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BARGAINING WITH AN AGENDA

by

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ABSTRACT. Gradual bargaining is represented by an agenda: a family of increasing sets of joint utilities, parameterized by time. A solution for gradual bargaining specifies an agreement at each time. We axiomatize an ordinal solution, i.e., one that is covariant with order-preserving transformations of utility. It can be viewed as the limit of a step-by-step bargaining in which the agreement of the last negotiation becomes the disagreement point for the next. The stepwise agreements may follow the Nash solution, the Kalai-Smorodinsky solution or many others.

1. INTRODUCTION

1.1. **Gradual bargaining.** Nash's pioneering paper on two-person bargaining, (1950), has led to two streams of research. One develops axiomatizations leading to Nash's solution or to later ones, while the second constructs plausible non-cooperative games behind the bargaining problem, then solves these games. Less attention has been paid to expanding the definition of what constitutes a bargaining problem.

This paper looks at bargaining as extended over time. Our primitive is a family of bargaining problems (each of which is a set of feasible agreements), rather than a single one as in Nash's conception. We refer to such a family of feasible sets as an *agenda*. A *gradual bargaining problem* is defined by its agenda and an initial point at which the bargaining starts. For clarity and simplicity we will consider a family of feasible sets that is ordered continuously by time.

Whereas a solution to a Nash bargaining problem specifies a single agreement, a solution for a gradual bargaining is a path of agreements – a *gradual agreement* – which specifies an agreement point for each point in time. We propose a solution for gradual bargaining, namely, a function that assigns to each gradual bargaining problem a certain

gradual agreement. We call this solution for a reason that should become clear, the *ordinal solution*.

Our model is meant to capture situations in which the parties are to reach agreements on several issues negotiated one after the other. These issues can have a natural order as is the case in labor contracts signed annually, or the division of profits determined at the end of each quarter. Alternatively, the issues can be ordered by the bargainers, as in a negotiation to end a political conflict in which territorial, economic and other issues are negotiated sequentially. In each of these cases, the possible agreements at each stage of the bargaining process cover *all* the issues raised until this stage. Thus, for example, in a case of profit sharing the possible agreements at a certain stage are all the possible sharing arrangements of the profit accumulated up to this stage.

1.2. **The axioms.** Our framework views the agreement reached at each stage as final for the issues on the table up to that time. This assumption is expressed in our axiomatic characterization by the time consistency axiom. It requires that taking the agreement reached on the solution path at a given point in time and applying the solution rule to the same agenda with this agreement as an initial point yields the same path. The assumption on the bargaining process also underlies the description of the ordinal solution as the limit of a step-by-step bargaining which we discuss later in the introduction.

The axiom of time consistency can be compared to the axiom of step-by-step negotiation in Kalai (1977), which is used to characterize the family of proportional solutions for Nash's bargaining problem. It considers that a bargain can be concluded in two steps. The first involves one with a smaller feasible set and the same disagreement point as the entire problem, and the second step uses the solution of the first as its new disagreement point. Kalai's axiom requires that the same agreement be reached independently of the choice of the feasible set in the first stage, and so is much stronger than ours. An axiom of agenda independence is also used by Ponsati and Watson (1997) to characterize the Nash solution. In contrast to Kalai's and Ponsati and Watson's axioms, our time independence axiom involves a given agenda.

The ordinal solution is characterized by five axioms. The time consistency axiom, which has already been discussed, is special to the gradual bargaining setup. So also is a directional continuity requirement which depends on the solution being a path of agreements. The other three axioms are analogous to axioms commonly used for Nash's bargaining problem. We require that the agreement reached at each point in time

must be efficient in the set of feasible utilities at that time. The solution of a symmetric problem must also be symmetric. Further, the solution must be invariant under positive linear transformations of a bargainer's utility.

The ordinal solution is described by a differential equation that is simple to interpret: at each point on the agreement path the ratio of players' marginal utility gains (with respect to time) is the rate of substitution of their utilities on the current efficient frontier. Thus, using the marginal rate of substitution to make an interpersonal comparison of utility, the gains of solving the next stage of the negotiation are divided in egalitarian way.

1.3. **Ordinality.** The name of the proposed solution, the ordinal solution, is suggested by the following two properties. First and most important, ordinality applies to utility representation. The ordinal solution is covariant with respect to order-preserving (i.e., monotonic) transformations of each bargainer's utility. Ordinality also refers to time: the solution depends only on the order of the agenda and not the precise timing of when issues are negotiated.

Covariance with respect to order-preserving transformations is desirable for a solution, since it means that the solution is based on the most elementary aspect of utility—the order of outcomes—and nothing else. Shapley (1969) demonstrated that the two-person Nash problem has no single-valued solution satisfying symmetry, efficiency, and covariance with respect to order-preserving transformations of the utility functions. He showed that three-person problem, however, has such a solution. Recently, Safra and Samet (2000) extended this ordinal solution to more than three players. The solution proposed here for gradual bargaining is ordinal for any number of players, even for two players.

