\ x_{j+1}$, and if $j = q$, then $x_q\ R^i\ x_1$. By Claim 5.4, $K(\{x_i, x_{i+1}\}, R^N) = \{x_i\}$, $i = 1, \ldots, q-1$, and $K(\{x_1, x_q\}, R^N) = \{x_q\}$. Thus, the strict social preference relation $P = P(R^N)$ is cyclic, which is the desired contradiction. □

The property of regular binary SCRs established in Theorem 5.5 can be seen as an extension to the present context (of SCRs and ERs) of the well-known condition of Nakamura (1979) for a social choice function to be a selection from the core of a simple game at the given profile. We shall refer to the condition as the *generalized condition of Nakamura:*

(GN) If $\{x_1, \ldots, x_q\} \subseteq A$, $\{x_i\} \in E(\{x_i, x_{i+1}\}, S_i)$, $i = 1, \ldots, q-1$, and $\{x_q\} \in E(\{x_1, x_q\}, S_q)$, then $\cap_{i=1}^q S_i \neq \emptyset$.

We shall now show that Theorem 5.5 is not valid without (IIA).

EXAMPLE 5.6. Let $A = \{a, b, c\}$ and let $N = \{1, 2, 3, 4, 5\}$. We now define a binary SCR K as follows. Clearly, we have only to define $K(\{x, y\}, R^N)$ for all $x, y \in A$ and $R^N \in L^N$. Let $K(\{x, y\}, R^N) = \{x\}$ if $R^i = (x, y, z)$ for all $i \in S$, where $S = S_1 = \{1, 2, 3\}$, or $S = S_2 = \{3, 4, 5\}$; also, $K(\{x, y\}, R^N) = \{x\}$ if $|\{i \mid x\ R^i\ y\}| \geq 4$; in all other cases $K(\{x, y\}, R^N) = \{x, y\}$.

Let $R^N \in L^N$ and let $P = P(R^N)$. As the reader may easily verify, P is acyclic. Thus, if we define

$$K(B, R^N) = \{x \in B \mid x\ R\ y \text{ for all } y \in B\},$$

where $R = R(R^N)$ is the social preference order, then we obtain a monotonic binary SCR that satisfies unanimity. However, $a \in E(\{a, b\}, S_1)$, $b \in E(\{b, c\}, S_2)$, and $c \in E(\{a, c\}, S_3)$, where $S_3 = \{1, 2, 4, 5\}$, $D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$, and $S_1 \cap S_2 \cap S_3 = \emptyset$.

5.2 Binary SCRs that are defined by binary ERs

In the preceding subsection, we introduced the ER associated with a binary SCR. It is now time to consider the converse construction. However, since binariness of ER was defined (see Definition 5.2) with reference to a binary SCR, we shall have to use another approach, to be described below.

Let $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ be a superadditive and monotonic ER. Moreover, assume that D satisfies the generalized condition of Nakamura: We define a regular binary SCR K from D in the following way. Let $R^N \in L^N$ be a profile. Clearly, to define the value of a binary SCR K at R^N it is enough to define the asymmetric part $P = P(R^N)$ of the binary relation underlying K. Therefore, let $x, y \in A$, $x \neq y$, and let $S = \{i \mid x\ R^i\ y\}$. Define

$$x\ P\ y \Leftrightarrow \{x\} \in E(\{x, y\}, S).$$

P is acyclic by (GN). For $a, b \in A$ define now: $a\ R\ b$ if not $b\ P\ a$. The desired SCR K is given by

$$K(B, R^N) = \{x \in B \mid x\ R\ y \text{ for all } y \in B\}.$$

As the reader may verify easily, K is a regular binary SCR. Note, however, that $D^K = \{\hat{E}(B, \cdot) \mid B \in P_1(A)\}$ may not coincide with D. Nevertheless, $E(B, \cdot) = \hat{E}(B, \cdot)$ for all $B \subseteq A$ such that $|B| = 2$.

