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# EFFICIENT MECHANISMS FOR MULTIPLE PUBLIC GOODS

by

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# Efficient Mechanisms for Multiple Public Goods*

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**Abstract**

We propose two sequential mechanisms for efficient production of public goods. Our analysis differs from the existing literature in allowing for the presence of multiple public goods and in also being "simple." While both mechanisms ensure efficiency, the payoffs in the first mechanism are asymmetric, being sensitive to the order in which agents move. The second mechanism corrects for this through a two-stage game where the order of moves in the second stage are randomly determined. The payoffs from the second mechanism correspond to the Shapley value of a well-defined game which summarizes the production opportunities available to coalitions in the economy.

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# 1 Introduction

Designing incentive compatible mechanisms that lead to efficient production of public goods has long been an important topic in the theoretical literature of public economics. Within this literature, various mechanisms have been proposed. Among others, Bagnoli and Lipman [3] proposed a mechanism that fully implements the core in a discrete and non-excludable public good economy and Jackson and Moulin [6] introduced a mechanism for the single indivisible public good economy to achieve efficient and equitable allocations. Abreu and Sen [1], Maniquet [7] and Moore and Repullo [9] construct mechanisms in general economic environments that under certain conditions can implement general social choice functions in the public goods framework.

Our paper is motivated by the same objectives but we depart from the existing literature in a number of ways. First, our framework is rather general. In contrast to the single public good case to which most of the literature confines itself, we are dealing with multiple public goods on which consumers' preferences are not necessarily separable. Our model can be interpreted in two different ways. On the one hand, we can think of the multiple public goods as involving intrinsically different goods like public school education, roads, national defence and so on. We can also interpret the model as that of a single public good with multiple attributes. For example, suppose that a common computer laboratory is required to serve various department of a firm or university. This means deciding on the amounts to be spent on each attribute like software, hardware, furniture, programmers and so on. Thus, even when the public good problem involves a single facility, the multiple public goods model might be the appropriate way of describing the problem.

Second, while existing literature distinguishes between the excludable public good case (for which agents' consumption can be restricted at zero cost, e.g. zoos, museums, cable TV etc.) and the non-excludable case (e.g. clean air, defence and security), our framework allows for both type of goods. It even allows some of the goods to be excludable and others non-excludable within the same problem.

Third, our concern is with simple mechanisms, where simplicity is meant to be in terms of the information that participants are requested to submit to the planner. Thus in contrast to standard mechanisms (like Groves-Clarke or Groves-Ledyard mechanisms) where players are required to submit a full utility function over all possible levels of the public good, in ours players are only required to announce their desirable level of the public good/s, and subscribe to a contribution that will be paid conditional on their demand being met. In addition to their simplicity we believe that the operation of our mechanism resemble the way real life collective decisions on public good production are often made. Given their efficiency, they become natural candidates for application in various concrete real life environments. Consider for instance, the problem of building a common infrastructure for use by cellular phone companies in a country.[1] The problem is one of deciding where to locate the transmission towers and how much to charge each company. Since each company has different preferences concerning location because of the different geographical profiles of its customers, we believe that this problem is best treated one involving multiple public goods. The government can divide the country into different geographical zones with each zone being treated as a public good. If our mechanism is used, each company would be required to submit a list of the public goods it requires along with a cost contribution. Another potential application, to which we have already alluded to, is the building of a common computer facility. In this case, each department would be required to submit a list of the attributes it requires along with a cost contribution.

We will be interested in two properties to be satisfied by the mechanism, efficiency and monotonicity with respect to individual preferences. The first property concerns the public good level while the second concerns the allocation of costs. By efficiency we require that the mechanism overcomes the free riding problem. Namely, the equilibrium behavior within the mechanism results in the socially optimal vector of public goods. The property of monotonicity requires that if player $i$'s marginal utility for every good at all points exceeds player $j$'s marginal utility for every good, then player $i$ ends up having a higher net utility in equilibrium. In particular, it requires that if two players share the same preferences regarding the public goods, then they will end up having the same net utility, and thus will also be making the same contribution. In addition to these two properties, we argue that in special cases, our mechanism satisfies a third property of core stability, i.e., the resulting outcome is immune to deviations by coalitions. Indeed this property will hold whenever the problem involves a single excludable public good (as shown by Bag and Winter [2]). We note that no mechanism satisfies this property on the whole domain as such outcomes may not exist in the general case.

This paper is related to the literature on subgame perfect implementation, specifically the papers of Abreu and Sen [1], Bag and Winter [2], Moore and Repullo [9] and Maniquet [7]. Abreu and Sen [1] and Moore and Repullo [9] have provided necessary and sufficient conditions for a general social choice function to be implementable in subgame perfect equilibrium. Their mechanisms are quite complex relying on constructs such as integer games.

Maniquet [7], in a recent contribution, characterizes the set of anonymous

[1] A common infrastructure is preferable to the alternative of having each company building it's own infrastructure on two counts. Firstly, it is efficient. Secondly, the potential health hazards are minimized by having a common infrastructure.

and individually rational social choice functions implementable through a "Divide and Challenge" mechanism. In the first stage of this $n$-stage mechanism, an agent (the "divider") proposes an allocation along with the preference profile of all agents. In subsequent stages, the other $n-1$ agents (the "choosers") can challenge the divider by stating the name of an agent for whom the stated preference is incorrect and proposing an alternate allocation. While this mechanism is applicable in our context as well, it needs at least three agents to be present. It also requires the announcement of the entire preference profile. In contrast, our mechanisms are valid even for the case of two agents. Also, an agent's announcement consists only of a desired vector of public goods and a cost contribution which is "simpler" than having to announce the entire preference profile.

Finally, our paper is related to that of Bag and Winter [2] who propose two related mechanisms for the single excludable public good case. While simpler, the proposed mechanism there relies on the presence of a single good which is excludable. Indeed, their mechanisms do not work if the public good is pure or if there are even multiple excludable public goods. Our mechanisms, in contrast, work even if the public good is pure, or if there are multiple public goods.

We organize the remainder of the paper as follows. In Section 2 we set up the model of general multiple public goods problem. In Section 3 we define two sequential mechanisms in which players move in turns by making bids regarding the requested level from each of public goods and conditional contribution. The first auxiliary mechanism (called $\Gamma_m$) yields an efficient outcome in equilibrium but the equilibrium outcome may not be equitable as players moving earlier have an advantage. The second mechanism (called $\Gamma_m^1$) corrects for this asymmetry by allowing the mechanism to be played twice with the last period order being chosen randomly. Section 4 contains the formal analysis of the two mechanisms defined in Section 3. We demonstrate the operation of these mechanisms in Section 5 on a prominent 3-agent, 3-good problem proposed by Moulin [10]. In Section 6 we discuss conditions under which the equilibrium outcome of the mechanism is core stable. In particular, we demonstrate that this property holds whenever the cost function is submodular and the utility functions are supermodular. We conclude in Section 7.

