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 byD. GRANOT, H. HAMERS, J. KUIPERS and M. MASCHLER

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## CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

# CHINESE POSTMAN GAMES ON A CLASS OF EULERIAN GRAPHS^ 

D. Granot, H. Hamers, J. Kuipers, M. Maschler


#### Abstract

The extended Chinese postman (CP) enterprize is induced by a connected and undirected graph $G$. A server is located at some fixed vertex of $G$, to be referred to as the post office. Each player resides in a single edge, and each edge contains at most one player. Thus, some of the edges can be "public". Each edge has a cost and a prize attached to it. The players need some service, e.g., mail delivery, which requires the server to travel from the post office and visit all edges wherein players reside, before returning to the post office. The server collects the prize attached to an edge upon the first traversal of this edge, but the cost of an edge is incurred every time it is traversed. The cost of a cheapest tour for each coalition defines a CP cost game. The issue is how to allocate, among the players, the cost that the server incurs. We study the class of extended CP enterprizes which are induced by Eulerian graphs satisfying two properties: The 4 -cut property (Definition 4.4) and completeness (Definition 4.8). For this class we prove that the core, resp., the nucleolus when the core is not empty, are Cartesian products of the cores, resp., nucleoli of CP enterprizes whose graphs are simple cycles generated from $G$ by identifying therein the end points of each elementary path (Definition 4.3). Finally, for the class of extended complete Eulerian graphs having the 4 -cut property, we are able to test core membership in $O(n)$ time, and when the core is not empty, we show how to calculate the nucleolus in $O\left(n^{2}\right)$ time, $n$ being the number of players.


[^0]D. Granot: Faculty of Commerce and Business Administration, University of British Columbia,
Vancouver, B. C., Canada V6T 1Y2.
e-mail: daniel.granot@commerce.ubc.ca.
Tel: +1-604-8228432. Fax: +1-604-8229574.
H. Hamers: Department of Econometrics and Operations Research, Tilburg University,
P.O. Box 90153, Tilburg, The Netherlands.
e-mail: H.J.M.Hamers@uvt.nl.
Tel: +31-13-4662660. Fax: $+31-13-4663280$.
J. Kuipers Department of Mathematics, Faculty of General Sciences, Maastricht University,
P.O. Box 616, 6200 MD Maastricht, The Netherlands.
e-mail: kuipers@math.unimaas.nl.
Tel: $+31-43-3883499$. Fax: $+31-43-3884910$.
M. Maschler: Department of Mathematics and

Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Jerusalem 92261, Israel.
e-mail: maschler@vms.huji.ac.il.
Tel: +972-2-5660869. Fax:+972-2-5618019.

## 1. Introduction

In the Chinese postman problem (Kwan [1962], Edmonds and Johnson [1973]), defined on a weighted undirected connected graph $G$, one seeks a least weighted tour which starts at some vertex $v_{0}$, of $G$, traverses all the edges in $G$ at least once and returns to $v_{0}$.

Hamers et al. [1999] formulated and analyzed a cost allocation problem associated with the Chinese postman problem which can be described as follows. A server is located at some fixed vertex of a graph, $G$, to be referred to as the post office, and each edge of $G$ belongs to a different player. The players need some service, e.g., mail delivery, and the nature of this service requires the server to travel from the post office and visit all edges, before returning to the post office. The cost allocation problem associated with this delivery problem is concerned with a fair allocation of the cost of a cheapest Chinese postman tour in the graph. That is, the cost of a cheapest tour, which starts at the post office, visits each edge of $G$ at least once and returns to the post office. Following what is by now an established line of research, Hamers et al. [1999] formulated this cost allocation problem as a cooperative game, $(N ; c)$, referred to as the Chinese postman (CP) game, where $N$ is the set of players (edges) in the graph, and $c: 2^{N} \rightarrow \mathcal{R}$ is the characteristic function. For each $S \subseteq N, c(S)$ is the cost of a cheapest tour, which starts at the post office, visits each edge in $S$ at least once and returns to the post office. Solution concepts in cooperative game theory were then evaluated as possible cost allocation schemes for the above delivery problem. In particular, Hamers et al. studied the core of such games; namely, the set of all allocations having the property that no subset of players can achieve the same service at a strictly lower cost. A cooperative game whose core is not empty is said to be balanced, and if the core of any subgame of it is nonempty, it is said to be totally balanced.

In general, a CP game associated with an undirected connected graph could have an empty core. However, Hamers et al. [1999] have shown that a CP game induced by a connected weakly Eulerian graph is balanced. Here, a graph $G$ is called weakly Eulerian if it consists of Eulerian components connected in a tree-like structure. Further, Hamers [1997] has shown that if a connected undirected graph is weakly cyclic, that is, every edge therein is contained in at most one cycle, then the associated CP game is convex, or submodular.

Granot et al. [1999] were interested in properties of CP games which hold for all non negative edge costs and all locations of the post office. Define a graph to be Chinese Postman-convex, Chinese Postman-totally balanced or Chinese Postman-balanced (or, for short, CP convex, CP-totally balanced and CP-balanced), if the corresponding CP game is convex, totally balanced, or balanced, respectively, for all non negative edge costs and all locations of the post office. Granot et al. [1999] proved that an undirected graph is CP-convex if and only if it is CP-totally balanced, which holds if and only if it is weakly cyclic. An undirected graph was shown by them to be CP-balanced if and only if it is weakly Eulerian. In contrast with the undirected case, they proved that any connected directed graph is CP-balanced. Further, it was proven by Granot et al. [1999] that a CP
game induced by a directed graph is convex if and only if the directed graph is weakly cyclic. (In a directed weakly cyclic graph each arc is contained in exactly one circuit.)

In this paper we study an extended model of a CP-enterprize which allows for public edges; namely, edges not belonging to any of the players, and prizes. The prizes, which are associated with the edges, are collected only upon the first traversal of the edges. Inclusion of prizes appeared already in the travelling salesman problem literature (see, e.g., Balas [1989] and Bienstock et al. [1993]), and it is interesting to study it also in the context of the CP problem. They occur naturally in our games, because the Davis and Maschler's reduced game [1965], when applied to a standard CP game, is not necessarily a model of a standard CP enterprize, but rather, it is a model of an extended CP enterprize. The objective of this paper is to study the structure of the core and the nucleolus of extended CP games and to find efficient ways to compute them. The employment of the reduced game for this purpose is very useful, because both the core and the prenucleolus satisfy the reduced game property (see Theorem 3.10).

In the present paper we study the class of extended CP games induced by an extended CP enterprize whose graph is Eulerian, having the 4 -cut property. ${ }^{1}$ We further require that the enterprize is complete. ${ }^{2}$ We prove that for this class, the core is a Cartesian product of cores of extended CP enterprizes generated from the original enterprize, whose graphs are simple cycles. These cyclic enterprizes are derived by identifying the endpoints of the elementary paths ${ }^{3}$ of the original graph to form new post offices. The nucleolus has a similar property whenever the core is not empty.

There remains to discuss the core and the nucleolus of extended CP enterprizes whose graphs are simple cycles. For this class we show that the core is not empty if and only if its characteristic function is non-negative, which holds if and only if the game is convex. We prove that the core of a CP enterprize defined on a simple cycle graph is determined by at most $2 n+1$ non-redundant constraints, where $n$ is the number of players, and that core membership can be tested in $O(n)$ time. Further, we develop an $O\left(n^{2}\right)$ algorithm which calculates the nucleolus of a CP enterprize defined on a simple cycle graph whose core is not empty. Thus, for our class of extended CP games, one can test core membership in linear time and if the core is not empty, the nucleolus can be computed in quadratic time.

## 2. Preliminary and Notation

We present in this section some elementary definitions in game theory and graph theory which are needed for subsequent analysis. A (cost) cooperative game is a pair $\Gamma=(N ; c)$, where $N$ is a finite set of players, $c$ is a function, and $c: 2^{N} \rightarrow \mathcal{R}$. In our applications, we will not require that $c(\emptyset)=0$ although this will be the case for games with a nonempty

[^1]core. A subset of $N$ will sometimes be referred to as a coalition. A function $c: 2^{N} \rightarrow \mathcal{R}$ is said to be submodular if
\[

$$
\begin{equation*}
c(T \cup\{j\})-c(T) \leq c(S \cup\{j\})-c(S) \tag{2.1}
\end{equation*}
$$

\]

for all $j \in N$ with $S \subset T \subseteq N \backslash\{j\}$. A game ( $N ; c$ ) is convex, or submodular, if $c: 2^{N} \rightarrow \mathcal{R}$ is submodular (Shapley 1971).