The derivation of the ordinality of utility from the axioms is somewhat surprising. The covariance axiom requires that the solution is covariant with respect to linear transformations only. Thus even if one insists on solving gradual bargaining problems using Von Neumann Morgenstern utilities, one is lead to the conclusion that using any other utility function would result in the same path of gradual agreements, provided one accepts the proposed axioms. It is easy to see why ordinality is implied by these axioms. From the directional continuity and time consistency axioms, the agreement reached at a certain time depends only on the local behavior of the agenda. Thus, the solution is covariant with respect to monotonic transformations of the utilities

that are locally linear. But every smooth monotonic transformation is, in an appropriate sense, locally linear.

Obviously, it is possible to strengthen the covariance axiom by requiring covariance with respect to monotonic transformations. Potentially, this could lead to a characterization of the ordinal solution in which one or more of the other axioms would be weakened. Unfortunately we could not find appealing weakened axioms.

1.4. **Step-by-step bargaining.** The ordinal solution has an interesting relation to various solutions of Nash's bargaining problem. Suppose an agreement is reached at a certain time, and using this agreement as a status quo point we solve the Nash bargaining problem with the set of utilities that are feasible after some time increment. Assume we solve this problem using the Nash solution. It turns out that when the time increment approaches zero, the utility gains of the players per unit time converge to those predicted by the ordinal solution. Thus, the ordinal solution is the limit of a discrete process in which each agreement serve as a status quo point for a Nash bargaining problem of the next stage, and the next agreement is the Nash solution for this problem. Somewhat surprisingly, if in the process described above we use the Kalai and Smorodinsky (1975) solution instead of the Nash solution, we also end up with the ordinal solution. Indeed, any solution may be used as long as it coincides with the Nash solution on linear problems[1] and satisfies a certain continuity condition that allows linearization of small feasible sets.

1.5. **Related works.** Continuous time processes in the context of the Nash bargaining model were used among others by the following authors. Maschler, Owen and Peleg (1988) characterized the Nash solution by means of a system of differential equations and interpreted the solution as a continuous process of moving within the feasible set of utilities. Livne (1989) and Peters and van Damme (1993) used a similar approach to characterize the continuous version of Raiffa's solution (Raiffa 1953). Zhou (1997) used a differential equation to extend the Nash solution to non-convex problems. In all these works bargaining is described by Nash bargaining problem, that is a single set of feasible utilities, and not as a family of Nash bargaining problems as here. Related papers studied discrete bargaining with multiple pies: Fershtman (1990) and John and Raith (1997) in a bilateral context, and Winter (1997) and Seidmann and Winter (1997) in a multilateral context.

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[1]This is the case when the solution is efficient, symmetric and covariant with respect to linear transformations of utilities.

Nicolò and Perea (2000) offered a different model of two-person bargaining that also leads to an ordinal solution. An ordinal solution, due to Shapley, to Nash's bargaining problem, in terms of a differential equation, can be found in Calvo and Peters (2001). In the theory of cost sharing, continuous time solutions are common since the introduction of the Aumann-Shapley pricing by Mirman and Tauman (1982). Recently, Sprumont (1998) introduced such a solution for ordinal cost sharing.

Bergman (1992) studied two-person non-cooperative bargaining over a shrinking pie, described by a family of Pareto surfaces (see also Binmore (1987)). He developed a differential equation that corresponds to our solution in the special case of two players, by taking the continuous time limit of the alternating offer bargaining game. In contrast, we motivate the ordinal solution axiomatically and by considering step-by-step cooperative bargaining.

In this paper the bargaining agenda is given exogenously. However a few authors studied the choice of agenda itself as a non-cooperative bargaining problem. An agenda in these studies is typically a finite set of issues. Negotiations are modeled as non-cooperative extensive form games and the results mainly concern the comparison of agendas (the ordering of issues) in terms of their prospects of yielding efficient outcomes. Examples are Fershtman (1990) and John and Raith (1997) in bilateral negotiations, and Winter (1997) in a multilateral framework. Thomson (1994) gives a concise survey of bargaining models derived from Nash's axioimatic approach, including Shapley's ordinal solution.

1.6. **The paper plan.** Section 2 formalizes the gradual bargaining problem, defines gradual agreements and describes solutions to gradual bargaining problems. The ordinal solution is introduced in Section 3 and axiomatized in Section 4. Section 5 shows the two ordinal properties of the solution. The relation of the gradual solution to other concepts in the Nash bargaining framework is in Section 6. Finally, the proofs appear in Section 7.

2. Gradual bargaining

2.1. **Gradual bargaining problems.** Consider a finite set N of n players. A gradual bargaining problem is one in which they negotiate the issues one after another. In term of utilities, it is described by feasible sets that expand over time. For each time t the set in t is the subset of utilities in $\mathbb{R}^N$ that correspond to possible agreements on the issued negotiated until t.

In our continuous time model the expanding feasible sets are described by an increasing function f on $\mathbb{R}^N$, the value of which is time. The set $\{x \in \mathbb{R}^N \mid f(x) \leq t\}$ is the set of utility vectors of possible agreements on issues negotiable up to time t.