## 2 The Model

Let $N = \{1, 2, \ldots, n\}$ denote the set of agents. There are $p$ public goods in the economy. The set of *feasible vectors* of public goods for the economy are located in $Y \subset \mathbb{R}_+^p$. We shall assume that $Y$ is non-empty, unbounded and contains $\mathbf{0} = (0, \ldots, 0)$. A generic element of $Y$ shall be denoted as $y$. Our formulation allows for the fact that some of the public goods may be "lumpy" in that they can be consumed in discrete units only. It also permits a given public good to be pure or excludable: the analysis is not affected. In addition to the public goods, there is also a private, perfectly divisible good in the economy which can be interpreted as "money."

The preferences of the individuals are quasi-linear taking the form $U_i(y, x_i) = v_i(y) - x_i$ where $v_i(y)$ is $i$'s valuation (in units of the private good) for the vector $y$ of public goods; then $x_i$ has a natural interpretation as his contribution towards the cost of production. The technology in the economy is given by a production function $f$ with an associated cost function $c$. The following restrictions are imposed on the preferences and the technology.

**Assumption 1** *1. The function $v_i$ is continuous, non-decreasing, bounded and normalized such that $v_i(\mathbf{0}) = 0$ for all $i \in N$.*

*2. The cost function $c$ is continuous, non-decreasing and satisfies $c(\mathbf{0}) = 0$. Also, $c(y) \to \infty$ as $y \to \infty$.*

**Remark 1** The assumption that the function $v_i$ is bounded may appear to be strong. However, it is reasonable if we assume that each agent in the economy has a finite endowment of the private good. One can weaken this requirement: basically, what we need are conditions which ensure that the optimal vector of public goods for any subset of agents is not unbounded. Since the analysis does not depend on this assumption, we do not pursue this matter further in this paper. Note also that the restrictions on the technology are minimal.

We end our discussion of the model by introducing some notation which will be used in this paper. The subset of players $\{k, k+1, \ldots, n\}$ is denoted $S_k$ and $S_{n+1}$ indicates the empty set. The announcement of agent $i$ in a mechanism is indicated by the use of a subscript as in $w_i, z_i$ and so on while the $n$-tuple of all agents' announcements is denoted as $w, z$ and so on. The term "announcement" is used interchangeably to indicate the announcement of a particular player and also to the vector of announcements of all players. This should not cause any confusion as the context will make it clear as to which usage is relevant.

# 3 The mechanisms $\Gamma_m$ and $\Gamma_m^1$

Implementation Theory distinguishes between two distinct scenarios, *complete information environments* and *incomplete information environments*. In both scenarios, the planner who wishes to implement a social choice function meeting her objectives is ignorant about the environment. In the *complete information* setting, all agents know the environment while in the *incomplete information* setting, agents may have private information so that an agent may not know the true environment either. In our setting, all agents including the planner know the technology while the preferences of an agent is known to all other agents excluding the planner: thus, the setting is one of complete information. The fact that agents know each other's preferences is used in a crucial way in our mechanisms.

We now define the mechanisms that we analyze subsequently. In the mechanism $\Gamma_m$, agents move sequentially and agent $k$, when it is her turn to move, chooses a tuple $w_k = (y^k, x_k)$ where $y^k$ is the vector of public goods she wishes to consume and $x_k$ is her contribution towards the cost of production conditional on her demands being met. We assume that the agents move in the "natural order," viz. first, agent 1 moves, then agent 2, and so on. As will become clear, the analysis can easily be reinterpreted to cover the case where the agents move in a different order.

**Notation**: Given a collection $\{y^k\}_{k\in S}$, we denote by $\hat{y}_S$ the vector $(\hat{y}_1, \ldots, \hat{y}_p)$ where $\hat{y}_i = \max_{k\in S} y_i^k$ for $i = 1, \ldots, p$.

**Definition 1** *The coalition $S$ is* compatible *with the announcement $w = \{(y^k, x_k)\}_{k\in N}$ if $\sum_{k\in S} x_k \geq c(\hat{y}_S)$.*

Thus, a coalition $S$ is compatible if the total contributions of the members of $S$ is sufficient to produce a vector of public goods such that the demands of all members of $S$ can be satisfied.

Once all players have announced, the planner selects the largest compatible coalition in the set $\{S_1, \ldots, S_{n+1}\}$ which is called the *maximal compatible coalition* and denoted $S^*(w)$.[2] He then announces that the vector $y^*(w) = \hat{y}_{S^*(w)}$ is to be produced (obviously, $y^*(w) = \mathbf{0}$ if $S^*(w) = \emptyset$) and charges the agents as follows:

$$x_k^*(w) = \begin{cases} x_k & \text{if } k \in S^*(w), \\ 0 & \text{otherwise.} \end{cases}$$

The mechanism $\Gamma_m^1$, which we define now, is a two-stage game. In the first stage, agents play the game $\Gamma_m$ according to a randomly selected order of the agents. At the end of the first stage, each agent is asked whether she would like to replay the game. If all agents answer "NO," then the game ends. If at least one agent answers "YES," then an order of the agents is chosen at random from the $n!$ possibilities, and the agents replay the game $\Gamma_m$ according to this randomly chosen order. The game ends after the optional second stage.

# 4 The Results

In this section we show that every subgame perfect equilibrium (hereafter, SPE) of the mechanisms $\Gamma_m$ and $\Gamma_m^1$ results in an efficient production of the

[2]Since $S_k \subset S_{k+1}$ for all $k = 1, \ldots, n+1$ and $S_{n+1} = \emptyset$ is compatible by definition, the maximal compatible coalition always exists.

public goods. For the mechanism $\Gamma_m^1$ we also show that the equilibrium payoffs correspond to the Shapley value of a cooperative game that summarizes the production opportunities of coalitions in the economy.

We start with the following preliminaries. Consider the optimization problem for any $S \subset N$:

$$\max_{y \in Y} \sum_{k \in S} v_k(y) - c(y) \tag{1}$$

Assumption 1 ensures that a solution exists for this problem. Since $v_k$ is non-negative and bounded for all $k \in N$, there exists $M > 0$ such that $v_k(y) \leq M$ for all $k \in N$ and all $y \in Y$. Let $Y^* = \{y \in Y | c(y) \in [0, nM]\}$. Since $c$ is continuous and $c(y) \to \infty$ as $y \to \infty$, it follows that $Y^*$ is both closed and bounded and therefore, compact.[3] The optimization problem (1) thus reduces to one where the maximum is taken over all $y \in Y^*$. The compactness of $Y^*$ and the continuity of $v_k, k = 1, \ldots, n$ and $c$ ensures that the maximum exists. We shall use the notation $y_S^*$ to signify an efficient vector of public goods for $S$, that is, $y_S^*$ solves (1). Of course, $y_S^*$ need not be unique.

**Definition 2** *The* stand alone *payoff to a coalition $S$ is defined as*

$$sa(S) = \begin{cases} \sum_{k \in S} v_k(y_S^*) - c(y_S^*) & \text{if } S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The stand-alone payoff summarizes the aggregate payoff possibilities for each coalition within the economy. We therefore have a natural associated cooperative game for our economy, given by $(N, sa)$.