An allocation $x=\left(x_{i}\right)_{i \in N} \in \mathcal{R}^{N}$ is a core-element of $(N ; c)$ if $\sum_{i \in N} x_{i}=c(N)$ and $\sum_{i \in S} x_{i} \leq c(S)$ for all $S \in 2^{N}$. These inequalities will be referred to as core constraints. The core of a game, $\mathcal{C}(N ; c)$, consists of all core-elements. A game is called balanced if its core is non-empty and it is totally balanced if for each $S \subseteq N, S \neq \emptyset,\left(S ; c_{S}\right)$ is balanced, where $c_{S}$ is the restriction of $c$ to the family of subsets of $S$.

The set of all pre-imputations (resp., imputations) of $\Gamma$ is denoted by $X^{\star}(\Gamma)$ (resp., $X(\Gamma)$ ). Thus, $X^{\star}(\Gamma)=\left\{x: \sum_{i=1}^{n} x_{i}=c(N)\right\}$ and $X(\Gamma)=\left\{x: \sum_{i=1}^{n} x_{i}=c(N), x_{i} \leq c(\{i\}), i=\right.$ $1, \ldots, n\}$. For a game $\Gamma=(N ; c)$ and $x \in \mathcal{R}^{N}$, let $\theta(x ; \Gamma)$ be the $\left|2^{N}\right|$-dimensional vector whose components are the excesses, $c(S)-x(S)$, for $S \in 2^{N}$, arranged in a nondecreasing order, where ${ }^{4} x(S) \equiv \sum_{k \in S} x_{k}$. Let $\succeq$ denote the lexicographically greater than relationship between vectors of the same dimension, and let $X_{0} \subseteq \mathcal{R}^{N}$. The nucleolus of $\Gamma$ with respect to $X_{0}$ is given by,

$$
\begin{equation*}
\nu\left(\Gamma, X_{0}\right)=\left\{x \in X_{0}: \theta(x ; \Gamma) \succeq \theta(y ; \Gamma), \text { for all } y \in X_{0}\right\} \tag{2.2}
\end{equation*}
$$

If $X_{0}=X^{\star}(\Gamma), \nu\left(\Gamma, X_{0}\right)$ is called the prenucleolus of $\Gamma$, and if $X_{0}=X(\Gamma), \nu\left(\Gamma, X_{0}\right)$ is called the nucleolus of $\Gamma$.

Schmeidler [1969] introduced the nucleolus and proved that if $X_{0}$ is nonempty and compact then $\nu\left(\Gamma, X_{0}\right) \neq \emptyset$, and if, furthermore, $X_{0}$ is a convex set, then $\nu\left(\Gamma, X_{0}\right)$ consists of a single point. Similarly, if $X_{0}$ is nonempty, closed and convex then the prenucleolus also consists of a unique point.

Let $G=(V(G), E(G))$ be an undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. A walk in $G$ is a finite sequence of vertices and edges of the form $v_{1}, e_{1}, v_{2}, \ldots, e_{k}, v_{k+1}$ with $k \geq 1, v_{1}, \ldots, v_{k+1} \in V(G), e_{1}, \ldots, e_{k} \in$ $E(G)$, such that $e_{j}=\left\{v_{j}, v_{j+1}\right\}$ for all $j \in\{1, \ldots, k\}$. Such a walk is said to be closed if $v_{1}=v_{k+1}$. The vertices $v_{1}$ and $v_{k+1}$ are called the extreme vertices of the walk. We will also refer to closed walks as tours. ${ }^{5}$

A path in $G$ is a walk in which all edges are distinct (but vertices may coincide). A closed path, or cycle is a path in which $v_{1}=v_{k}$. We use the terms simple path and simple cycle to indicate paths and cycles in which the vertices are distinct, except for the extreme vertices in the case of a cycle. Finally, we refer to a graph induced by a simple path as a chain.

[^2]
## 3. The Model

The extended Chinese postman enterprize is given by $\Gamma=\left(G, v_{0}, a, p, N\right)$, where $G$ is a connected undirected graph, containing a special vertex $v_{0}$ called post-office, a nonnegative cost function $a(\cdot)$ defined on the edges of $G$, a nonnegative prize function $p(\cdot)$, defined on the edges of $G$, a set of players $N$, each residing and occupying an edge of $G$, so that each edge contains at most one player. Edges without players residing in them are called public edges. The postman travels along a tour, starting at the post office and ending there. While making the tours the postman pays the cost of the tour which is the sum of the costs of the edges he traverses minus the sum of the prizes encountered. However, there is only a one-time prize per edge, so if during the tour he traverses an edge, say twice, he pays twice the cost of that edge and collects the prize just once. The players are expected to reimburse the postman for his expenses, and we are concerned with various allocations of his expenses among the players.

For each Chinese postman enterprize $\Gamma=\left(G, v_{0}, a, p, N\right)$ we associate a cost game ( $N ; c$ ), where $N$ is the set of players and the cost function $c: 2^{N} \rightarrow \mathcal{R}$, is defined by

$$
\begin{equation*}
c(S)=\text { the cost of a least expensive tour that serves all members }{ }^{6} \text { of } S . \tag{3.1}
\end{equation*}
$$

We will refer to $(N ; c)$ as the Chinese postman (CP) game. We will use the notation $\left(N ; c_{\Gamma}\right)$ if there is a need to distinguish between various CP cost games.

Note that if prizes are large enough it may well be that $c(\emptyset)<0$, because it would benefit the postman to serve the empty coalition by traversing a non-empty tour.

The following lemma follows directly from the definition.

Lemma 3.1. The Chinese postman game is monotonic; namely, if $S \subseteq T$ then $c(S) \leq$ $c(T)$.

Proof. The coalition $S$ can use the cheapest tour that serves all members of $T$. The cheapest tour for $S$ can only be cheaper or equally as expensive.

Corollary 3.2. The condition $c(S) \geq 0$, for all $S$, is equivalent to the condition $c(\emptyset)=0$.

The following lemma is a direct consequence of the previous one:

[^3]Lemma 3.3. Any core element of a CP game is non-negative.

Proof. Let $x$ be a core element. Then, for each player $i, x(N \backslash\{i\}) \leq c(N \backslash\{i\})$. However, $c(N)=x(N)=x_{i}+x(N \backslash\{i\}) \leq x_{i}+c(N \backslash\{i\})$. Thus, by Lemma 3.1, $x_{i} \geq 0$.

In this paper we assume that the graph $G$ associated with an extended Chinese postman enterprize is Eulerian; namely, the degree (valency) of each vertex is even. One reason for this restriction is the following theorem due to Hamers et al. [1999].

Theorem 3.4. If the graph of a Chinese postman enterprize is Eulerian and if there are no public edges and no prizes then the core is not empty.

This theorem ceases to be true if public edges and/or prizes are introduced, as the following examples show.

Example 3.5. $G$ is an "onion" with four edges. Each edge is occupied by a single player. One edge has a prize equal to 2 and the cost of each edge is 1 (Figure 1). Here, $c(\{1\})=$ $c(\{2\})=c(\{3\})=c(\{4\})=0$, and $c(\{1,2,3,4\})=2$, which implies that the core is empty.

Example 3.6. $G$ is an "onion" with a single public edge whose cost is equal to 1 . The other edges are occupied by players $1,2,3$ and each costs 2 (Figure 2). We have $c(\{1,2,3\})=7$, whereas $c(\{1,2\})=c(\{2,3\})=c(\{1,3\})=4$ and the core is empty.