Let $K : P_1(A) \times L^N \to P_1(A)$ be a regular binary SCR and let $D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$ be the corresponding ER. If K^* is the SCR which is defined by D^K, then as the reader may verify, $K = K^*$. Hence, if two regular binary SCRs have the same ER, then they coincide.

The approach described above enables us to construct all regular binary SCRs from superadditive and monotonic ERs.

If a regular binary SCR is neutral, that is, it is covariant under permutation of the alternatives, then it is the core of a (proper and monotonic) simple game (see Theorem 4.4.14 in Peleg (1984)).

It is obvious that (GN) is not sufficient to characterize regular binary EFs. A complete characterization of regular binary ERs will be given in the next subsection. It depends on some results on von Neumann-Morgenstern solutions of almost transitive binary relations that are presented in an appendix.

5.3 Complete characterization of regular binary ERs

We find first an additional neccessary condition which is satisfied by regular binary ERs. The new condition and the old (GN) are sufficient for a complete characterization.

Let $K : P_1(A) \times L^N \to P_1(A)$ be a regular binary SCR and let $D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$ be the associated regular binary ER. For each $S \in P_1(N)$ we define a binary relation $\succ_S$ on A as follows,

$$x \succ_S y \Leftrightarrow x \neq y \text{ and } \{x\} \in E(\{x, y\}, S).$$

$\succ_S^*$ will denote the almost transitive closure of $\succ_S$ (see Example A.2).

Example 5.7. Let $A = \{a, b, c\}$, let $N = \{1, 2, 3\}$, let $W = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\})$, let $G = (N, W)$, and let $K(B, R^N) = \mathrm{C}(G, B, R^N)$ for all $B \in P_1(A)$ and $R^N \in L^N$. Then $\succ_S = \{(x, y) \mid x \neq y,\ x, y \in A\}$ if $S \in W$, and $\succ_S = \emptyset$ otherwise.

We are now ready for the following theorem.

Theorem 5.8. *Let $K : P_1(A) \times L^N \to P_1(A)$ be a regular binary SCR, and let $D^K = \{E(B, \cdot) \mid B \in P_1(A)\}$ be the associated regular binary ER. Then for every $S \in P_1(N)$ and $B \in P_1(A)$*

$$E(B, S) = \{B' \in P_1(B) \mid B' \textit{ contains a vNM solution of } (B, \succ_S^*)\}.$$

Thus, the minimal (under inclusion) members of $E(B, S)$ are the vNM solutions of $(B, \succ_S^*)$.

Proof: The proof consists of several steps.

Step (a): Let $B \in P_1(A)$ and $S \in P_1(N)$. If U_0 is a vNM solution of $(B, \succ_S^*)$, then $U_0 \in E(B, S)$.

Let $\{U_0, U_1, \ldots, U_h\}$ be the partition of B associated with U_0 (see Remark A.4). Choose $R^S \in L^S$ such that $x\ R^i\ y$ for all $i \in S$, $x \in U_j$, and $y \in U_{j+1}$, $j = 0, 1, \ldots, h-1$. By (IIA), Claim 5.4, and (PRJ),

$$K(B, (R^S, Q^{N\setminus S})) \subseteq U_0, \text{ for all } Q^{N\setminus S} \in L^{N\setminus S}.$$

Thus, $U_0 \in E(B, S)$.

Step (b): $K(B, \cdot)$ is Maskin monotonic for every $B \in P_1(A)$.

Let $B \subseteq A$, $R^N \in L^N$, and $x \in K(B, R^N)$. If $Q^N \in L^N$ satisfies $[x\ R^i\ y \Rightarrow x\ Q^i\ y]$, for all $i \in N$ and $y \in B$, then, by monotonicity, $x\ R\ y \Rightarrow x\ Q\ y$ for all $y \in B$. (Here Q is the social preference induced by Q^N.) As $x \in K(B, R^N)$, $x\ R\ y$ for all $y \in B$. Hence, $x\ Q\ y$ for all $y \in B$. By (RPO), $x \in K(B, Q^N)$.