**Definition 3** *The "marginal contribution" of agent $k$, $k = 1, \ldots, n$ is $u_k^* = sa(S_k) - sa(S_{k+1})$.*

**Remark 2** Assumption 1 implies that $0 \leq sa(S) \leq sa(T)$ if $S \subset T$. The coalition $T$ can always produce $y_S^*$, and thus by Assumption 1, $sa(T) \geq \sum_{i \in T} v_i(y_S^*) - c(y_S^*) \geq sa(S)$. The first inequality follows because $\mathbf{0}$ is a feasible production plan for any coalition and therefore, using Assumption 1 once more, $sa(S) \geq \sum_{i \in S} v_i(\mathbf{0}) - c(\mathbf{0}) \geq 0$.

The main results concerning the mechanisms $\Gamma_m$ and $\Gamma_m^1$ can be summarized in the following theorems.

**Theorem 1** *In all SPE of $\Gamma_m$, an efficient vector of public goods is produced. Also, if $y^*$ is an efficient vector of public goods, then there exists a SPE which supports $y^*$ as an equilibrium outcome. Furthermore, the net payoffs of the agents in all SPE are uniquely given by $(u_1^*, \ldots, u_n^*)$.*

[3] Note also that since $c(\mathbf{0}) = 0$, $\mathbf{0} \in Y^*$.

**Theorem 2** *In all SPE of $\Gamma_m^1$, an efficient vector of public goods is produced. Also, if $y^*$ is an efficient vector of public goods, then there exists a SPE which supports $y^*$ as an equilibrium outcome. Furthermore, the net payoffs to the agents in all SPE are given uniquely by the Shapley value of $(N, sa)$.*

Let us first consider the mechanism $\Gamma_m$. Suppose that agents in $\{1, \ldots, j\}$ have already announced $\{w_k = (y^k, x_k)\}_{k=1}^{j}$. Let $i \in \{1, \ldots, j\}$ be such that

$$\sum_{k=j+1}^{n} v_k(\tilde{y}^j) - c(\tilde{y}^j) + \sum_{k=i}^{j} x_k > sa(S_{j+1}) \tag{2}$$

for some $\tilde{y}^j$ such that $\tilde{y}^j \geq y^k$ for all $k = i, \ldots, j$.

To understand what's being said here, note that agents in $S_{j+1}$ can expect a collective net payoff of at most $sa(S_{j+1})$ without the cooperation of other agents. If an $i \leq j+1$ satisfying (2) exists, then it means that the agents in $S_{j+1}$ can obtain a collective payoff strictly greater than $sa(S_{j+1})$ by cooperating with agents in $\{i, i+1, \ldots, j\}$.

Let $i_j$ be the largest integer in $\{1, \ldots, j\}$ satisfying (2). Note that $i_j$ need not exist at all. However, if $i_j$ exists, then $S_{i_j}$ is the *minimal* set of agents who have to be included in the maximal compatible coalition if the agents following $j$ are to obtain a collective payoff greater than $sa(S_{j+1})$.[4] In other words, if it turns out that the resulting maximal compatible coalition in some SPE of the subgame following $j$'s announcement is not a superset of $S_{i_j}$), then the agents in $S_{j+1}$ must be receiving a net payoff less than or equal to $sa(S_{j+1})$.

The following lemma says that if $i_j$ exists, then all SPE of the subgame following $j$'s announcement will be such that the resulting maximal compatible coalition is a superset of $S_{i_j}$.

**Lemma 1** *Suppose that agents in $\{1, \ldots, j\}$ have announced $\{(y^k, x_k)\}_{k=1}^{j}$ in the mechanism $\Gamma_m$ and that $i_j$ exists. Then, all SPE of the subgame following $j$'s announcement will be such that the resulting maximal compatible coalition is a superset of $S_{i_j}$.*

**Proof**: We proceed by induction on $j$. We will show that if $S_{i_j}$ is not a compatible coalition, then there exists a profitable deviation for some agent in $S_{j+1}$ in the subgame following $j$'s announcement.

If $j = n$, then there is no subgame following $n$'s announcement.[5] It is easy to verify that (2) implies that $S_{i_n}$ is a compatible coalition in this case.

[4] Compatibility requires that if any of the agents preceding $j+1$ are part of the maximal compatible coalition, then they necessarily must form a "connected" set of agents.

[5] Remember that agents are announcing in the order $1, 2, \ldots, n$ so that $n$ is the last to announce.

It follows that that the resulting maximal compatible coalition is a superset of $S_{i_n}$.

Assume that the lemma is true for all $j > J$ and suppose that $i_J$ exists but that the maximal compatible coalition is not a superset of $S_{i_J}$ in some SPE of the subgame following $J$'s announcement. Let $(u_{J+1}, \ldots, u_n)$ be the corresponding net payoffs to the agents following agent $J$. Since the maximal compatible coalition is not a superset of $S_{i_J}$, we must have $\sum_{k=J+1}^{n} u_k \leq sa(S_{J+1})$. We can distinguish between two cases.

**Case 1**: There exists $k > J$ such that $u_k < u_k^*$.

Let agent $k$ deviate by announcing $(y_{S_k}^*, x_k')$ where[6]

$$sa(S_{k+1}) - \sum_{i=k+1}^{n} v_i(y_{S_k}^*) + c(y_{S_k}^*) < x_k' < v_k(y_{S_k}^*) - u_k. \quad (3)$$

The upper bound on $x_k'$ is strictly greater than the lower bound if and only if

$$v_k(y_{S_k}^*) - u_k > sa(S_{k+1}) - \sum_{i=k+1}^{n} v_i(y_{S_k}^*) + c(y_{S_k}^*). \quad (4)$$

Since $\sum_{i=k}^{n} v_i(y_{S_k}^*) - c(y_{S_k}^*) = sa(S_k)$, it follows that (4) is true if and only if $u_k < sa(S_k) - sa(S_{k+1}) = u_k^*$ which is true by assumption. Thus, a value of $x_k'$ satisfying (3) exists.

Since $\sum_{i=k+1}^{n} v_i(y_{S_k}^*) - c(y_{S_k}^*) + x_k' > sa(S_{k+1})$, it follows that $i_k = k$ after $k$'s deviation. Since $k > J$, the induction hypothesis implies that the resulting maximal compatible coalition in any SPE of the subgame following $k$'s deviation will be a superset of $S_{i_k} = S_k$. Compatibility implies that the resulting vector of public goods, say $y$, will be such that $y \geq y_{S_k}^*$. By Assumption 1, we have $v_k(y) - x_k' \geq v_k(y_{S_k}^*) - x_k' > u_k$. Thus, $k$ has a profitable deviation.

**Case 2**: $u_k = u_k^*$ for all $k = J+1, \ldots, n$.

Let agent $J+1$ deviate by announcing $(\tilde{y}^J, x_{J+1}')$ where $x_{J+1}'$ satisfies the following conditions.[7]

1. $sa(S_{J+2}) - \sum_{i=J+2}^{n} v_i(\tilde{y}^J) + c(\tilde{y}^J) - \sum_{i=i_J}^{J} x_i < x_{J+1}'$,

2. $x_{J+1}' < v_{J+1}(\tilde{y}^J) - sa(S_{J+1}) + sa(S_{J+2})$.