An important tool for the study of the core and the nucleolus of a cost game $(N ; c)$ is the reduced game $\left(S ; \hat{c}_{S}\right)$ on $S$ at $x$, which was first studied by Davis and Maschler [1965]. Here $S$ is a non-empty subset of $N$ and $\hat{c}_{S}(R)$, for $x$ in the core, is defined to be:

$$
\begin{equation*}
\hat{c}_{S}(R)=\min _{Q \subseteq S^{c}}[c(R \cup Q)-x(Q)], \quad \text { for all } R \subseteq S \tag{3.2}
\end{equation*}
$$

Here, $S^{c}:=N \backslash S$. We employ the following notation: Let $G=(V(G), E(G))$ be a graph with costs, $a(e)$, and prizes, $p(e), e \in E(G)$, and let $Q=(V(Q), E(Q))$ be a "multisubgraph" of $G$, where $V(Q) \subseteq V(G)$, and where $E(Q)$ may contain several copies ${ }^{7}$ of the same edge in $E(G)$. We denote by $k(Q)$ the cost associated with $Q$, i.e., $k(Q)=$ $\sum_{e \in E(Q)} a(e)-\sum_{e \in E^{1}(Q)} p(e)$, where $E^{1}(Q)$ is the set of all distinct edges in $E(Q)$.

[^4]

Fig. 1. CP with prizes and empty core


Fig. 2. CP with public edges
and empty core

Theorem 3.7. If $(N ; c)$ is a CP game corresponding to an extended Chinese postman enterprize $\Gamma=\left(G, v_{0}, a, p, N\right)$, and $x \in \mathcal{C}\left(N ; c_{\Gamma}\right)$, then the reduced game $\left(S ; \hat{c}_{S}\right)$ corresponds to the enterprize $\hat{\Gamma}=\left(G, v_{0}, a, p+\hat{x}, S\right)$, where $\hat{x}_{i}=0$ if $i \in S$, and $\hat{x}_{i}=x_{i}$ if $i \notin S$. Thus, $\hat{\Gamma}$ is obtained from $\Gamma$ by making the edges, which were previously occupied by members of $N \backslash S$, public, and the prize of each such edge $i$ increases by $x_{i}$.

Proof. Denote by $(S ; \tilde{c})$ the game that corresponds to the enterprize $\hat{\Gamma}$. We have to show that $\tilde{c}(R)=\hat{c}(R)$ for every coalition $R, R \subseteq S$. Let $R$ be a subset of $S$. Let

$$
\begin{equation*}
\hat{c}_{S}(R)=c\left(R \cup Q_{0}\right)-x\left(Q_{0}\right), \quad Q_{0} \subseteq S^{c} \tag{3.3}
\end{equation*}
$$

Here, $Q_{0}$ is a set in $N \backslash S$ for which the minimum in (3.2) is achieved. Take a tour, $\hat{w}$, whose cost in $\Gamma$ is equal to $c\left(R \cup Q_{0}\right)$. In view of Lemma 3.3, and the minimum requirement in (3.1), we can assume that $Q_{0}$ is the set of all the players that reside in edges traversed by $\hat{w}$ and belong to $N \backslash S$. The tour $\hat{w}$ is a valid tour also in $\hat{\Gamma}$ and its cost there is equal to $\hat{c}_{S}(R)$, because the prize collected in each $j$ in $N \backslash S$ is $x_{j}$-higher in $\hat{\Gamma}$. Thus, $\tilde{c}(R) \leq \hat{c}(R)$ for all $R \subseteq S$. Conversely, let $\tilde{w}$ be a tour whose cost in $\hat{\Gamma}$ is equal to $\tilde{c}(R)$. Then, it is a closed walk in $\hat{\Gamma}$ and its cost is equal to $k(\tilde{w})-x(Q)$, where $Q$ are the players that reside in edges traversed by $\tilde{w}$ and are in $S^{c}$ and $k(\tilde{w})$ is the cost of $\tilde{w}$ in $\Gamma . \tilde{w}$ is a valid tour also in $\Gamma$ for the coalition $R$, and its cost there is higher by $x(Q)$; consequently, it is a candidate for the minimum problem in (3.2) and so, $\tilde{c}(R) \geq \hat{c}(R)$. It follows that $\tilde{c}(R)=\hat{c}_{S}(R)$.

We call $\hat{\Gamma}$ of the last theorem the reduced enterprize of $\Gamma$ on $S$, at $x$. When needed, we denote it as $\hat{\Gamma}_{S}^{x}$.

In view of the above discussion we have:

Corollary 3.8. The extended class of Chinese postman enterprizes is closed under the reduced enterprize operation, provided that it is taken at a core vector $x$.

Remark 3.9. If $x$ is not a core point, the reduced game is somewhat more complicated. It still corresponds to a certain Chinese postman enterprize, but this enterprize need not belong to the class treated in this paper.

We close this section by reminding some well known facts concerning the core and the nucleolus of a game (see, e.g., Maschler et al [1979]). For the literature concerning these results see Maschler [1992].

Theorem 3.10. The core and the prenucleolus of a cooperative game satisfy the reduced game property; namely, if $x$ is a core/prenucleolus point for a game ( $N ; c$ ) then its restriction $\left.x\right|_{S}$ to a nonempty subset $S$ of $N$ belongs to the core/prenucleolus of the reduced game on $S$ at $x$. Moreover, the prenucleolus coincides with the nucleolus if the core is not empty, because it is a core point and therefore an imputation.

## 4. Chinese Postman on Eulerian Graphs Having the 4 -cut Property

In this section we study the class of extended (defined in Section 3) and complete (Definition 4.8) CP enterprizes induced by Eulerian graphs having the 4 -cut property (Definition 4.4). We show that the core and the nucleolus of such CP games is the Cartesian product of the cores and the nucleoli of extended CP games, whose graphs are simple cycles that are generated from the original graph. In the next section we show that the core of an extended CP game whose graph is a simple cycle has only a linear number of non redundant core constraints and we develop therein an algorithm which computes the nucleolus of such games in quadratic time. ${ }^{8}$ Thus, results we obtain in this section, coupled with those obtained in the next section, imply that for an extended CP game belonging to our class, verifying whether a given vector is in the core can be done in linear time and computing the nucleolus, if the core is not empty, can be done in quadratic time.

Theorem 4.1. Let $(N ; c)$ be the game corresponding to an extended Chinese postman enterprize $\Gamma=\left(G, v_{0}, a, p, N\right)$ in which $G$ is a simple cycle. The following statements are equivalent:
(1) $c(S) \geq 0$ for all $S \subseteq N$.
(2) $\mathcal{C}(N ; c) \neq \emptyset$.
(3) $(N ; c)$ is a convex game.

Proof. By Shapley [1971], (3) implies (2). Further, (2) implies (1) since, if for some $S$, $c(S)<0$, then for all $x \in \mathcal{C}(N ; c), x_{j}<0$ for some $j \in S$, contradicting Lemma 3.3. It remains to show that (1) implies (3). It is known (Shapley [1971]) that convexity is equivalent to the condition: $c(T \cup\{i\})-c(T) \leq c(S \cup\{i\})-c(S)$ whenever $S \subseteq T$ and

[^5]$i \notin T$, provided that $c(\emptyset)=0$. By Corollary 3.2, (1) is equivalent to $c(\emptyset)=0$. Let $S \subset T$ and $i \notin T$. Denote by $V(Q)$ a maximal, under inclusion, vertex set of an optimal Chinese postman (CP) tour for $Q$. Since we are dealing with a cycle graph, $V(Q)$ is either the whole cycle, or a connected part of it, in which every edge in the optimal path is traversed twice. In both cases, $V(S) \subseteq V(T)$.

Now, if $c(T \cup\{i\})=c(T)$, then, $0=c(T \cup\{i\})-c(T) \leq c(S \cup\{i\})-c(S)$, where the inequality follows from the monotonicity of $(N ; c)$ (Lemma 3.1). So, assume that $c(T \cup\{i\}) \neq c(T)$. If $c(S \cup\{i\}) \neq c(N)$, denote by $C_{1}$ the closed walk in the cycle graph, whose addition to the CP tour corresponding to $S$ would result with the CP tour corresponding to $S \cup\{i\}$ (see Figure 3).