Definition 1. *An* **agenda** *is a real-valued function f on $\mathbb{R}^N$. The agenda f defines for each time t the* **feasible set** $S_t^f = \{x \in \mathbb{R}^N \mid f(x) \leq t\}$. *We assume that f satisfies the following conditions.*

1. *f is continuously differentiable.*
2. *$\nabla f > 0$.*
3. *∇f is locally Lipschitz, i.e, for each bounded subset of $\mathbb{R}^N$ there is a constant K, such that for each x and y in the subset, $\|\nabla f(x) - \nabla f(y)\| < K\|x - y\|$.*

We denote by $\mathscr{F}$ the set of all agendas. By the strict monotonicity of f (condition 2), for $t < t'$, $S_t^f \subset S_{t'}^f$. The set $\{x \in \mathbb{R}^N \mid f(x) = t\}$ is the Pareto frontier of S_t^f.

Definition 2. *A* **gradual bargaining problem** (*or* **problem**, *for short*) *is a pair (f, a), in $\mathscr{F} \times \mathbb{R}^N$, of an agenda f and an* **initial (status quo) point** a.

2.2. **Solutions.** A gradual bargaining problem results in interim agreements, one for each point in time t, which are given as n-tuples of utilities. A specification of these agreements is called a gradual agreement.

Definition 3. *A* **gradual agreement** ϕ *is a continuously differentiable path in $\mathbb{R}^N$, $\phi \colon \mathbb{R} \to \mathbb{R}^N$.*

For each time t, $\phi(t)$ is the vector of utilities determined by the interim agreement at time t. The set of all gradual agreements is denoted by $\mathscr{P}$.

Definition 4. *A* **solution** *for gradual bargaining problems is a function*

$$\Phi : \mathscr{F} \times \mathbb{R}^N \to \mathscr{P},$$

such that for each problem (f, a), the gradual agreement $\phi = \Phi(f, a)$ is feasible at each time, that is, $f(\phi(t)) \leq t$, for all t.

The initial point of a bargaining problem (f, a) is the status quo agreement at the beginning of the bargaining process at time $f(a)$. However, the solution of a bargaining problem (f, a) specifies agreement points even for times prior to $f(a)$. This has been done for reasons of simplicity only. The results of this paper can be formulated for solutions that are defined only for times later than that of the initial

point. Yet, the agreements prior to $f(a)$ are meaningful. In light of the axiom of time consistency below each point along the solution of a bargaining problem, when considered as an initial point, results in the same path. Thus, agreements on this path at times prior to $f(a)$ are status quo points that would have led to the point a.

3. The ordinal solution

The ordinal solution is determined by a differential equation. In Section 4 we characterize this equation axiomatically, but here we outline a derivation of it by reducing the solution of a gradual bargaining problem to the solution of a sequence of Nash bargaining problems.

Equipped with a solution for Nash bargaining problems, one may approach a gradual bargaining as follows. At each stage solve the Nash bargaining problem that consists of the feasible set at this stage with the agreement of the previous stage as a status quo point. In our continuous setup, where there is no "previous" stage, we require an appropriate limit process, which we describe next.

Suppose that at time t an efficient agreement x is reached for the agenda f. Thus, $f(x) = t$. Consider the Nash bargaining problem that consists of the status quo point x, and the feasible set at time $t + \Delta t$, that is, the set $\{y \mid f(y) \le t + \Delta t\}$. An efficient solution for this Nash bargaining problem is a point $y = x + \Delta x$ on the Pareto frontier of the feasible set, that is $f(x + \Delta x) = t + \Delta t$. For the left hand side we take the first order approximation $f(x) + \sum_i f_i \Delta x_i$, where the f_i's are the partial derivatives of f at x. Using this approximation, the requirement that $x + \Delta x$ is on the frontier is given by $\sum_i f_i \Delta x_i = \Delta t$.

Suppose we apply the Nash solution to this problem. Then, Δx is the maximizer of the function $h(\Delta x) = \prod_i \Delta x_i$ subject to $\sum_i f_i \Delta x_i = \Delta t$. This constrained optimization problem is solved by the vector Δx that satisfies for some λ (the Lagrangian multiplier of the constraint) $h_i = \lambda f_i$, for each i. As $h_i = h/\Delta x_i$, we conclude that $\Delta x_i = (h/\lambda)(1/f_i)$. Substituting the right hand side in the constraint, we find that $h/\lambda = \Delta t/n$. Thus, for each i, $\Delta x_i = \Delta t/(nf_i)$. This leads to the differential equation described next.

As we see in Section 6 the same differential equation results if we use the Kalai-Smorodinsky solution rather than the Nash solution. Moreover any solution that coincides with these solutions on linear bargaining problems and satisfies a simple continuity property gives rise to the same equation.

Definition 5. *The* **ordinal solution** *for gradual bargaining problems associates with each problem (f, a) the unique gradual agreement ϕ that*

solves the simultaneous differential equations

$$(1) \qquad \phi_i'(t) = \left[n \frac{\partial f}{\partial x_i} \big(\phi(t) \big) \right]^{-1}, \quad i \in N$$

with the initial condition $\phi(f(a)) = a$.