The upper bound on $x_{J+1}'$ is strictly greater than the lower bound if and only if

$$\sum_{i=J+1}^{n} v_i(\tilde{y}^J) - c(\tilde{y}^J) + \sum_{i=i_J}^{J} x_i > sa(S_{J+1}) \quad (5)$$

[6] Recall that for any $S \subset N$, $y_S^*$ denotes an efficient vector of public goods for $S$.

[7] Recall that $\tilde{y}^J$ is a vector of public goods satisfying (2) and such that $\tilde{y}^J \geq y^k$ for $k = i_j, \ldots, j$.

which is true because $\tilde{y}^J$ satisfies (2).

Since $\sum_{i=J+2}^{n} v_i(\tilde{y}^J) - c(\tilde{y}^J) + \sum_{i=i_J}^{J} x_i + x'_{J+1} > sa(S_{J+2})$, it follows that $i_{J+1}$ exists after $J+1$'s deviation. The induction hypothesis implies that the resulting maximal compatible coalition in any SPE of the subgame following $J+1$'s deviation will be a superset of $S_{i_{J+1}}$. Since $i_{J+1} \leq J+1$, it follows that $J+1$ is a member of the resulting maximal compatible coalition. If $y$ is the resulting vector of public goods, then compatibility implies that $y \geq \tilde{y}^J$. By Assumption 1, we have $v_{J+1}(y) - x'_{J+1} \geq v_{J+1}(\tilde{y}^J) - x'_{J+1} > sa(S_{J+1}) - sa(S_{J+2}) = u^*_{J+1}$. Thus, $J+1$ has a profitable deviation. This completes the proof of the lemma. ■

**Corollary 1** *Let $(u_1, \ldots, u_n)$ be the net payoffs to the agents in some SPE of $\Gamma_m$. Then, $u_k \geq u^*_k$ for all $k \in N$.*

**Proof**: Suppose that $u_k < u^*_k$ for some $k$. Let agent $k$ deviate by announcing $(y^*_{S_k}, x_k)$ where

$$sa(S_{k+1}) - \sum_{j=k+1}^{n} v_j(y^*_{S_k}) + c(y^*_{S_k}) < x_k < v_k(y^*_{S_k}) - u_k \tag{6}$$

The upper bound on $x_k$ is strictly greater than the lower bound if and only if $u_k < sa(S_k) - sa(S_{k+1})$ which is true by assumption. Since $\sum_{j=k+1}^{n} v_j(y^*_{S_k}) - c(y^*_{S_k}) + x_k > sa(S_{k+1})$, it follows that (2) is satisfied for $i_k = k$. Lemma 1 implies that the maximal compatible coalition in any SPE resulting after $k$'s deviation must include $S_k$. If $y$ is the resulting vector of public goods, then by compatibility, $y \geq y^*_{S_k}$. Thus, by Assumption 1, we have $v_k(y) - x_k \geq v_k(y^*_{S_k}) - x_k > u_k$. This shows that $k$ has a profitable deviation, which is a contradiction. ■

We are now in a position to prove Theorems 1 and 2.

**Proof of Theorem 1**: Let $(u_1, \ldots, u_n)$ be the net payoffs to the agents in some SPE of the game $\Gamma_m$. Let $y^*$ be the corresponding vector of public goods produced. Corollary 1 implies that $u_i \geq u^*_i$ for all $i \in N$ in any SPE of $\Gamma_m$. If $u_i > u^*_i$ for some $i$ then we must have $\sum_{j=1}^{n} u_j = \sum_{j=1}^{n} v_j(y^*) - c(y^*) > sa(N)$. However, Remark 2 which follows from Assumption 1 shows that the maximum possible stand-alone payoff is $sa(N)$. The contradiction shows that $u_i = u^*_i$ for all $i \in N$. The fact that the efficient vector of public goods is produced follows trivially from the observation that $\sum_{i=1}^{n} u^*_i = sa(N)$.

Suppose now that $y^*$ is an efficient vector of public goods. Consider the strategy profile $w$ where $w_i = (y^*, x^*_i)$, $x^*_i = v_i(y^*) - u^*_i$ for all $i = 1, \ldots, n$. We claim that $w$ is a SPE of $\Gamma_m$. Suppose not. Then, there exists $j$ such that all agents prior to $j$ announce $w_i, (i = 1, \ldots, j-1)$, and $j$ announces $w'_j \neq w_j$. Following $j$'s announcement, we then have a subgame. Let $(v_1, \ldots, v_n)$ be the net payoffs of the agents in the equilibrium resulting from this subgame.

Note that we can replicate the arguments in Lemma 1 to conclude that $v_k \geq u_k^*$ for all $k > j$. If $v_j \leq u_j^*$, then clearly, $j$ has not benefited from the deviation. So suppose that $v_j > u_j^*$. Then, we have $\sum_{l=j}^{n} v_l > sa(S_j)$. Now, if the maximal compatible coalition is also $S_j$, then we have a contradiction. So, suppose that the maximal compatible coalition is $S_t$ for some $t < j$. Note that $v_l \geq u_l^*$ for all $t \leq l < j$.[8] Thus, we have $\sum_{l=t}^{n} v_l > sa(S_t)$ which is once again, a contradiction. This completes the proof of the theorem. ■

**Proof of Theorem 2**: Let $(\phi_1(sa), \ldots, \phi_n(sa))$ denote the Shapley value payoffs to the agents in $(N, sa)$. Suppose the game enters Stage 2 of $\Gamma_m^1$. We know from Theorem 1 that for any order selected by the planner, the SPE payoffs will be given by the "marginal contribution" vector. Since each order is equally likely, it follows that the expected payoff to any agent at the beginning of the second stage is exactly his Shapley value in the game $(N, sa)$.

Consider now the agents' decisions at the beginning of Stage 1. If any agent gets less than her Shapley value payoff at the end of Stage 1, then she will force the game into the second stage. It thus follows that the strategy profile $(\{w_i^* = (y_N^*, x_i^*)\}, \text{"NO"})_{i=1}^n$ where $x_i^* = v_i(y_N^*) - \phi_i(sa)$ constitutes a SPE of the game $\Gamma_m^1$. However, this may not be a unique SPE as it is possible that some agent is indifferent between the game ending in the first stage and getting his Shapley value payoff and getting the same payoff in expected terms in Stage 2. (Note that the efficient vector of public goods is produced in either case.) If we assume that all agents have a lexicographic preference for the game ending in the first stage, then all SPE of the game $\Gamma_m^1$ will end in the first stage with the production of the efficient vector of public goods, and the agents getting their Shapley value payoffs.■

**Remark 3** If agents discount the future instead of having a lexicographic preference for the game ending in stage 1, then the game $\Gamma_m^1$ always ends in the first stage and the payoffs approach the Shapley value as the discount factor approaches one.