Let $C_{1}^{\prime}$ be the closed walk which is contained in $C_{1}$, but is not contained in the tour corresponding to $c(T)$ (see Figure 4). Clearly, the cost $k\left(C_{1}^{\prime}\right)$ of $C_{1}^{\prime}$, is smaller than or equal to the cost $k\left(C_{1}\right)$, of $C_{1}$. Otherwise, $k\left(C_{1} \backslash C_{1}^{\prime}\right)$ is strictly negative, implying that the tour corresponding to $S$ was not optimal. Now, $c(T \cup\{i\})-c(T) \leq k\left(C_{1}^{\prime}\right) \leq k\left(C_{1}\right)=$ $c(S \cup\{i\})-c(S)$.

Finally, if $c(S \cup\{i\})=c(N)$, then $c(T \cup\{i\})=c(N)$, and $c(T \cup\{i\})-c(T)=c(N)-c(T)=$ $c(S \cup\{i\})-c(T) \leq c(S \cup\{i\})-c(S)$, where the last inequality follows by monotonicity.


Fig. 3. Tours for $S$ and $S \cup\{i\}$


Fig. 4. Tours for $T$ and $T \cup\{i\}$

We need the following notation:

Notation 4.2. Let $H=(V(H), E(H))$ be a subgraph of $G=(V, E)$. We denote:

$$
\begin{equation*}
G \backslash H:=(V \backslash V(H), E \backslash E(H)) \tag{4.1}
\end{equation*}
$$

$\overline{G \backslash H}:=$ the subgraph of $G$, whose edge set is $E \backslash E(H)$.
Thus, $\overline{G \backslash H}$ is the closure of $G \backslash H$. For $T=(V(T), E(T))$, we also denote

$$
\begin{equation*}
H \cup T=(V(H) \cup V(T), E(H) \cup E(T)) \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
H \cap T=(V(H) \cap V(T), E(H) \cap E(T)) \tag{4.4}
\end{equation*}
$$

The concept of an elementary path is used extensively in this section.

Definition 4.3. A path in a graph $G$ whose interior vertices have degree 2 and is maximal under this property is called an elementary path.

Definition 4.4. A connected graph $G$ is said to have the 4-cut property, if the cardinality of every edge-cut set that intersects each elementary path at most once is at least 4. (Thus, by the max flow min cut theorem (Ford and Fulkerson [1962]), between every pair of extreme vertices there exist at least four edge disjoint paths.)

Lemma 4.5. Let $G$ be a connected graph having the 4 -cut property. Let $P$ be an elementary path in $G$. There exists two cycles in $G$ whose intersection is $P$.

Proof. If $P$ is a simple cycle, we take $P$ itself for both cycles. Suppose $P$ is not a simple cycle. Then, since $G$ is connected and has the 4 -cut property, there exists at least three edge disjoint paths between the two extreme vertices of $P$ which do not use edges in $P$. Two of these three paths, together with $P$, form the two requires cycles.

Lemma 4.6. Let $G$ be a connected Eulerian graph having the 4-cut property. Denote by $v_{0}$ one of its vertices which is an extreme vertex of an elementary path in $G$. Let $P$ be an elementary path in $G$. There exist two cycles in $G$ which traverse $v_{0}$ and whose intersection is $P$.

Proof. The proof is by induction on the elementary-path distance $r$ of $P$ from $v_{0}$; namely the minimal number $r$ of elementary paths needed to connect $v_{0}$ with $P$. The case $r=0$, was proved in Lemma 4.5. Suppose the current lemma is true for elementary paths whose distance from $v_{0}$ is $r-1$. Let $P=[B C]$ be an elementary path whose distance from $v_{0}$ is $r, r \geq 1$. Denote by $Q=[A B]$ the preceding elementary path on the path connecting $v_{0}$ to $P$, whose distance from $B$ to $v_{0}$ is $r-1$. By the induction hypothesis, there exist two cycles in $G$, which traverse $v_{0}$ and whose intersection is $Q=[A B]$. Thus, there exist four edge-disjoint paths $\alpha, \beta, \gamma$ and $\delta$, connecting $v_{0}$ to the extreme vertices of $Q$ - two to each extreme vertex. Suppose $P$ is not a simple cycle. By Lemma 4.5, there exist two edge-disjoint paths $\eta$ and $\zeta$, connecting $B$ and $C$. Figures 5 to 9 show these paths, which are drawn only until they first vertex-intersect one of the other four paths, and cover all the various possibilities. In all cases we can identify a cycle which contains $P$ and traverses $v_{0}$. The cycle is denoted by a heavy line and, as it will turn out, it is the first of the two cycles required by the lemma. Removing this cycle, we obtain a possibly disconnected graph, in which $B, C$ and $v_{0}$ belong to the same component. Indeed, the remaining paths
described above exhibit the connection to the origin in all the cases. This component is Eulerian; therefore, there exists a cycle $C_{1}$ in this component which traverses $C, B$ and $v_{0}$. Indeed, since the component is Eulerian, $C_{1}$ could be the tour which traverses all the edges therein precisely once. Now $C_{1}$ contains at least two paths, $a_{1}$ and $a_{2}$, between $B$ and $C$. If, say, $a_{1}$ does not traverse $v_{0}$, then upon replacing $a_{1}$ with $P$ in $C_{1}$, we derive the second cycle which contains $P$ and traverses $v_{0}$, as required. If both $a_{1}$ and $a_{2}$ traverse $v_{0}$, then replacing any one of them with $P$ in $C_{1}$ would produce the second cycle as required. If $P$ is a simple cycle, we do not need $\eta$ and $\zeta$ and the relevant figure is Figure 10. One cycle is composed of $\alpha, \beta$, the elementary path $[\mathrm{AB}]$ and $P$. Removing it, ${ }^{9}$ we are left with a Eulerian graph with a connected component $C_{1}$, containing $B$ and $v_{0}$. This component, together with $P$ is the second required cycle. -


Fig. 5. Paths $\eta$ and $\zeta$ initially intersect path $\beta$


Fig. 7. Paths $\eta$ and $\zeta$ initially intersect paths $\beta$ and $\delta$


Fig. 6. Paths $\eta$ and $\zeta$ initially intersect paths $\beta$ and $\alpha$


Fig. 8. Paths $\eta$ and $\zeta$ initially intersect paths $\alpha$ and $\gamma$

[^6]

Fig. 9. Paths $\eta$ and $\zeta$ initially intersect path $\gamma$


Fig. 10. $P$ is a simple cycle.

For a subgraph $H$ of $G$ we denote by $N(H)$ the set of all players that reside in edges of $H$.

Remark 4.7. Obviously, Lemma 4.6 is incorrect when $v_{0}$ is an interior point of an elementary path, different from $P$.

Consider an extended CP enterprize $\Gamma=\left(G, v_{0}, a, p, N\right)$. For each elementary path $P$ in $G$ which contains players in some of its edges, we define an extended CP enterprize $\Gamma_{P}:=\left(G_{P}, v_{P}, a_{\mid P}, p_{\mid P}, N(P)\right)$, where $G_{P}$ is a simple cycle whose edges, their costs and prizes are in 1-1 correspondence with those in $G$ and are, respectively, occupied by the players in $P$. Here, the endpoints $\hat{v}_{P}$ and $\hat{w}_{P}$ of $P$ correspond to a single post office $v_{P}$ of $G_{P}$. Thus, $\Gamma_{P}$ is obtained from $P$ by identifying its end points to become the post office of $\Gamma_{P}$.

Definition 4.8. A CP enterprize $\Gamma=\left(G, v_{0}, a, p, N\right)$ is called complete, if $^{10}$
(1) The post office is an extreme vertex of an elementary path.
(2) Every elementary path $P$ for which $k(P)>0$ has at least one resident.
(3) For each $N(P)$ there exists an optimal tour in $G_{P}$, which traverses all the edges of $G_{P}$.
A CP game induced by an extended and complete CP enterprize will also be called complete.

Lemma 4.9. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be an extended complete $C P$ enterprize induced by $a$ Eulerian graph having the 4-cut property. Under these conditions, $c(N)=k(G)$.