By condition 2 in Definition 1, the right hand side of (1) is well de-fined. Let $I : \mathbb{R}^N \backslash 0 \to \mathbb{R}^N$ be the function $I(x_1 \ldots, x_n) = (x_1^{-1}, \ldots x_n^{-1})$. Then, the set of equations (1) can be written:

$$\phi'(t) = I\Big(n \nabla f(\phi(t)) \Big).$$

Since I is Lipschitz on any domain that is bounded away from 0, it follows by condition 2 that the right hand side of (1) is locally Lipschitz. It then follows from conditions 1 and 3, that (1) has a unique solution (see for example Hartman (1982)).

We show later that for each t, $f(\phi(t)) = t$. That is, the agreement at time t belongs to the Pareto frontier of S_t^f, the feasible set at t. In light of this, the interpretation of the ordinal solution is straightforward. The ratio of players i's and j's marginal increments of utility at time t, $\phi_i'(t)/\phi_j'(t)$, is, according to (1), the marginal rate of substitution of i's and j's utilities at $\phi(t)$ along the Pareto frontier of S_t^f. Thus the ordinal solution equates players' gains according to the appropriate substitution rate of their utilities.

4. AXIOMATIC CHARACTERIZATION

We now consider a set of axioms that characterize the ordinal solu-tion. The first three are analogous to axioms in many characterizations of solutions of Nash bargaining problems.

We require first that no feasible outcome at time t dominate the agreement point at t.

Axiom 1. (Efficiency) *For each t, if $x > \Phi(f, a)(t)$, then $f(x) > t$.*

Since the Pareto surface of the S_t^f is $\{x \mid f(x) = t\}$, and since solu-tions are required to be feasible, this axiom is equivalent to requiring that for each t, $f(\Phi(f, a)(t)) = t$.

The next axiom corresponds to the standard symmetry condition used for several solutions of Nash's problem. For a permutation of N, $\pi\colon N \to N$ and $x = (x_i)_{i \in N}$ in $\mathbb{R}^N$, we denote $\pi x = (x_{\pi(i)})_{i \in N}$. A problem (f, a) is *symmetric* if for any permutation π and $x \in \mathbb{R}^N$, $f(x) = f(\pi x)$ and $a = \pi(a)$ (i.e., all coordinates of a are the same.)

Axiom 2. (Symmetry) *If (f, a) is symmetric, then for each t, $\Phi(f, a)(t)$ is symmetric, i.e., all its coordinates are the same.*

The following axiom requires that the solution be covariant with respect to positive linear transformations of utility. Let $s = (s_i)_{i \in N}$ be a vector of linear transformations of $\mathbb{R}$. For x in $\mathbb{R}^N$, we denote $s(x) = (s_i(x_i))_{i \in N}$. For each function f on $\mathbb{R}^N$, the function fs is defined by $(fs)(x) = f(s(x))$.

Consider two bargaining problems (g, b) and (f, a), the first formulated in terms of the utilities before the transformation s and the second in utilities after the transformation. That is, $fs = g$, and $a = s(b)$. The covariance axiom requires that the solution of (f, a) be the one obtained by applying the transformation s to the solution of (g, b).

Axiom 3. (Covariance) *Let $s = (s_i)_{i \in N}$ be a vector of linear transformations. If for the pair of problems (f, a) and (g, b), $g = fs$, and $a = s(b)$, then $\Phi(f, a) = s(\Phi(g, b))$.*

The next two axioms are special to the gradual bargaining context. The first expresses the essence of gradual bargaining: bargaining restarts at each point in time with the "last" agreement serving as a status quo point. The axiom requires that taking any of the interim agreements as the initial status quo results in the same path of agreements.

Axiom 4. (Time consistency) *If $\Phi(f, a)(t) = x$, then $\Phi(f, x) = \Phi(f, a)$*

Next we require that the solution be continuous in the following sense. If the agendas in the two problems (f,a) and (g,a) are close in a neighborhood of a, then the rates of utility gains at a for these two problems are also close.

Distance of agendas in the neighborhood of a cannot be measured by the difference $|f(a) - g(a)|$, since it reflects only differences in time measuring. Thus, for example, the rates of utility gains at a for the two agendas f and $g = f + T$, for any scalar T, should be exactly the same (see also the property of time ordinality below). What matters is the way f and g *change* in the neighborhood of a, which leads to the following definition.

For a bounded neighborhood B of a we defined a pseudo-metric d_B on the set of agendas $\mathscr{F}$, such that for each f and g, $d_B(f, g) = \sup_{x \in B} ||\nabla f(x) - \nabla g(x)||$.

Axiom 5. (Directional continuity) *For any problem (f, a) and a neighborhood B of a, the functional*

$$f \to \Phi'(f, a)(f(a))$$

is continuous with respect to the metric d_B on $\mathscr{F}$.

Theorem 1. *The ordinal solution is the unique solution that satisfies axioms 1– 5. Furthermore, these axioms are independent.*

5. ORDINALITY

We justify the name of the solution by showing that it is ordinal with respect to both utilities and time.

The first property considerably strengthens the covariance axiom by requiring that the solution be covariant not only with linear transformations of utility, but with monotonic transformations of utility. The notation is the same as used above for the covariance axiom.