Let $0 < \beta < 1$ be the discount factor. If the game $\Gamma_m^1$ reaches the second stage, then the expected payoffs to the agents at the beginning of Stage 2 are obviously $\beta(\phi_1(sa), \ldots, \phi_n(sa))$. Thus, if $i$'s payoff at the end of stage 1 is strictly less than $\beta\phi_i(sa)$, then she will move the game to the second stage. Therefore, the optimal strategy for player $i$ in stage 1 is to announce so as to leave exactly the second stage payoffs $(\sum_{j=i+1}^{n} \beta\phi_j(sa))$ to the agents following her. Using the argument recursively, it follows that agent 1 will expropriate the entire surplus that accrues on account of time discounting. Note that the game cannot go to the second stage because agent 1 would prefer to concede a little to the agents following her rather than having the

[8]The vector of public goods produced must be $y \geq y^*$ since all members in $\{t, \ldots, j-1\}$ announce $y^*$ as the desired vector of public goods. Since the utility functions are non-decreasing, $v_l(y) - x_l \geq v_l(y^*) - x_l = u_l^*$ for all $l = t, \ldots, j-1$.

game go to the second stage.[9] Therefore, with discounting, the game $\Gamma_m^1$ always ends in the first stage with production of an efficient vector of public goods and net payoffs $(sa(N)-\beta\sum_{j=2}^{n}\phi_j(sa), \beta\phi_2(sa), \ldots, \beta\phi_n(sa))$. Thus, as $\beta \to 1$, the payoffs converge to the Shapley value.

**Remark 4** The symmetry property of the Shapley value implies that if two agents have the same preferences, then they receive the same net utility from the mechanism $\Gamma_m^1$. More generally, the mechanism $\Gamma_m^1$ has the following "monotonicity property": if agents $i$ and $j$ are such that $i$'s marginal utility at all points is at least as high as $j$'s marginal utility, then $i$'s net payoff must be at least as high as $j$'s net payoff. It is easy to check that if $i$'s marginal utility is higher than $j$'s marginal utility at all points, then this implies that $z(S \cup \{i\}) - z(S) \geq z(S \cup \{j\}) - z(S)$ for all $S \subset N \backslash \{i, j\}$. A simple computation using the formula for the Shapley value then implies that $i$'s payoff is at least as large as $j$'s payoff.

# 5 An Example

We now consider an example due to Moulin [10] which illustrates the operation of the two mechanisms considered above. Let $N = \{1, 2, 3\}$ and let the set of public goods be given by $\{a, b, c\}$. Following Moulin [10], we can interpret the public goods as being "street-lights." Let $a$ be the street-light on the road between 1 and 2, $b$ be the street-light on the road between 1 and 3, and $c$ the street-light on the road between 2 and 3. We shall say that the street-light $x$ is adjacent to agent $i$ if $x$ is located on a road connecting $i$ and some other agent $j$. The utility function of agent $i$ is given as follows:

$$U_i = \begin{cases} 30 & \text{if there is one street-light adjacent to } i, \\ 45 & \text{if there are two street-lights adjacent to } i, \\ 0 & \text{otherwise.} \end{cases}$$

The cost of production is a uniform \$40 per street-light. It is easily seen that in this example $sa(\{i\}) = 0, sa(\{i, j\}) = 20$ and $sa(N) = 25$. This follows because the optimal action for an individual agent is not to construct any street-light; for any two agents to construct the street-light on the road connecting them; and for the grand coalition to construct *any two* street-lights. Observe that the optimal vector of public goods is not uniquely defined for the grand coalition.

Consider the mechanism $\Gamma_m$ where the agents move in the order 1, 2, 3. Using Theorem 1, it follows that the equilibrium net payoffs to the

[9] If the game does go to a second stage, then agent 1 can deviate by conceding a fraction $0 < \alpha < 1$ of the surplus $(S = sa(N) - \beta\sum_{j=2}^{n}\phi_j(sa))$ to agents following her, agent 2 can follow by conceding some of $(1-\alpha)S$ to the subsequent agents and so on. This ensures that the game ends in the first stage and agent 1 is strictly better off from the deviation. This argument has been used before in Bag and Winter [2].

agents are $(u_1^*, u_2^*, u_3^*) = (5, 20, 0)$. Similarly, using Theorem 2, it follows that the equilibrium net payoffs in the game $\Gamma_m^1$ are given by $(v_1^*, v_2^*, v_3^*) = (8\frac{1}{3}, 8\frac{1}{3}, 8\frac{1}{3})$.

Note that the equilibrium *strategies* cannot be uniquely specified in the two mechanisms. For instance, in the mechanism $\Gamma_m$, the strategy profiles $\{\{(a,b), 40\}, \{(a,b), 10\}, \{(a,b), 30\}\}$ and $\{\{(a,c), 25\}, \{(a,c), 25\}, \{(a,c), 30\}\}$ are both equilibrium strategy profiles.[10] Similarly, in the mechanism $\Gamma_m^1$, the strategy profiles $\{\{(a,b), 36\frac{2}{3}\}, \{(a,b), 21\frac{2}{3}\}, \{(a,b), 21\frac{2}{3}\}\}$ and $\{\{(a,c), 21\frac{2}{3}\}, \{(a,c), 36\frac{2}{3}\}, \{(a,c), 21\frac{2}{3}\}\}$ are both equilibrium strategy profiles. It is a feature of both the $\Gamma_m$ and $\Gamma_m^1$ mechanisms that even though the equilibrium strategies cannot be specified uniquely, the equilibrium payoffs are nonetheless unique. Furthermore, the vector of public goods produced in equilibrium is always optimal.

Note also that the core of the game $(N, sa)$ is empty. This brings forward an important point: the mechanisms that we have proposed may not be immune to coalitional deviations. To see this more clearly, imagine that the public goods in this example are excludable and that the technology is freely available to all coalitions. Observe that the coalition $\{1, 3\}$ collectively gets a payoff of 5 in any SPE of $\Gamma_m$ (assuming that agents move in the order 1, 2, 3). However, the coalition $\{1, 3\}$ can do better by opting to produce the good $b$ by themselves and sharing the cost equally. This leaves both 1 and 3 with a surplus of 10 which is more than what they get in any SPE of $\Gamma_m$. Similarly, in the mechanism $\Gamma_m^1$, any pair of agents can deviate because the total payoff to $S = \{i, j\}$ in any SPE of $\Gamma_m^1$ is $16\frac{2}{3} < sa(S) = 20$.

In the following section we discuss conditions under which the mechanisms $\Gamma_m$ and $\Gamma_m^1$ are also coalition stable.

# 6 Coalition Stability

We confine our discussion of coalition stability of our mechanisms to the case where all public goods are excludable. The primary reason for this is that it is only in this case that coalition stability is well-defined through the notion of *stand alone core*. This is defined as follows. First, note that an allocation for an economy with (multiple) excludable public goods is $(y, x_1, \ldots, x_n)$ where $y$ is a vector of public goods and $x_i$ the contribution of agent $i$, satisfying $c(y) \leq \sum_{i=1}^n x_i$.

**Definition 4** *An allocation $(y, x_1, \ldots, x_n)$ is in the stand alone core of the excludable public goods economy if there does not exist $\emptyset \neq S \subset N$, $(y', (x'_i)_{i\in S})$ satisfying*

*1. $c(y') \leq \sum_{i\in S} x'_i$,*

[10]These two strategy profiles do not exhaust the set of equilibrium strategies in this example.

*2.* $v_i(y') - x'_i \geq v_i(y) - x_i$ *for all* $i \in S$ *with a strict inequality for at least one* $i \in S$.

**Remark 5** It is easily confirmed that an allocation $(y, x_1, \ldots, x_n)$ is in the stand alone core of the excludable good economy if and only if it is in the core of the TU-game $(N, sa)$.