Proof. Since $G$ is Eulerian, there exists a path that traverses all edges precisely once. Therefore, $c(N) \leq k(G)$. It remains to show that all other $N$-tours do not cost strictly less. Let $\tau$ be any optimal $N$-tour. By (2), it must contain at least one edge in each elementary path of positive cost. It may also intersect other elementary paths. Let $P$ be

[^7]an elementary path with $A$ and $B$ being its extreme points, which $\tau$ intersects. Denote by $\tau_{P}$ the multi-subgraph of the multigraph $\tau$ whose edges lie in $P$. It need not be connected; however, one can enter $P$ only via $A$, or $B$, or both. So, identifying $A$ and $B$ to get $G_{P}, \tau_{P}$ becomes an $N(P)$-tour for all the players in $G_{P}$. Thus, ${ }^{11}$ by (3), $k\left(\tau_{P}\right) \geq k(P)$. Consequently, $k(\tau)=\sum\left\{k\left(\tau_{P}\right): \tau\right.$ intersects $\left.P\right\} \geq \sum\{k(P): \tau$ intersects P$\}=k(G)$. The last equality follows from the fact that paths that $\tau$ does not intersect have zero cost.

Theorem 4.10. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be a complete extended CP enterprize in which $G$ is a connected Eulerian graph having the 4-cut property. Let $(N ; c)$ be the CP game induced by $\Gamma$. Then, for every elementary path $P$ in $G$ and every core vector $x, x(N(P))=k(P)$.

Proof ${ }^{12}$. Denote by $A$ and $B$ the extreme vertices of $P$. By Lemma 4.6, there exists two paths $\bar{\alpha}$ and $\bar{\beta}$ joining $A$ with $v_{0}$ and two paths $\bar{\gamma}$ and $\bar{\delta}$ joining $B$ with $v_{0}$, such that all these paths are edge-disjoint and neither of them contains $P$. These paths, together with $P$ are drawn in heavy lines in Figure 11. The graph $G \backslash(\bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma} \cup \bar{\delta})$ is Eulerian and may be composed of several connected components. Call those components, that are not connected to either $v_{0}$, or $A$ or $B, \eta$-components. Denote by $\alpha$ the union of $\bar{\alpha}$ with those $\eta$-components that have a vertex in common with $\bar{\alpha}$. Denote by $\beta$ the union of $\bar{\beta}$ with those $\eta$-components that have a vertex in common with $\bar{\beta}$ and do not intersect $\alpha$. Denote by $\gamma$ the union of $\bar{\gamma}$ with those $\eta$-components that have a vertex in common with $\bar{\gamma}$ and do not intersect $\alpha \cup \beta$. Denote by $\delta$ the union of $\bar{\delta}$ with those $\eta$-components that have a vertex in common with $\bar{\delta}$ and do not intersect $\alpha \cup \beta \cup \gamma$. The graphs $\alpha, \beta, \gamma$, and $\delta$ are Eulerian, connected, edge disjoint and neither of them contains $P$. Let $H=\alpha \cup \beta \cup \gamma \cup \delta$, then its complement $\bar{H}^{c}$ is a Eulerian subgraph having at most two components, one connected to $v_{0}$ and the other connected to both $A$ and $B$. Indeed, the original graph was Eulerian and connected, so, after removing $\alpha \cup \beta \cup \gamma \cup \delta$, each component is Eulerian and connected to either $v_{0}$, or $A$, or $B$. Those components that are connected to $A$ [resp. to $B$ ] are also connected to $B$ [resp. to $A$ ], because $P$ is in $\bar{H}^{c}$.

Let $x$ be a core point of $(N ; c)$. Since $\alpha \cup \beta$ is Eulerian, connected and contains $v_{0}$, it follows that

$$
\begin{equation*}
x(N(\alpha))+x(N(\beta)) \leq k(\alpha \cup \beta) \tag{4.5}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
x(N(\gamma))+x(N(\delta)) \leq k(\gamma \cup \delta) \tag{4.6}
\end{equation*}
$$

$\bar{H}^{c} \cup \alpha \cup \beta$ is Eulerian, connected and contains $v_{0}$, therefore,

$$
\begin{equation*}
x(N(\alpha))+x(N(\beta))+x\left(N\left(\bar{H}^{c}\right)\right) \leq k\left(\alpha \cup \beta \cup \bar{H}^{c}\right) . \tag{4.7}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
x(N(\gamma))+x(N(\delta))+x\left(N\left(\bar{H}^{c}\right)\right) \leq k\left(\gamma \cup \delta \cup \bar{H}^{c}\right) \tag{4.8}
\end{equation*}
$$

${ }^{11}$ This is true even if $P$ contains no players. Indeed, by $(3), k\left(G_{P}\right)=0$ and so, by $(3), k\left(\tau_{P}\right) \geq 0$.
${ }^{12}$ Note that the proofs of this theorem as well as the proofs of Corollary 4.12 and Theorem 4.13 are valid also if $P$ is a simple cylcle.


Fig. 11. The determination of the $\eta$-components
Adding (4.5) - (4.8) we obtain $2 x(N) \leq 2 k(G)$. However, by Lemma 4.9, the last inequality must be satisfied as an equality, because $x(N)=c(N)$, so all inequalities in (4.5) - (4.8) are, in fact, satisfied as equalities.

Similar arguments show that

$$
\begin{gather*}
x(N(P))+x(N(\alpha))+x(N(\delta)) \leq k(P \cup \alpha \cup \delta),  \tag{4.9}\\
\left.x\left(N\left(H^{c} \backslash P\right)\right)+x(N(\beta))+x(N(\gamma)) \leq k\left(\overline{\left(\bar{H}^{c} \backslash P\right.}\right) \cup \beta \cup \gamma\right), \tag{4.10}
\end{gather*}
$$

because $P \cup \alpha \cup \delta$ as well as its complement $\overline{\left(\bar{H}^{c} \backslash P\right)} \cup \beta \cup \gamma$ are Eulerian, connected and contain $v_{0}$. Adding these inequalities we conclude that they are, in fact, satisfied as equalities.

In particular, we have proved that

$$
\begin{equation*}
x(N(P))+x(N(\alpha))+x(N(\delta))=k(P \cup \alpha \cup \delta) . \tag{4.11}
\end{equation*}
$$

In a similar fashion we prove that

$$
\begin{equation*}
x(N(P))+x(N(\beta))+x(N(\gamma))=k(P \cup \beta \cup \gamma) . \tag{4.12}
\end{equation*}
$$

Adding (4.11) and (4.12), taking into account the previous equalities, we obtain: $2 x(N(P))=$ $2 k(P)$.

Remark 4.11. Following the proof of Theorem 4.10, we see that we could have replaced Conditions (2) and (3) of Definition 4.8 by the requirement that $c(N)=k(G)$. This is a somewhat more general result. However, it also follows from the proof that these conditions plus non-emptiness of the core imply that every elementary path of positive cost must contain players. Indeed, all inequalities and equalities of the proof are valid, so, for an elementary path without players, $0=x(N(P))=k(P)$ and $k(P)$ cannot be positive.

An important consequence of Theorem 4.10 is that for games satisfying the condition of the theorem, it is not strictly profitable for an $S$-tour to traverse twice a proper subset of an elementary path that does not contain members of $S$ :

Corollary 4.12. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be a complete and extended CP enterprize, in which $G$ is a connected Eulerian graph having the 4-cut property. Let $(N ; c)$ be the $C P$ game induced by $\Gamma$. Suppose that the core of $(N ; c)$ is not empty. Then, for every closed walk $T$ in an elementary path $P$, which originates and terminates at one of the two extreme vertices of $P, k(T) \geq 0$.

Proof. Let $x$ be a core element. Denote by $w$ the vertex at which $T$ originates. If $w=v_{0}$, the result follows from Theorem 4.1. Suppose $w \neq v_{0}$, and assume that $k(T)<0$. Consider a cycle $C$ which traverses $w$ and $v_{0}$, which is edge-disjoint from $P$. Such a cycle exists since $G$ has the 4 -cut property. Since $C$ is a union of edge-disjoint elementary paths, it follows from Theorem 4.10 that $x(N(C))=k(C)$. Note that $C \cup T$ is a feasible $N(C)$-tour. A contradiction now follows from $x(N(C))=k(C)>k(C \cup T)$.