Step (c): If $B^* \in E(B, S)$ is minimal for inclusion (in $E(B, S)$), then B^* is a vNM solution of $(B, \succ_S^*)$.

Let $R^S \in L^S$ satisfy $K(B, (R^S, Q^{N \backslash S})) \subseteq B^*$, for all $Q^{N \backslash S} \in L^{N \backslash S}$. By Maskin monotonicity, we may assume that $B^*\ R^S\ B \backslash B^*$. We claim now that B^* is externally stable (with respect to $(B, \succ_S^*)$). Let $\mathrm{Dom}(B^*, \succ_S^*) = \{x \in B \mid \exists y \in B^* \text{ such that } y \succ_S^* x\}$. Assume, on the contrary, that $B_1 = B \backslash [B^* \cup \mathrm{Dom}(B^*, \succ_S^*)]$ is non-empty. Define a binary relation $\succ$ on B by

$$x \succ y \Leftrightarrow x \succ_T y \text{ and } x\ R^i\ y \text{ for } i \in T \text{ for some } \emptyset \neq T \subseteq S.$$

Clearly, $x \succ y \Rightarrow x \succ_S y$. Hence, if $y \in B_1$ and $x \in B^* \cup \mathrm{Dom}(B^*, \succ_S^*)$, then $x \not\succ y$. As K is binary, $\succ$ is acyclic. Let $\bar{x}$ be a $\succ$-maximal member of B_1. If $Q_0^{N \backslash S} \in L^{N \backslash S}$ satisfies $\bar{x}\ Q_0^i\ y$ for all $i \notin S$ and $y \in B$, then $\bar{x} \in K(B(R^S, Q_0^{N \backslash S}))$ by (RPO), which is the desired contradiction.

We conclude that $B^* \supseteq B^{**}$, where B^{**} is a vNM solution of $(B, \succ_S^*)$ (see the proof of Theorem A.3). By Step (a), $B^{**} \in E(B, S)$. Hence, by the minimality of B^*, $B^* = B^{**}$. □

Let $D = \{E(B, \cdot) \mid B \in P_1(A)\}$ be an ER. D satisfies the condition (called the vNM condition) of Theorem 5.8 if:

(vNM) For each $S \in P_1(N)$ and each $B \in P_1(A)$,

$$E(B, S) = \{B' \in P_1(B) \mid B' \text{ contains a vNM solution of } (B, \succ_S^*)\}$$

We summarize now the results of this section.

Corollary 5.9. *A monotonic and superadditive ER D is a regular binary ER if and only if it satisfies (GN) and (vNM).*

Proof: If D is a regular binary ER then it satisfies (GN) and (vNM) respectively by Theorems 5.5 and 5.8. If D satisfies (GN), then it defines a regular binary SCR K. By Theorem 5.8, $D = D^K$. □

6 Concluding remarks

We have shown first in this paper that every monotonic and superadditive EF can be obtained as the EF of some SCC. This result motivated our definiiton of binary ERs. A binary ER is the ER of some binary SCR. Binary ERs proved to be too difficult to characterize (see Example 5.6). Therefore we restricted our investigation to regular binary ERs. A regular binary ER is the ER of some regular binary

SCR, that is, a binary SCR that is monotonic and satisfies unanimity and (IIA). Regular binary ERs were characterized by two properties: (GN) and (vNM). (GN) is the generalization of Nakamura's condition for the non-emptiness of the core of a simple games to regular binary ERs (notice that a regular binary SCR that satisfies neutrality must be the core correspondence of some simple game, by Section 4.4 in Peleg (1984)). The second condition, (vNM), is more involved and it uses certain von Neumann-Morgenstern solutions in the characterization of regular binary ERs (see Subsection 5.3).