The stand-alone core is equivalent to the well-known notion of the $\alpha$-core. Note that in defining the stand-alone core, we have assumed that a deviating coalition $S$ does not have access to the public goods produced by $N\backslash S$. Another way of looking at this is that the coalition $N\backslash S$ acts so as to minimize the (joint) payoff of the coalition $S$. We argue that this assumption is reasonable because there is no reason why a coalition should make available the set of (excludable) public goods produced by it to free-riding outsiders. Our definition would make less sense if there are pure public goods in the economy because a deviating coalition cannot be excluded from the set of pure public goods produced by others. In the pure public good case, Definition 4 implies that in the event of a deviation by the coalition $S$, the coalition $N\backslash S$ will not produce any public goods whatsoever. This is obviously very strong and Carraro and Siniscalco [4] and Chander and Tulkens [5] have proposed weaker behavioral assumptions. Our impression, though, is that this is still an unsettled issue and we therefore, confine ourselves to the excludable public goods case.

In the case of a single excludable public good economy, Bag and Winter [2] and Moulin [10] show that the stand alone core is non-empty under weak assumptions. Indeed, it turns out that the TU-game $(N, sa)$ is convex in this case. The mechanisms proposed by Bag and Winter [2] for the single excludable public good economy are coalition stable because they implement payoff vectors which are in the stand alone core. In the multiple excludable public goods case, the stand alone core may be empty as the example in the previous section demonstrates. (Note that preferences are convex and marginal cost constant in that example!) Moulin [10] gives sufficient conditions for the convexity of the TU-game $(N, sa)$ which ensures that the stand alone core is non-empty. We turn to these conditions now.

Let $y, y' \in \mathbb{R}^p_+$. Then, we let $y \vee y' = \max\{y, y'\}$ and $y \wedge y' = \min\{y, y'\}$ where the maximum and minimum are taken coordinate-wise.

**Definition 5** *A function* $f : \mathbb{R}^P \rightarrow \mathbb{R}$ *is* supermodular [submodular] *if for all* $y, y'$, $f(y \vee y') + f(y \wedge y') \geq [\leq] f(y) + f(y')$.

The proof of the following lemma is straightforward and is omitted.

**Lemma 2** *Suppose that* $v_i$ *satisfies the conditions in Assumption 1 and is also supermodular for all* $i = 1, \ldots, n$. *Also, let the cost function* $c$ *satisfy the conditions in Assumption 1 as well as submodularity. Then, the game* $(N, sa)$ *is convex.*

This now leads to the following result.

**Theorem 3** *Suppose that $v_i$ is supermodular for $i = 1, \ldots, n$ and $c$ is submodular. Then, the mechanisms $\Gamma_m$ and $\Gamma_m^1$ are coalition stable.*

**Proof**: By Remark 5 it suffices to show that the allocations achieved by $\Gamma_m$ and $\Gamma_m^1$ are in the core of $(N, sa)$. By Lemma 2, $(N, sa)$ is convex. The coalition stability of $\Gamma_m$ thus follows from the observation following the proof of Lemma 2. Since the Shapley value is simply a convex combination of the "marginal contribution" vectors, the coalition stability of $\Gamma_m^1$ follows also. ■

**Remark 6** Observe that supermodularity and submodularity impose no restrictions when there is only one public good. Lemma 2 thus confirms the result of Moulin [10] that in the case of a single excludable public good, the game $(N, sa)$ is convex under weak assumptions.

**Remark 7** Topkis [11] has shown that if a function is smooth, then supermodularity [submodularity] is equivalent to the condition that all cross-partials are non-negative [non-positive]. Thus, on the preference side, supermodularity implies complementarity between the public goods in the sense that the marginal utility from a public good does not decrease when the amount of some other public good increases. A number of well-known utility functions are supermodular; for instance, the Cobb-Douglas, separable, and Leontief functions are all supermodular.

Submodularity of the cost function suggests a similar complementarity: the marginal cost of a public good does not increase when the quantity produced of some other public good increases. Such complementarities may not be unrealistic: an example of such a scenario is when one (excludable) public good is public education and the other is a direct intervention to reduce poverty. Examples of submodular cost functions include $c(y) = \sum_{i=1}^{p} p_i y_i$ (where $p_i$ is the unit cost of good $i$) and $c(y) = \ln(1 + \sum_{i=1}^{p} y_i)$.[11]

# 7 The non-quasilinear case

One important restriction of our analysis has been the assumption that the preferences are quasi-linear. We now show that we do get efficiency and equity when the preferences are not quasi-linear but the equitable outcome cannot be characterized in terms of the Shapley value of a cooperative game.

As in the quasi-linear case, we assume that the preferences of agent $i$ is given by a utility function $u_i(x, y)$ where $x$ is the quantity of the private good and $y$ a vector of public goods. We assume that agent $i$ has a strictly postive

[11]Note that $c(y) = \sum_{i=1}^{p} p_i y_i$ is only weakly submodular while $c(y) = \ln(1 + \sum_{i=1}^{p} y_i)$ is strictly submodular. Also, observe that the former cost function is a "natural" specification in many contexts.

endowment of the private good, denoted by $w_i$. The following restrictions are imposed on an agent's utility function and the cost function.

**Assumption 2** *The function $u_i(x, y)$ is continuous and strictly monotone in all arguments and is normalized such that $u_i(0, \mathbf{0}) = 0$. Furthermore, the private good is* indispensable*: if $x > 0$, then for all $y, y'$, $u_i(x, y) > u_i(x, y')$.*[12] *The cost function c is continuous, non-decreasing, satisfies $c(y) \geq c(\mathbf{0}) = 0$ for all $y$ and $c(y) \to \infty$ as $y \to \infty$.*

Assume as before that the agents announce in the order $1, \ldots, n$. We now define the vector of "marginal utilities" as follows. Let

$$
\begin{aligned}
& u_n^* = \max u_n(x, y) \\
\text{subject to (i)} \quad & x \geq 0, \\
\text{(ii)} \quad & c(y) \leq w_n - x.
\end{aligned}
$$

be the stand alone payoff of agent $n$. Suppose now that $u_k^*$ has been defined for all $k > K$. Then, define $u_K^*$ as follows.

$$
\begin{aligned}
& u_K^* = \max u_K(x, y) \\
\text{subject to (i)} \quad & x \geq 0, x_j \geq 0 \text{ for all } j > K, \\
\text{(ii)} \quad & c(y) \leq w_K - x + \textstyle\sum_{j=K+1}^{n}(w_j - x_j), \\
\text{(iii)} \quad & u_j^* \leq u_j(x_j, y) \text{ for all } j = K+1, \ldots, n.
\end{aligned}
$$

**Remark 8** Observe that indispensability implies that the solution to the optimization problem defining $u_k^*$ – say $(x_k, y)$ – must involve $x_k > 0$.

An allocation for the coalition $S$ is a vector $(\{x_i\}_{i\in S}, y)$ where $y$ is a vector of public goods and $x_i$ the quantity of the private good consumed by $i$. An allocation for $S$ is feasible if $x_i \geq 0$ for all $i \in S$ and $\sum_{i\in S}(w_i - x_i) \geq c(y)$.