We can now demonstrate that the core and the nucleolus, when the core is not empty, of an extended complete Chinese postman enterprize defined on a Eulerian graph $G$ having the 4-cut property is the Cartesian product of the cores/nucleoli of the extended CP games defined on cycle graphs which are induced by the elementary paths of $G$.

Theorem 4.13. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be a complete extended CP enterprize in which $G$ is a connected Eulerian graph having the 4 -cut property. Let $(N ; c)$ be the game that corresponds to $\Gamma$. Then, the core of $\Gamma$ is equal to the Cartesian product of the cores of $\Gamma_{P}$; namely, $\mathcal{C}(\Gamma)=\chi_{P \in \mathcal{P}} \mathcal{C}\left(\Gamma_{P}\right)$, where $\chi$ denotes the Cartesian product and $\mathcal{P}$ is the set of all elementary paths of $G$.

Proof. A. Let $x \in \mathcal{C}(\Gamma)$. Then, by Theorem 4.10, $x(Q)=k(Q)$ for every elementary path $Q$ of $G$. Let $P$ be a single elementary path of $G$ which contains players at some of its edges and let $\Gamma_{P}$ be the CP enterprize corresponding to the cycle that is induced by $P$. We propose to show that $x_{\mid P}$ belongs to the core of $\Gamma_{P}$. By Theorem 3.10 it certainly is a core element of the reduced game $\hat{\Gamma}_{P}:=\hat{\Gamma}_{P}^{x}$ which, by Theorem 3.7, is obtained by making all edges outside $P$ public and increasing the prize by $x_{i}$ for each edge occupied by a player $i$, which does not reside in $P$. Thus, these prizes cancel the cost of an elementary path if it is traversed once. It remains to show that $\Gamma_{P}$ and $\hat{\Gamma}_{P}$ are isomorphic. This will be the case if we show that for every coalition $S$ whose members reside in $P$, a least-cost $S$-tour, $t^{S}$, in $\hat{\Gamma}_{P}$ has zero cost outside $P$.

First, let us show that an $S$-tour exists, whose cost is zero outside $P$. Since $G$ has the 4 -cut property, for every elementary path $P$ with extreme vertices $u$ and $v$, there exists at least two ${ }^{13}$ edge disjoint paths, say $\alpha$ and $\beta$, between $u$ and the post office $v_{0}$, and two edge disjoint paths, say $\gamma$ and $\delta$, between $v$ and $v_{0}$, and all are also edge disjoint from $P$. If the tour for coalition $S$, inside $P$, originates and terminates at $u$, (resp. $v$ ), the tour

[^8]outside $P$ would consist of $\alpha$ and $\beta$ (resp. $\gamma$ and $\delta$ ), and if it originates in $P$, say, at $u$ and terminates at $v$, then the tour outside $P$ would consist of, say, $\alpha$ and $\gamma$. In both cases, it follows from Theorem 4.10 that the cost of the tour outside $P$ is zero.

Now, by Corollary 4.12, a least-cost $S$-tour, $t^{S}$, in $\hat{\Gamma}_{P}$, would only consist of elementary paths outside $P$. It is not cheaper to artificially traverse an elementary path, or part of it, more than once, since prizes are collected only upon that first traversal of each edge. Thus, if it is feasible, such a tour $t^{S}$ will only traverse once, and completely, elementary paths once outside $P$ and we already proved that such tours exist.
B. Let $x \in X_{P \in \mathcal{P}} \mathcal{C}\left(\Gamma_{P}\right)$. We have to show that $x \in \mathcal{C}(\Gamma)$. Note that, since $x \geq 0$ (Lemma 3.3), the cost of every coalition in $\Gamma_{P}$ in non-negative. Therefore, it does not pay to traverse unnecessarily edges merely in order to benefit from the prizes.

Consider a coalition $S, S \subseteq N$, and let $t^{S}$ be an optimal tour for $S$. Denote by $T$ the set of players that occupy the edges of $t^{S}$, then $c(T)=c(S)$. It is sufficient to prove that $c(T)-x(T) \geq 0$, because $x \geq 0$.

Since $x \in X_{P \in \mathcal{P}} \mathcal{C}\left(\Gamma_{P}\right)$ and $\Gamma$ is assumed to be complete, for every elementary path $P$ traversed by $t^{S}, x(N(P))=k(P)$. This is true also for elementary paths that contain no player. Further, for any closed walk, $Z$, which is contained in some elementary path, $Q, Q \in t^{S}, x(N(Z)) \leq k(Z)$. Again, this is true also if $Q$ contains no player. Now, the set of players in $T$, corresponding to $t^{S}$, consists of all players contained in the set, $\mathcal{P}^{S}$, of elementary paths traversed by $t^{S}$, and all players in the set, $\mathcal{Z}^{S}$, of closed walks at elementary paths which are contained in $t^{S}$. Then

$$
\begin{align*}
x(T) & =\sum_{P \in \mathcal{P}^{S}} x(N(P))+\sum_{Z \in \mathcal{Z}^{S}} x(N(Z)) \\
& \leq \sum_{P \in \mathcal{P}^{S}} k(P)+\sum_{Z \in \mathcal{Z}^{S}} k(Z)  \tag{4.7}\\
& \leq c(T)
\end{align*}
$$

where the last inequality follows since it is possible that an elementary path is traversed by $t^{S}$ more than once, and the cost of traversing an elementary path, even on the first time, is non-negative. Since $G$ is Eulerian, and prizes are collected only upon the first traversal of edges, $x(N)=\sum\{k(P)$ : all elementary paths in $G\}=k(G)$, by Lemma 4.9. This concludes the proof that $x \in \mathcal{C}(\Gamma)$.

Corollary 4.14. If $\mathcal{C}(\Gamma)=\emptyset$ then $\mathcal{C}\left(\Gamma_{P}\right)=\emptyset$ for some elementary path $P$.

Remark 4.15. Note that for part B of the proof of Theorem 4.13, we did not make use of the 4-cut property. So, $X_{P \in \mathcal{P}} \mathcal{C}\left(\Gamma_{P}\right)$ is always a subset of the core for a complete extended CP-enterprize in which $G$ is a connected Eulerian Graph.

Theorem 4.16. Under the notation and conditions of Theorem 4.13, if the core of $\Gamma$ is not empty then the nucleolus of $\Gamma$ is the Cartesian product of the nucleoli of $\Gamma_{P}$ :

$$
\begin{equation*}
\nu(\Gamma)=X_{P \in \mathcal{P}} \nu\left(\Gamma_{P}\right) \tag{4.8}
\end{equation*}
$$

Proof. Since $\Gamma$ has a nonempty core, $\nu(\Gamma)$ is a core point and the nucleolus coincides with the prenucleolus. However, the prenucleolus satisfies consistency (reduced game property), therefore $\nu(\Gamma)_{\mid P}$ is the prenucleolus of $\hat{\Gamma}_{P}$. It is also a core point of $\hat{\Gamma}_{P}$, and therefore this game has a non-empty core and therefore it is isomorphic to $\Gamma_{P}$. Consequently, $\nu(\Gamma)_{\mid P}$ is the nucleolus of $\Gamma_{P}$. $■$

Corollary 4.17. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be a complete extended Chinese postman enterprize in which $G=(V, E)$ is the complete graph $G_{2 n+1}, n \geq 2$ and $a(e)-p(e)=0$ for edges $e$ that contain no player. Then, $\mathcal{C}(\Gamma)$ is not empty if and only if a $(e)-p(e) \geq 0$ for all $e \in E$. Further, if the latter condition is satisfied, then $\nu(\Gamma)=(a(e)-p(e))_{e \in E}$.

Proof. $G_{2 n+1}, n \geq 2$, is Eulerian and has the 4 -cut property. Further, $\Gamma$ is a complete extended CP enterprize. Thus, by Theorem 4.13, if the core is not empty then each elementary path $P$ has a non-empty core. However, $\Gamma_{P}$ is a single-edge $e$, single-player game, so, by Lemma 3.3, $a(e)-p(e) \geq 0$. Conversely, if $a(e)-p(e) \geq 0$ for every edge $e$, the vector which assigns $a(e)-p(e)$ to the player residing in $e$, for each $e$, is a core point, by Theorem 4.13. In fact, it is the unique core point, therefore it is the nucleolus.