Property 1. (Utility ordinality) *Let $s = (s_i)_{i \in N}$ be a vector of strictly increasing transformations of $\mathbb{R}$. If for the pair of problems (f, a) and (g, b), $g = fs$, and $a = s(b)$, then $\Phi(f, a) = s(\Phi(g, b))$.*

Feasible sets of utility, S_t^f, as well as agreements along the solution path have been parameterized here by time. The next property says that only the order of the feasible sets and agreements matters, not their precise timing. To formulate this exactly, consider a **time transformation**, which is simply an increasing function $\lambda \colon \mathbb{R} \to \mathbb{R}$. When time is transformed by λ, agendas and gradual agreements change correspondingly. The agenda f is represented, after the transformation, by λf, which is defined by $(\lambda f)(x) = \lambda(f(x))$. Similarly, a gradual solution ϕ is represented by $\phi \lambda$ which is defined by $(\phi \lambda)(t) = \phi(\lambda(t))$.

Property 2. (Time ordinality) *Let f be an agenda, and λ a time transformation. If λf is an agenda and $\phi = \Phi(\lambda f, a)$ then $\Phi(f, a) = \phi \lambda$.*

Theorem 2. *The ordinal solution satisfies the properties of utility ordinality and time ordinality.*

6. GRADUAL AND ONE-SHOT BARGAINING

A *one-shot bargaining problem* is a pair (S, d), where S, the feasible set, is a subset of $\mathbb{R}^N$, and d, the status quo (or disagreement point), is a point in S. Let $\mathscr{D}$ be a set of one-shot bargaining problems. A *solution* for $\mathscr{D}$ is a function $\sigma \colon \mathscr{D} \to \mathbb{R}^N$, such that for each problem $(S, d) \in \mathscr{D}$, $\sigma(S, d) \in S$.

An agenda defines a continuum of feasible sets. We are interested in agendas for which any of these sets, in combination with a disagreement point, belongs to the domain $\mathscr{D}$ over which σ is defined.

Definition 6. *An agenda f, is* compatible *with $\mathscr{D}$ if for each t and $d \in S_t^f$, $(S_t^f, d) \in \mathscr{D}$.*

A gradual agreement can be thought of as the limit of agreements achieved in discrete time when the time intervals between agreements tend to zero. Let ϕ be a gradual agreement, and $\phi(t)$ be the interim agreement at time t. Suppose the next agreement is reached at time $t' > t$. The feasible set at time t' is $S_{t'}^f$, and the status quo point is the most recent agreement $\phi(t)$. Applying the solution σ to this one-shot bargaining problems results in the agreement $\sigma(S_{t'}^f, \phi(t))$. Dividing the utility gains of this agreement, $\sigma(S_{t'}^f, \phi(t)) - \phi(t)$, by the elapsed time $t' - t$ yields the rate of change in utility gains. If the limit of this rate, when t' converges to t, is $\phi'(t)$, for each t, then we say the gradual agreement ϕ is compatible with the solution σ. The following definition formalizes the idea that a gradual solution Φ is compatible with a solution σ for one-shot bargaining problems in the way just described.

Definition 7. *Let σ be a solution on $\mathscr{D}$, and Φ be a solution for gradual bargaining problems. We say that Φ is* **compatible** *with σ if for each agenda f that is compatible with $\mathscr{D}$, and for each a, the gradual agreement $\phi = \Phi(f, a)$ satisfies for each t,*

$$(2) \qquad \phi'(t) = \lim_{t' \downarrow t} \frac{\sigma(S_{t'}^f, \phi(t)) - \phi(t)}{t' - t}.$$

Thus the rate of utility gains at a point $\phi(t)$ on the ordinal solution path is the limit of the rate of gains, according to the solution σ, for small problems with $\phi(t)$ being the status quo point.

We now consider two properties of a solution σ of one-shot bargaining problems that guarantee that the ordinal solution be compatible with σ.

A problem (S, d) is *linear* if S is of the form $\{x \mid c(x - d) \le \gamma\}$, for some $c > 0$ in $\mathbb{R}^N$ and positive real number γ. We assume that the domain of σ, $\mathscr{D}$, contains all linear problems. The first property concerns solutions for linear bargaining problems.

Property 1. (Solutions for linear problems) *If $S = \{x \mid c(x-d) \le \gamma\}$, then $\sigma(S, d) - d = \gamma I(nc)$.*

This property is possessed by any solution that is efficient, symmetric and covariant with respect to linear transformation of utility, such as the Nash and the Kalai-Smorodinsky solutions.

The next property concerns the approximation of bargaining problems by linear problems. Consider an agenda f that is compatible with $\mathscr{D}$, and a problem $(S_{t'}^f, d)$. Let $f(d) = t$, and note that $S_{t'}^f = \{x \mid f(x) - f(d) \leq t' - t\}$. We approximate $(S_{t'}^f, d)$ by the linear problem $(\hat{S}_{t'}^f, d)$ where $\hat{S}_{t'}^f = \{x \mid (\nabla f)(d)(x - d) \leq t' - t\}$.