**Definition 6** *A utility vector $(u_i)_{i\in S}$ is feasible for $S$ if there exists a feasible allocation $(\{x_i\}_{i\in S}, y)$ such that $u_i = u_i(x_i, y)$ for all $i \in S$.*

**Definition 7** *A utility vector $(u_i)_{i\in S}$ is efficient for $S$ if it is feasible for $S$ and if there does not exist another feasible utility vector $(v_i)_{i\in S}$ such that $v_i > u_i$ for all $i \in S$.*

**Definition 8** *A vector of public goods $y$ is efficient for $S$ if there exists a feasible allocation for $S$ $(\{x_i\}_{i\in S}, y)$ giving rise to an efficient utility vector $(u_i)_{i\in S}$.*

**Theorem 4** *Suppose that Assumption 2 is satisfied. Then, an efficient vector of public goods for $N$ is produced in all SPE of the mechanism $\Gamma_m$. If $(u_1, \ldots, u_n)$ are the net payoffs to the agents in a SPE of $\Gamma_m$ then it must be an efficient utility vector for $N$ and furthermore, $u_i = u_i^*$ for $i = 1, \ldots, n$.*

[12] The assumption of indispensability is taken from Mas-Colell [8].

The proof of Theorem 4 follows the same methodology as that involved in proving Theorem 1. Suppose that agents in $\{1, \ldots, k\}$ have announced $(y^j, \bar{x}_j)_{j=1}^k$. Let $i_k$ be the largest integer in $\{1, \ldots, k\}$ such that there exists a vector $(x_{k+1}, \ldots, x_n, y)$ satisfying

$$y \geq y^j \text{ for all } j = i_k, \ldots, k, \text{ and } x_j \geq 0 \text{ for all } j = k+1, \ldots, n, \tag{7}$$

$$c(y) \leq \sum_{j=i_k}^{k} \bar{x}_j + \sum_{j=k+1}^{n} (w_j - x_j), \tag{8}$$

$$u_{k+1}(x_{k+1}, y) > u_{k+1}^* \text{and } u_j(x_j, y) \geq u_j^* \text{ for all } j = k+2, \ldots, n. \tag{9}$$

Observe that if agent $k+1$ is to obtain a payoff greater than $u_{k+1}^*$, then he must have the cooperation of some of the agents preceding him and all the agents following him. Note also that the definition of a maximal compatible coalition requires that the set of agents preceding $k+1$ must necessarily be a "connected" one. The fact that $i_k$ is the largest integer in $\{1, \ldots, k\}$ satisfying (7)-(9) means that if $k+1$'s net payoff is strictly greater than $u_{k+1}^*$ and each $j$ following $k+1$ receives at least $u_j^*$, then the maximal compatible coalition must be a superset of $S_{i_k}$. Therefore, if the maximal compatible coalition is not a superset of $S_{i_k}$, then we must have either (i) the net payoff to $k+1$ does not exceed $u_{k+1}^*$, or (ii) the net payoff to some $j > k+1$ is strictly less than $u_j^*$.

The following lemma is crucial to proving Theorem 4.

**Lemma 3** *Let Assumption 2 be satisfied. Suppose that agents in $\{1, \ldots, k\}$ have announced $(y^j, \bar{x}_j)_{j=1}^k$. Suppose that $i_k$ exists. Then all SPE of the subgame following $k$'s announcement must be such that the resulting maximal compatible coalition is a superset of $S_{i_k}$.*

**Proof**: We show by induction that if the lemma is not true then there exists a credible deviation for some agent in $\{k+1, \ldots, n\}$. The lemma is clearly true for $k = n$. So suppose that it is true for all $k > K$ and consider the case $k = K$. Suppose that the maximal compatible coalition in some SPE of the subgame following $K$'s announcement is not a superset of $S_{i_K}$. Let $(v_{K+1}, \ldots, v_n)$ be the payoffs to the agents following $K$ in this SPE. We can distinguish between two cases.

**Case 1**: There exists $j > K$ such that $v_j < u_j^*$.

By the definition of $u_j^*$, there exists a feasible allocation for $S_j$, $(x_j, \ldots, x_n, y)$ such that $u_j(x_j, y) = u_j^*$ and $u_i(x_i, y) \geq u_i^*$ for all $i > j$. Let $j$ deviate by announcing $(y, w_j - \tilde{x}_j)$ where $x_j > \tilde{x}_j > 0$ and $u_j(\tilde{x}_j, y) > v_j$.[13] Now consider the allocation $(\tilde{x}_j, \tilde{x}_{j+1}, \ldots, \tilde{x}_n, y)$ where $\tilde{x}_j$ is as defined above and for $i > j$, $\tilde{x}_i = x_i + (x_j - \tilde{x}_j)/(n-j)$. It follows that this allocation is also feasible for $S_j$ and that $u_i(\tilde{x}_i, y) > u_i^*$ (by strict monotonicity of the

[13] The existence of $\tilde{x}_j$ follows from the continuity of the utility function. Note also that indispensability implies that $x_j > 0$.

utility function in the private good) for all $i > j$. Note now that $i_j = j$ after $j$'s deviation. It follows by the induction hypothesis that the maximal compatible coalition in any SPE of the subgame following $j$'s deviation will be a superset of $S_{i_j} = S_j$. Thus, $j$ will always be a member of the maximal compatible coalition in any SPE of the subgame resulting after his deviation. This, though, shows that $j$ has a credible deviation.

**Case 2**: $v_{K+1} = u^*_{K+1}$ and for all $j > K+1$, $v_j \geq u^*_j$.

Since $i_K$ exists, there is a vector $(x_{K+1}, \ldots x_n, y)$ satisfying (7)-(9). Choose $0 < \tilde{x}_{K+1} < x_{K+1}$ such that $u_{K+1}(\tilde{x}_{K+1}, y) > u^*_{K+1}$. Such a choice is possible since $u_{K+1}(x_{K+1}, y) > u^*_{K+1}$, $u_{K+1}(.,.)$ is continuous and the private good is indispensable. Consider the vector $(\tilde{x}_{K+2}, \ldots, \tilde{x}_n, y)$ where $\tilde{x}_j = x_j + (x_{K+1} - \tilde{x}_{K+1})/(n - K - 1)$. Note that by strict monotonicity of the utility function, we have $u_j(\tilde{x}_j, y) > u^*_j$ for all $j > K+1$.