Finally, we note that one can test very efficiently whether a graph $G$ has the 4-cut property. Indeed, all interior vertices of the elementary paths of $G$ can be removed in $\mathrm{O}(|V|)$ time, to produce a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ in which all elementary paths are edges. One can then verify in $\mathrm{O}\left(\left|V^{\prime}\right|\right)$ time if the resulting graph is Eulerian and then employ Gomory and Hu [1961] algorithm to solve $\left|V^{\prime}\right|-1$ maximum flow problems in order to check whether $G^{\prime}$, and thus $G$, has the 4 -cut property.

In the main theorems of this section we insisted that the enterprize is complete. We conclude this section by counter examples that show that this requirement is necessary.

Example 4.18. Figure 12 shows a 4 -person extended CP enterprize whose graph is an "onion". Three elementary paths consist of a single edge occupied by the players 1,2 and 3. The fourth elementary path consists of two edges and only the edge which is not incident to the post office is occupied by player 4. The other edge is public. The costs of the edges are denoted by boxed numbers. There are no prizes. The fourth elementary path gives rise to an extended 1-person CP cyclic enterprize, $G_{P}$, whose game satisfies $c_{P}(\{4\})=2 \neq k(P)=3$. Thus, the enterprize is not complete. If Theorem 4.13 were true without the requirement of completeness, then $x=(4,4,4,2)$ would have been the only core point. But it is not, because $c(N)=15>x(N)$.

Example 4.19. Figure 13 shows a 5 -person extended CP enterprize in which $v_{0}$ is an interior point of an elementary path. There are no public edges but there are prizes that are encircled and in bold font. Here $c(N)=4$ and, in fact, $c(S)=4$ for all nonempty coalitions, because it pays to travel all the edges in order to benefit from the prizes. Thus, all the players are symmetric and (. $8, .8, .8, .8, .8$ ) is the nucleolus. This example contradicts Theorem 4.10. It also contradicts Theorems 4.13 and 4.16 , because the Cartesian product of the cores of the cyclic games induced by elementary paths is the empty set.


Fig. 12. Incomplete CP


Fig. 13. $v_{0}$ is an interior point

## 5. Chinese Postman Game on Simple Cycle Graphs

In the previous section we showed that both the core and the nucleolus of a CP game induced by a complete extended Eulerian enterprize having the 4 -cut property are Cartesian products of the cores and the nucleoli of CP games induced by simple cycle graphs derived from elementary paths in the original graph. ${ }^{14}$ It remains to study how to compute these solutions concepts for CP games induced by simple cycles.

In this section we shall not require that the cyclic enterprize is complete, since it is not needed for the proofs. Moreover, the proof with several simplifications can be adapted to the case when $G$ is a chain.

## I. The core of a CP game induced by an extended cyclic enterprize.

The following definitions will prove helpful to state our results.

Definition 5.1. A coalition $S$ in a CP enterprise is called saturated if an optimal $S$-tour exists, whose residents comprise of the members of $S$ only. ${ }^{15}$

[^9]Definition 5.2. A coalition $S$ in a CP cyclic enterprise is called one-sided if an optimal $S$-tour exists that does not contain the post office, except as an origin or a terminal vertex.

Lemma 5.3. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be an extended $C P$ enterprize whose graph $G$ is a simple cycle. Let $(N ; c)$ denote its induced game. Then the core of $(N ; c)$ is determined by $x(N)=c(N)$ and $x(S) \leq c(S)$ for all coalitions that are saturated and one-sided and all coalitions of size $n-1$, where $n$ is the number of players.

Proof. The efficiency $x(N)=c(N)$ and the constraint $x(N \backslash\{i\}) \leq c(N \backslash\{i\})$ imply $x_{i} \geq c(N)-c(N \backslash\{i\}) \geq 0$ for all $i \in N$. Let $S$ be a non-saturated coalition and let $T$ be the set of players that reside in the edges of an optimal $S$-tour. Then $S \subset T, T$ is saturated and $c(S)=c(T)$. Consequently, $x(S) \leq x(T) \leq c(T)=c(S)$, hence the core constraint for $S$ is implied by the core constraint for $T$ together with the nonnegativity constraints. Now let $S$ be a saturated coalition which is not one-sided. Then $S=S_{1} \cup S_{2}$, where the coalitions $S_{j}$ are non-empty, disjoint, saturated and one-sided. Thus, $x(S)=$ $x\left(S_{1}\right)+x\left(S_{2}\right) \leq c\left(S_{1}\right)+c\left(S_{2}\right)=c(S)$ if the core constraints are satisfied for saturated one-sided coalitions. Thus, core constraints corresponding to saturated coalitions that are not one-sided are redundant.

Note that there are at most $2 n-1$ saturated and one-sided coalitions. Of these, we can eliminate all coalitions whose values are equal to $c(N)$, because their core constraint is automatically satisfied if $x$ is non-negative and satisfies $x(N)=c(N)$. In fact, there are at most $n$ relevant saturated and one sided coalitions, as Lemma 5.4 shows.

Lemma 5.4. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be an extended cyclic enterprize, where $G=(V, E)$. Let $(N ; c)$ denote its induced game. Let $S$ and $T$ be two saturated and one-sided coalitions whose cost is different from $c(N)$. Then, either $S \subseteq T$, or $T \subseteq S$, or $S \cap T=\emptyset$.

Proof. Let $S$ and $T$ be two such coalitions and suppose that $S \nsubseteq T, T \nsubseteq S$ and $S \cap T \neq \emptyset$. Then $E(S) \cup E(T)=E$.

Since $S$ is saturated and one-sided and since $c(S)<c(N)$, an optimal $S$-tour is a closed walk, which traverses twice the edges in $E(S)$. Then,

$$
\begin{align*}
& \sum\{a(e): e \in E(S)\} \\
& <\sum\{a(e)-p(e): e \in E \backslash E(S)\}  \tag{5.1}\\
& \leq \sum\{a(e): e \in E \backslash E(S)\} \\
& \leq \sum\{a(e): e \in E(T)\},
\end{align*}
$$

where the first inequality follows since travelling back to the post-office along the edges in $E(S)$ (i.e. without collecting prizes) is strictly cheaper than travelling back to the postoffice along the edges in $E \backslash E(S)$. The second inequality follows since $p(e) \geq 0$ for all $e$,
and the third inequality follows since $E \backslash E(S) \subseteq E(T)$. In the same way we prove that $\sum\{a(e): e \in E(T)\}<\sum\{a(e): e \in E(S)\}$. This contradiction shows that no such pair of coalitions exists.

Denote by $\mathcal{B}$ the set of one-sided saturated coalitions whose value is different from $c(N)$ and by $\overline{\mathcal{B}}$ the union of this set with the coalitions $N \backslash\{i\}, i \in N$. We can summarize the finding by:

Theorem 5.5. Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be an extended CP enterprize whose graph $G$ is a simple cycle. Let $(N ; c)$ denote its induced game. Then the core of $(N ; c)$ is determined by $x(N)=c(N)$ and $x(S) \leq c(S)$ for all coalitions in $\overline{\mathcal{B}}$.

## II. A prenucleolus algorithm.

Let $\Gamma=\left(G, v_{0}, a, p, N\right)$ be an extended enterprize whose graph $G$ is a simple cycle. We assume that its core is not empty. By Theorem 4.1, the induced game ( $N ; c$ ) is convex. Thus, its kernel is a unique point and coincides with the nucleolus (Maschler et al. [1972]). However, since the kernel intersection with the core is a locus of the core; namely, it does not change when we change the game, as long as the core remains unchanged (Maschler et al. [1972]), we can compute the nucleolus of the original game by computing the nucleolus of another game having the same core. We find it convenient to do so. Specifically, we enlarge the values of all coalitions not in $\overline{\mathcal{B}}$ and different from $N$ so much that their excesses at any imputation $x$ will always be larger than the excesses of the coalitions of $\overline{\mathcal{B}}$. Thus, they will be irrelevant for the computation of the nucleolus. ${ }^{16}$

In order to proceed, we have to determine the relevant minimal balanced collections; namely, minimal balanced collections with coalitions taken from $\overline{\mathcal{B}}$.