We relate now our analysis to (part of) the existing literature. Theorem 3.1 relies heavily on some results in Chapter 7 of Abdou and Keiding (1991). A survey of binary SCRs may be found in Sen (1986, pp. 1097 – 8). Our terminology is somewhat different from Sen's, and it is borrowed from Pattanaik (1978). The core of simple games is treated, for example, in Section 2.6 of Peleg (1984).

The results obtained may be used also as a characterization of social welfare functions, since the basic idea of a binary SCR is to select the maximal elements of a suitable binary relation depending on the profile. Previously, results on acyclic social welfare functions satisfying IIA, monotonicty, and neutrality (anonymity) were characterized by Nakamura (1979) (Moulin (1985)). Our results generalize both by leaving out both neutrality and anonymity.

It is worthwhile to mention that it is possible to use a less demanding definition of regularity of binary SCRs in Section 5. An SCR $K : P_1(A) \times L^N \to P_1(A)$ is monotonic (respectively satisfies unanimity, satisfies (IIA)) in the limited sense if every SCC $K(B, \cdot)$ with $|B| = 2$ is monotonic (respectively satisfies unanimity, satisfies (IIA)). A binary SCR is regular in the limited sense if it is monotonic in the limited sense and satisfies unanimity and (IIA) in the limited sense. As the reader may verify, if a binary SCR is regular in the limited sense, then it is regular. Thus, we could have used binary SCRs that are regular in the limited sense in Section 5 instead of regular binary SCRs.

7 Appendix. Von Neumann-Morgenstern solutions of almost transitive binary relations

Let B be a finite set and let $\succ$ be an irreflexive binary relation on B. The binary relation $\succ$ is *almost transitive* if for all $x, y, z \in B$:

$$[x \succ y, y \succ z, \text{ and } x \neq z] \Rightarrow x \succ z.$$

EXAMPLE A.1. Let $\succ^*$ be an irreflexive and transitive binary relation on B. Then $\succ^*$ is almost transitive.

EXAMPLE A.2. Let $\succ^*$ be an irreflexive binary relation on B. Define the *almost transitive closure* $\succ_1$ of $\succ^*$ as follows. Let $x, y \in B$, $x \neq y$. Then $x \succ_1 y$ if there exist distinct members of B, $z_0, z_1, \ldots, z_{q+1}$, $q \geq 0$, such that $z_0 = x$, $z_{q+1} = y$, and $z_i \succ^* z_{i+1}$ for $i = 0, 1, \ldots, q$. As the reader may verify easily, $\succ_1$ is almost transitive.

We recall now that $B_e \subseteq B$ is *externally stable* if for all $x \in B \backslash B_e$ there exists $y \in B_e$ such that $y \succ x$. $B_i \subseteq B$ is *internally stable* if there exists no pair $x, y \in B_i$ such that $x \succ y$. A set $B_0 \subseteq B$ is *stable* if it is both externally and internally stable. Stable sets are also called von Neumann-Morgenstern (vNM) solutions (of $(B, \succ)$).

The following theorem is used in subsection 5.3.

THEOREM A.3. *Let B be a finite set and let $\succ$ be an (irreflexive and) almost transitive binary relation on B. Then there exists a stable set $B_0 \subseteq B$.*

PROOF: B is externally stable. Therefore, there exists a minimal externally stable set B_0. If $x, y \in B_0$ and $x \succ y$, then $B_0 \backslash \{y\}$ is externally stable. Hence, B_0 must also be internally stable. □

We conclude with the following remark.

REMARK A.4. Let B be a finite set and let $\succ$ be an irreflexive binary relation on B. Let $\succ^*$ be the almost transitive closure of $\succ$, let $U = U_0$ be a stable set of $(B, \succ^*)$, and define

$$U_{k+1} = \{x \in B \backslash \cup_{i=0}^{k} U_i \mid \exists y \in U_k \text{ such that } y \succ x\},$$

$k = 0, 1, \ldots$. Then there exists $h \geq 0$ such that $U_h \neq \emptyset$, $U_k = \emptyset$, $k > h$, and $\{U_0, \ldots, U_h\}$ is a partition of B.

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