Let agent $K+1$ deviate by announcing $(y, w_{K+1} - \tilde{x}_{K+1})$. From the discussion in the preceding paragraph, it follows that $i_{K+1}$ exists after $K+1$'s deviation: indeed, we must have $i_{K+1} \geq i_K$.[14] By the induction hypothesis, the resulting maximal compatible coalition must be a superset of $S_{i_{K+1}}$. Since $i_{K+1} \leq K+1$, $K+1$ is always a member of this coalition. It follows that $K+1$ has a credible deviation since $u_{K+1}(\tilde{x}_{K+1}, y) > u^*_{K+1} = v_{K+1}$. This concludes the proof of the lemma. ∎

**Corollary 2** *Let Assumption 2 be satisfied. If $(u_1, \ldots, u_n)$ are the net payoffs to the agents in a SPE of $\Gamma_m$, then we must have $u_i \geq u^*_i$ for all $i = 1, \ldots, n$.*

**Proof**: Suppose not: then there exists $j$ such that $u_j < u^*_j$. Then, agent $j$ can credibly deviate using the same strategy discussed in Case 1 of Lemma 3. ∎

**Proof of the theorem**: Let $(u_1, \ldots, u_n)$ be the net payoffs to the agents in some SPE of $\Gamma_m$. By Corollary 2, $u_i \geq u^*_i$ for $i = 1, \ldots, n$. Suppose that $(u_1, \ldots, u_n)$ is not efficient: then there exists a feasible utility vector $(v_1, \ldots, v_n)$ such that $v_i > u_i$ for all $i$. However, since $v_i > u_i \geq u^*_i$ for $i = 2, \ldots, n$, and $v_1 > u_1 = u^*_1$, we find ourselves contradicting the definition of $u^*_1$. Therefore, $(u_1, \ldots, u_n)$ must be an efficient utility vector. The fact that an efficient vector of public goods is produced follows trivially from the the observation that $(u_1, \ldots, u_n)$ is an efficient utility vector.

To complete the proof, we need to show that $u_i = u^*_i$ for $i = 1, \ldots, n$. The argument in the previous paragraph shows that $u_1 = u^*_1$. So, suppose that $u_j > u^*_j$ for some $j > 1$. Let $\bar{y}$ be the vector of public goods produced and $(w_1 - \bar{x}_1, \ldots, w_n - \bar{x}_n)$ the contributions of the agents in the SPE. We thus have $u_i = u_i(\bar{x}_i, \bar{y})$ for $i = 1, \ldots, n$ and $c(\bar{y}) \leq \sum_{i=1}^n (w_i - \bar{x}_i)$. Consider a deviation by agent $j-1$ announcing $(\bar{y}, w_{j-1} - \bar{x}_{j-1} - \epsilon)$ where $\epsilon > 0$ is such that $u_j > u_j(\bar{x}_j - \epsilon, \bar{y}) > u^*_j$. Such an $\epsilon$ exists by the continuity of $u_j(.,.)$.

[14] Obviously, a value of $i = i_K$ satisfies (7)-(9). Therefore, we must have $i_{K+1} \geq i_K$.

Note also that $u_{j-1}(\bar{x}_{j-1} + \epsilon, \bar{y}) > u_{j-1}$ by the strict monotonicity of the utility function in the private good. Finally, observe that after $j$'s deviation $i_j$ exists: the vector $(w_j - \bar{x}_j + \epsilon, \bar{x}_{j+1}, \ldots, \bar{x}_n, \bar{y})$ satisfies (7)-(9). We can now invoke Lemma 3 to show that $j-1$'s deviation is strictly profitable. This is a contradiction since we assumed that $(u_1, \ldots, u_n)$ are the payoffs in a SPE of $\Gamma_m$. ■

With regard to the two-stage mechanism $\Gamma_m^1$, we note that the expected net payoff to player $i$ at the beginning of stage 2 is simply the average of his net payoff under all possible orderings of the agents. Let $(\phi_1, \ldots, \phi_n)$ be the vector of expected net payoffs. In contrast to the quasi-linear case, this vector is not necessarily efficient. The following example illustrates this point.

**Example 1** Let $N = \{1, 2\}$ and there be one public good only. Let $u_1(x, y) = xy, u_2(x, y) = x + y, w_1 = w_2 = 1$ and $c(y) = y$. The stand-alone utilities are given as $u_1^s = 1/4, u_2^s = 1$. When the agents announce in the order $1, 2$, the optimal strategy involves 1 announcing $(1, 0)$ and 2 announcing $(1, 1)$. Therefore, 1 unit of the public good is produced. The corresponding payoffs are $(u_1, u_2) = (1, 1)$. When the agents announce in the order $2, 1$, the optimal strategies involve 1 announcing $((2 + \sqrt{3})/2, 1)$, 1 announcing $((2+\sqrt{3})/2, \sqrt{3}/2)$. In this case, $(2+\sqrt{3})/2$ units of the public good is produced and the utility vector is $(u_1, u_2) = (1/4, (2+\sqrt{3})/2)$. The average payoff is thus $(\bar{u}_1, \bar{u}_2) = (5/8, (4 + \sqrt{3})/4)$. This vector is however not efficient. To see this, consider the allocation $(x_1, x_2, y) = ((4 - \sqrt{3})/4, 0, (4 + \sqrt{3})/4)$. It is trivial to check that this is a feasible allocation and gives rise to the utility vector $(u_1, u_2) = (13/16, (4+\sqrt{3})/4)$. This utility vector weakly dominates the vector $(\bar{u}_1, \bar{u}_2)$ but one can easily modify the allocation to make both agents strictly better off.

If the two agents do play the mechanism $\Gamma_m^1$, then the order in which the agents announce in Stage 1 is still important. Consider what happens when agents announce in the order $2, 1$. If agent 2 wants the game to end in Stage 1 itself, then she has to ensure agent 1 a payoff of $5/8$. The optimal strategy involves agent 2 announcing $((4 + \sqrt{6})/4, 1)$, agent 1 announcing $((4+\sqrt{6})/4, (4-\sqrt{6})/4)$ which gives rise to the payoffs $(u_1, u_2) = (5/8, (4+\sqrt{6})/4)$. On the other hand, when the agents announce in the order $1, 2$, the optimal strategies involve agent 1 announcing $((4 + \sqrt{3})/4, \sqrt{3}/4)$, agent 2 announcing $((4 + \sqrt{3})/4, 1)$ which give corresponding payoffs of $(u_1, u_2) = (13/16, (4 + \sqrt{3})/4)$.

Example 1 suggests that the randomization technique used by us in the quasi-linear case to achieve order independence is not of much use here. The problem, clearly, is that while the Pareto frontier is a straight line in the quasi-linear case, it can be convex on the more general domain of preferences. In such a situation, one option is to use more complicated mechanisms like those in the papers of Abreu and Sen [1], Moore and Repullo

[9] and Maniquet [7]. Another option, suitable in our context, is to perform the randomization beforehand. In other words, we choose an order randomly and then allow the agents to play the mechanism $\Gamma_m$ according to the chosen order. This, though, will give equity only in expected terms.[15]

# 8 Conclusion

We have examined a model of cost sharing of multiple public goods in this paper under weak restrictions on the preferences and the technology. Two mechanisms have been proposed here, in both, agents announce sequentially. An agent's announcement is restricted to a vector of public goods that she wishes to consume and a contribution to the cost of production, conditional on her demands being met. Both mechanisms ensure efficiency; however in the first mechanism, the payoffs are asymmetrical and depend crucially on the order in which players move. The second mechanism corrects for this by having players play two rounds with the order of play in the second round being randomly determined. This mechanism ensures that the payoffs of the agents are equitable in that they correspond to the Shapley value of a well-defined TU-game. Furthermore, it is *monotonic* with respect to the players preferences: if player $i$'s marginal utility is greater than player $j$'s at all points, then player $i$ gets a higher net utility from the mechanism.

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