Property 2. (Linear approximation) *Let f be an agenda that is compatible with $\mathscr{D}$. Then*

$$\lim_{t' \downarrow t} \frac{\sigma(S_{t'}^f, d) - d}{t' - t} = \lim_{t' \downarrow t} \frac{\sigma(\hat{S}_{t'}^f, d) - d}{t' - t}.$$

Theorem 3. *The ordinal solution is compatible with any solution σ satisfying Properties 1 and 2.*

The proof of this theorem is straightforward. If a solution σ satisfies Property 1 then $\sigma(\hat{S}_{t'}^f, d) - d = (t' - t)I((n\nabla f)(d))$. Therefore, Property 2 is equivalent, in this case, to requiring that the ordinal solution is compatible with σ.

We prove that the Nash and the Kalai-Smorodinsky satisfy Properties 1 and 2 and hence the following theorem follows.

Theorem 4. *The ordinal solution is compatible with the Nash and the Kalai-Smorodinsky solutions.*

7. Proofs

Proof of Theorem 1. Let Φ be the ordinal solution. To see that it is efficient, let $\phi = \Phi(f, a)$, and denote $\lambda(t) = f(\phi(t))$. Then, by (1) $\lambda'(t) = \sum_{i \in N} \partial f / \partial x_i(\phi(t)) \phi_i'(t) = 1$. Also, $\lambda(f(a)) = f(\phi(f(a))) = f(a)$. Therefore $\lambda(t) = t$.

Next we show that Φ satisfies the ordinal utility axiom, which is stronger than the covariance axiom. Assume that (f, a) and (g, b) are as described in the ordinal utility axiom. Since $g = fs$, it follows that s_i is continuously differentiable for each i. Let $\psi = \Phi(g, b)$. We need to show that $\phi = s\psi$ solves (1) for (f, a). The gradual agreement ψ solves

$$\psi_i'(t) \;=\; \left[n\frac{\partial g}{\partial x_i}(\psi(t))\right]^{-1}$$

$$=\; \left[n\frac{\partial f}{\partial x_i}(s(\psi(t)))\frac{\mathrm{d}s_i}{\mathrm{d}x_i}(\psi_i(t))\right]^{-1}$$

with the initial condition $\psi(g(b)) = b$. Multiplying both sides of the differential equation by $(\mathrm{d}s_i/\mathrm{d}x_i)(\psi_i(t))$ shows that $\phi = s\psi$ solves (1) for (f, a) with the initial condition $\phi(f(a)) = s\psi(f(s(b)) = s\psi(g(b)) = s(b) = a$.

To see why the consistency axiom is satisfied, note that the differential equations for (f, a) and (f, x) differ only in the initial condition. Suppose that for $\phi = \Phi(f, a)$, $\phi(t) = x$. Since the ordinal solution is efficient, $f(x) = t$. It is enough to show that ϕ satisfies the initial condition of (1) for (f, x). Indeed, $\phi(f(x)) = \phi(t) = x$.

It is easy to see that the ordinal solution satisfies the axioms of symmetry and directional continuity.

Conversely, let Φ be a solution that satisfies axioms 1-5. We show that it is the ordinal solution.

(a) If $\alpha \in \mathbb{R}^N$ is symmetric and has positive coordinates, and Φ is a solution that satisfies the symmetry and efficiency axioms, then for the linear function $h(x) = \alpha x + c$, where αx is the scalar product and c a real number, $\Phi'(h, 0)(h(0)) = I(n\alpha)$

Indeed, h is a symmetric agenda, and thus $\Phi(h, 0)$ is symmetric. Fix t, and let $\Phi(h, 0)(t) = (x, \ldots, x)$. By efficiency, $h(x, \ldots, x) = t$. Thus, $x = t(n\,\alpha_1)^{-1}$. Hence, $\Phi'(h, 0)(t) = I(n\alpha)$ for all t, and in particular this holds for $t = h(0)$.

(b) Let g be an agenda for which $\nabla g(0)$ is symmetric. If Φ is a solution that satisfies the axioms of efficiency, symmetry, and directional continuity, then $\Phi'(g, 0)(g(0)) = I\Big(n\nabla g(0)\Big)$.

To see this, let $h(x) = \nabla g(0)x + g(0)$. Then by (a), $\Phi'(h, 0)(h(0)) = I\Big(n\nabla g(0)\Big)$ and therefore it is enough to show that

$$\Phi'(g, 0)(g(0)) = \Phi'(h, 0)(h(0)).$$

To show this we need to use the directional continuity axiom. Fix a neighborhood B of 0 and $\varepsilon > 0$. By the directional continuity of Φ at $(h, 0)$, there exists $\delta > 0$ such that if for each $i \in N$, and $x \in B$,

(3) $$|(\partial f/\partial x_i)(x) - (\partial h/\partial x_i)(x)| \le \delta$$

then $|\Phi'(f, 0)(f(0)) - \Phi'(h, 0)(h(0))| \le \varepsilon$.