Lemma 5.6. The minimal balanced collections composed by coalitions from $\overline{\mathcal{B}}$ are:
(1) $\{N \backslash\{i\}: i \in N\}$,
(2) A partition $\left\{S_{1}, S_{2}\right\}$ of $N$, if such coalitions exist in $\overline{\mathcal{B}}$.
(3) $\{S\} \cup\{N \backslash\{i\}: i \in S\}$, for every $S$ in $\mathcal{B}$.

Proof. The collections (1) - (3) are certainly minimally balanced. There is at most one partition of type (2). Consider any minimal balanced collection $\mathcal{D}$ composed of coalitions from $\overline{\mathcal{B}}$ and pay attention to $\mathcal{E}=\mathcal{D} \cap \mathcal{B}$. Since the characteristic vectors in $\mathcal{E}$ are either vectors of the type $(1,1, \ldots, 0,0, \ldots 0)$ or $(0,0, \ldots, 1,1, \ldots, 1)$, we consider the coalitions $S$ and $T$ in this collection which are maximal under inclusion from each side. We write $S=\emptyset$ or $T=\emptyset$ if $S$, resp., $T$ is not a member of $\mathcal{D}$. All other coalitions in $\mathcal{E}$ are subsets of these coalitions. If the union of the two maximal coalitions form a partition, this partition

[^10]is already balanced so no coalition of type $N \backslash\{i\}$ is contained in $\mathcal{D}$. This was covered under (2).

We are left with collections such that, the incidence matrix of $\mathcal{E}$ contains at least one column $k$ of zeroes. Without loss of generality we assume that $S \neq \emptyset$. Then, each column whose index is a member of $S$ dominates column $k$. Since $\mathcal{D}$ is balanced, for each $i$ in $S$ there must be a coalition in $\mathcal{D} \backslash \mathcal{B}$ that contains 0 in a Column $i$ and 1 in Column $k$. This coalition can only be $N \backslash\{i\}$. We have proved that a subset of the minimal balanced collection $\mathcal{D}$ contains a balanced collection of type (3), so $\mathcal{D}$ contains no other coalitions.

We shall now decrease the values of coalitions in $\overline{\mathcal{B}}$ in such a way that the core will shrink but remain non-empty and the nucleolus will not change. The procedure is similar to the method of finding the nucleolus by looking for the lexicographic center (see Maschler et al. [1979]).

By Lemma 5.6, the relevant Bondareva-Shapley conditions are:

$$
\begin{align*}
& \sum_{i \in N} c(N \backslash\{i\}) \geq(|N|-1) c(N) \\
& c(T)+c(N \backslash T) \geq c(N), \text { if }\{T, N \backslash T\} \text { is a partition of coalitions from } \mathcal{B},  \tag{5.2}\\
& c(B)+\sum_{i \in B} c(N \backslash\{i\}) \geq|B| c(N) \text { for all } B \in \mathcal{B}
\end{align*}
$$

Here, $|S|$ denotes the number of players in $S$. Let

$$
\begin{align*}
t_{0} & =\frac{1}{|N|} \sum_{i \in N} c(N \backslash\{i\})-\frac{|N|-1}{|N|} c(N), \\
t_{1} & =\frac{1}{2}[c(N)-c(T)-c(N \backslash T)]  \tag{5.3}\\
t_{B} & =\frac{1}{|B|+1} \sum_{i \in B} c(N \backslash\{i\})+\frac{1}{|B|+1}[c(B)-|B| c(N)], \quad B \in \mathcal{B} .
\end{align*}
$$

If we reduce each coalition of a balanced collection by its corresponding $t$, the corresponding inequality becomes equality.

Choose $t=\min \left\{t_{0}, t_{1}, t_{B}: B \in \mathcal{B}\right\}$ and reduce each coalition of $\overline{\mathcal{B}}$ by $t$. Then at least one inequality becomes an equality while the other inequalities in (5.1) remain valid. Thus, the core of the resulting game remains non-empty, and by Kohlberg theorem (Kohlberg [1971]) the nucleolus remains unchanged, because, for imputations $x$, the order of the first $2 n+1$ coalitions of $\theta(x)$ is not effected by such a reduction, and these are sufficient in order to determine the lexicographic order of $\theta(x)$.

Remove all inequalities in (5.1), for which equality now prevails, fix the values of coalitions that appear in inequalities that became equalities, and perform a similar reduction on the remaining coalitions, whose value was just reduced by $t$. Again, at least one inequality will become equality, the core will remain nonempty and the nucleolus will remain unchanged.

After at most $n+2$ repetitions of this process, all inequalities will become equalities, and the core will reduce to a single point. Indeed, $x(N)=c(N)$ and $\sum_{i \in N} \tilde{c}(N \backslash\{i\})=$ $(|N|-1) c(N)$ imply $x_{i}=c(N)-\tilde{c}(N \backslash\{i\})$ for every $x$ in the core of the last obtained game ${ }^{17}(N ; \tilde{c})$.

The complexity of this algorithm is $O\left(|N|^{2}\right)$. Indeed, it takes linear time to determine $\mathcal{B}$, the algorithm will terminate after at most $|\mathcal{B}|+2$ iterations, and by Lemma $5.4,|\mathcal{B}| \leq n$. The values of coalitions that appear in (5.2), and the values of the left-hand-side of the constraints in (5.2), which, in turn, are used to compute $t_{0}, t_{1}$ and $t_{B}, B \in \mathcal{B}$, in (5.3), can be done in quadratic time. Finding the values of $t, t_{0}$ and $t_{B}, B \in \mathcal{B}$ and the minimum value $t$ can be done in linear time. Finally, the values of the left-hand-side of the constraints in (5.2) that are satisfied as strict inequalities at any iteration, can be obtained in linear time, from their values in the previous iteration. Indeed, this is done for each constraint by subtracting the value of $t$ in the previous iteration times the number of coalitions in that constraint whose values were not fixed in previous iterations.

Conclusion 5.7. Recall that by Theorem 4.16, the nucleolus of an extended complete CP game defined on a Eulerian graph, $G$, having the 4 -cut property, whose core is not empty, is the Cartesian product of the nucleoli of the CP games defined on the cycle graphs induced by all the elementary paths of $G$. Since all elementary paths of a Eulerian graph having the 4 -cut property can be found in linear time, it follows that the nucleolus of such a CP game can be computed in $\mathcal{O}\left(|N|^{2}\right)$ time.

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[^1]:    ${ }^{1}$ Definition 4.4.
    ${ }^{2}$ Definition 4.8.
    ${ }^{3}$ Definition 4.3.

[^2]:    ${ }^{4}$ By convention, $x(\emptyset)=0$.
    ${ }^{5}$ The term "tour" is usually used if the walk is related to a coalition.

[^3]:    ${ }^{6}$ Namely, a tour contains all edges in which members of $S$ reside.

[^4]:    ${ }^{7}$ Hence the title "multigraph". Usually $Q$ will be a walk.

[^5]:    ${ }^{8}$ Similar results can be proved for CP enterprizes whose graph is a chain.

[^6]:    ${ }^{9}$ If $A B$ is itself a simple cycle, the two edge-disjoint cycles that exist by the induction hypothesis satisfy the requirement of the theorem when we replace $A B$ by $P$.

[^7]:    ${ }^{10}$ With proper conventions, Condition (2) is implied by Condition (3).

[^8]:    ${ }^{13}$ If $u=v_{0}$ or $v=v_{0}$, both of these paths are degenerate and consist of $v_{0}$.

[^9]:    ${ }^{14}$ The result for the nucleolus is valid only to games in which the core is not empty.
    ${ }^{15}$ The optimal $S$-tour may contain public edges.

[^10]:    ${ }^{16}$ In fact, we will compute the $\overline{\mathcal{B}}$ nucleolus over the core (see Maschler et al. [1992]).

[^11]:    ${ }^{17}$ Observe that $\tilde{c}(N)=c(N)$, because the value of $c(N)$ is not modified in the above calculations.