We construct an agenda f that satisfies (3) and coincides with g in some neighborhood of 0. By the directional continuity axiom $\Phi'(f,0)(f(0)) = \Phi'(g,0)(g(0))$. Therefore $|\Phi'(g,0)(g(0)) - \Phi'(h,0)(h(0))| \leq \varepsilon$. Since this is true for arbitrary ε, the required equality is established.

To complete the proof of (b) we construct the agenda f. Let q be a continuously differentiable function on $\mathbb{R}$ such that $0 \leq q \leq 1$, $q(r) = 0$ for each $r \leq 0$, and $q(r) = 1$ for each $r \geq 1$. Let M be a uniform bound on $|q'|$ such that $M \geq 1$.

Choose $c > 0$ such that for each $||x|| \leq 2c$, $|(\partial g/\partial x_i)(x) - (\partial g/\partial x_i)(0)| \leq \delta/(4nM)$, for each $i \in N$. Consider the function

$$f(x) = \Big(1 - q(||x||/c - 1)\Big)g(x) + q(||x||/c - 1)h(x).$$

Then, $f(x) - h(x) = \Big(1 - q(||x||/c - 1)\Big)\Big(g(x) - h(x)\Big)$. By the definition of q, f coincides with g for $||x|| \leq c$ and with h for $||x|| \geq 2c$.

We evaluate the size of the terms on the left hand side of this equality, as well as the derivatives of these terms. It is enough to consider only $||x|| \leq 2c$, since for $||x|| \geq 2c$ the difference vanishes. By the definition of q, and since $|\partial||x||/\partial x_i| \leq 1$,

$$\left| \frac{\partial\Big(1 - q(||x||/c - 1)\Big)}{\partial x_i} \right| \leq \frac{M}{c}.$$

For the derivative of the other term,

$$\left| \frac{\partial(g - h)}{\partial x_i}(x) \right| = \left| \frac{\partial g}{\partial x_i}(x) - \frac{\partial g}{\partial x_i}(0) \right| \leq \frac{\delta}{4nM}.$$

Also, $|1 - q(||x||/c - 1)| \leq 1$. Finally, since $g(0) - h(0) = 0$, $|g(x) - h(x)| = ||\nabla g(x') - \nabla h(x')|| \, ||x||$ for some x' with $||x'|| \leq 2c$, and therefore this term is bounded by $\Big(n\delta/(4nM)\Big)(2c)$. Thus,

$$\left| \frac{\partial(f - h)}{\partial x_i}(x) \right| \leq \frac{M}{c} \frac{n\delta}{4nM} 2c + 1 \frac{\delta}{4nM} \leq \delta.$$

This completes the proof of (b).

(c) If Φ satisfies the axioms of symmetry, efficiency, directional continuity, and covariance, then for each problem (f, a), $\Phi'(f, a)(f(a)) = I\Big(n\nabla f(a)\Big)$.

For each $i \in N$, define a linear transformation

$$(4) \qquad\qquad s_i(x_i) = a_i + \left(\frac{\partial f}{\partial x_i}(a) \right)^{-1} x_i.$$

Let $g = fs$. Since $s(0) = a$, it follows by the covariance axiom that

$$(5) \qquad \Phi(f, a) = s\Big(\Phi(g, 0)\Big).$$

It is easy to check that

$$(6) \qquad \nabla g(0) = (1, \ldots, 1),$$

and therefore by (b),

$$(7) \qquad \Phi'(g, 0)(g(0)) = I\Big(n\nabla g(0)\Big)$$

Applying, in this order, (5), (4), (7), and (6), we conclude

$$\begin{aligned}
\Phi'_i(f, a)(f(a)) &= \frac{\mathrm{d}s_i}{\mathrm{d}x_i}\Big(\Phi(g, 0)\Big)(f(a))\Phi'_i(g, 0)(g(0)) \\
&= \left(\frac{\partial f}{\partial x_i}(a)\right)^{-1}\left(n\frac{\partial g}{\partial x_i}(0)\right)^{-1} \\
&= \left(\frac{\partial f}{\partial x_i}(a)\right)^{-1} n^{-1}
\end{aligned}$$

(d) If Φ satisfies axioms 1-5, and $\phi = \Phi(f, a)$, then ϕ satisfies (1).

Let $\phi(t) = x$. By efficiency, $f(x) = t$. By (c), $\Phi'(f, x)(f(x)) = I\big(n\nabla f(x)\big)$. By the consistency axiom $\Phi(f, x) = \Phi(f, a)$, and therefore $\phi'(f(x)) = I\big(n\nabla f(x)\big)$. Substituting t for $f(x)$ in the left hand side, and $\phi(t)$ for x in the right hand side yields (1).

To prove the independence of the axioms, we provide for each a solution that violates it but satisfies the rest. Details are omitted.

Efficiency. Let Φ be the ordinal solution. Fix a real number $c > 0$, and define a solution Ψ by $\Psi(f, a)(t) = \Phi(f, a)(t - c)$. Since the derivative of ϕ is positive, it is strictly increasing. Thus, $f(\Psi(f, a)(t)) < f(\Phi(f, a)(t)) = t$, and Ψ does not satisfy efficiency.

Covariance. Let $g(x) = f(x, \ldots, x)$. Define a solution Ψ by $\Psi(f, a)(t) = (g^{-1}(t), \ldots, g^{-1}(t))$. Obviously Ψ is a solution, and it is easy to see that it satisfies all axioms but covariance.

Symmetry. Let $w = \{w_i\}_{i \in N}$ be a set of non-negative numbers (*weights*) such that $\sum_{i \in N} w_i = 1$. For each problem (f, a) let $\Phi^w(f, a)$ be the solution of $\phi'_i(t) = w_i(\frac{\partial f}{\partial x_i}(\phi(t))^{-1}$ with initial conditions $\phi(f(a)) = a$. Then, by applying Φ^w to the problem $(f, 0)$, where $f(x) = \sum_{i \in N} x_i$, it is easy to see that it satisfies the symmetry axiom if and only if $w_i = 1$ for each i. All other axioms are satisfied by Φ^w for each w.

Directional Continuity. Fix a non-symmetric weight vector w. Consider a solution that coincides with the ordinal solution for each problem (f, a) if there exists a monotonic transformation s, as in the axiom of ordinal utility, such that fs is a symmetric function. For all other problems (and there are indeed such problems) the solution is represented by Φ^w. This solution satisfies all axioms but directional continuity.

Consistency. Define a solution Ψ such that for each problem (f, a) and time t, $\Psi(f, a)(t)$ is the Nash solution for the one-shot problem (S_t^f, a). ∎

Proof of Theorem 2. In the proof of Theorem 1 we showed that the ordinal solution satisfies the ordinal utility axiom.

Let f be an agenda, and λ an increasing real valued function such that λf is also an agenda. Then, λ must be differentiable. Denote $g = \lambda f$, and let $\phi = \Phi(g, a)$.

We show that $\psi(t) = \phi(\lambda(t))$ solves (1) for f. Note that for each t, $g(\phi(t)) = t$, which implies that $f(\psi(t)) = t$. Now,

$$
\begin{aligned}
\psi'(t) &= \phi'(\lambda(t))\lambda'(t) \\
&= \lambda'(t)I\Big(n\nabla(g(\psi(t)))\Big) \\
&= \lambda'(t)I\Big(n\lambda'(f(\psi(t))\nabla(f(\psi(t)))\Big) \\
&= I\Big(n\nabla(f(\psi(t)))\Big).
\end{aligned}
$$

Also, $\psi(f(a)) = \phi(\lambda(f(a))) = \phi(\lambda(\lambda^{-1}(g(a)))) = a$. ∎

Proof of Theorem 4. Consider an agenda f and a point d such that $f(d) = t$.

We first examine the case that σ is the Nash solution. For $t' > t$, $\sigma(S_{t'}^f, d)$ is the point x at which the function $g(y) = \Pi_{i=1}^n(y_i - d_i)$ attains a maximum on $S_{t'}^f$. At the point x, $f(x) = t'$, and the direction of the gradients of g and f coincide. As the gradient of g at x is $g(x)I(x - d)$, it follows that $x - d$ is in the same direction as $I(\nabla f(x))$. Hence, there is an $\alpha = \alpha(t')$ such that $x = d + \alpha I(\nabla f(x))$. Therefore $f(d + \alpha I(\nabla f(x))) - f(d) = t' - t$. We conclude that $\alpha \nabla f(d)I(\nabla f(x)) + o(t' - t) = t' - t$. As $\nabla f(x) \to_{t' \downarrow t} \nabla f(d)$, it follows that $\alpha/(t' - t) \to_{t' \downarrow t} = 1/n$. Finally,

$$
[\sigma(S_{t'}^f, d) - d]/(t' - t) = \alpha I(\nabla f(x))/(t' - t) \to_{t' \downarrow t} I(n\nabla f(d))),
$$

which establishes (2) for the the Nash solution.

Assume now that σ is the Kalai-Smorodinsky solution. For each i, let b_i be the number that satisfies

$$(8) \qquad\qquad f(d + b_i e_i) = t',$$

where e_i is a unit vector along the i axis. Let $b = \sum_{i \in N} b_i e_i$. Then $\sigma(S_{t'}^f, d)$ is the point on the efficient frontier of S_t^f in the direction b from d. Therefore, $\sigma(S_{t'}^f, d) - d = \alpha b$ for some number α, such that

$$(9) \qquad\qquad f(d + \alpha b) = t'.$$

By (8), $f(d + b_i e_i) - f(d) = t' - t$, and therefore $b_i(\partial f / \partial x_i)(d) + o(t' - t) = t' - t$. Hence, $b \rightarrow_{t' \downarrow t} I(\nabla f(d))$.

By (9), $f(d + \alpha b) - f(d) = t' - t$, and therefore, $\alpha \nabla f(d) b + o(t' - t) = t' - t$. Hence, $\alpha / (t' - t) \rightarrow_{t' \downarrow t} 1/n$. Finally, $[\sigma(S_{t'}^f, d) - d] / (t' - t) = \alpha b / (t' - t) \rightarrow_{t' \downarrow t} I(n \nabla f(d)))$, as was to be to shown. $\blacksquare$

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