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ON THE TOPOLOGICAL SOCIAL CHOICE PROBLEM

by

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On the topological social choice problem.

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Abstract: Extending earlier work of Chichilnisky and Heal, we show that any connected space of the homotopy type of a finite complex admitting a continuous symmetric choice function respecting unanimity is contractible for any fixed finite number (>1) of agents. On the other hand, removing the finiteness condition on the homotopy type, we show that there are a number of non-contractible spaces that do admit such choice functions, for any number of agents, and, characterize precisely those spaces.

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Introduction and Statement of Results.

The major goal of this paper is to further analyze the topological social choice model, studied by Chichilnisky and Chichilnisky-Heal [2,4] and others.

We imagine k agents each picking elements out of X, the choice space. The problem is to give an aggregation of their choices $A(x_1....x_k)$ continuously in these variables and subject to two axioms:

1. $A(x,..x) = x$ (unanimity)
2. $A(x_1....x_k)$ is independent of the ordering (anonymity).

It is also useful to consider weaker symmetry conditions. One that we like is:

2 $A(x_1 ....x_k)$ is invariant under a transitive group of permutations (weak anonymity).

Without further assumption, we shall assume that our choice space is connected and of the homotopy type of a CW complex (or, equivalently, a simplicial complex). The second assumption is a common one in algebraic topology and includes all manifolds, algebraic varieties, many function spaces, (see [7]). Connectedness may not be such a natural assumption in the economic application, but one can readily see that under the hypothesis of being a CW complex, one can find such an A for X iff one can do so for every component. (However, the study of aggregate of such A s does not reduce to their study on the individual components.)

<u>Theorem 1</u>: Suppose that X has the homotopy type of a connected finite CW complex (e.g. a compact connected manifold, or polyhedron). If for some $k>1$, X has an aggregation operator A as above (even weakly anonymous), then X is contractible.

This should be compared with the well-known result of [4], which asserts that if X is a parafinite cell complex (i.e. up to homotopy equivalence, has finitely many cells in every dimension), and such an A exists <u>for all k</u>, X is contractible. Theorem 1 shows that the nonexistence of aggregations is not due to assuming something about populations of arbitrary sizes. It holds quite generally for any particular size.

The Chichlinisky-Heal result does give contractibility under weaker finiteness conditions. We now consider what happens on weakening their hypotheses in some other directions. We shall see that there are parafinite CW complexes which have such A s for infinitely many k, and there are finite dimensional complexes (which are not parafinite) for which such A s exist for all k. Whether these models have any economic significance waits to be seen.

Example 1. $X = (H-\{0\})/\sim$ where H is an infinite dimensional Euclidean space (with the CW topology, as in [8]) and $x \sim ux$, where $u = \exp(2\pi i/r)$. This is an infinite dimensional lens space (x R). It has (strongly anonymous) k-aggregators iff k is relatively prime to r. (When k is not relatively prime to r, even weakly anonymous aggregators do not exist. Using products of these, one can arrange for arbitrary multiplicative submonoids of the positive integers to be the sets of k for which aggregators exist.) This example shows that in the Chichilnisky-Heal theorem, one needs the hypothesis that for all k, the aggregator exists — not merely for infinitely many k. (It does suffice, however, to assume that the result holds for all prime numbers.) The proof of this comes from considering explicit models of classifying spaces of finite groups, and will not be discussed here.

Example 2. Let X be the union of the mapping cylinders the maps $S^1 \to S^1$ sending z to z^n, the top of the n-th cylinder glued to the bottom of the (n-1)st. This union has $\pi_1 = \mathbf{Q}$, the rational numbers, and has contractible universal cover. It has strongly anonymous k aggregators for all k. (Of course, it is not parafinite.) The proof of this example is given below in the proof of Corollary 2.

Example 3/ Proposition: If X has such an averaging function and Y is homotopy equivalent to X, then so does Y.

This is a simple application of the homotopy extension principle (see [8].).

To give our (partial) solution to this problem, we need some standard algebraic-topological terminology.

Definition. Z is an H-space (or Hopf space) if it has a continuous multiplication f:ZxZ -> Z such that for any point e in Z, the functions, f(e,z) and f(z,e) of z are homotopic to the identity as maps of Z to itself.

Assuming, as we shall, that Z is connected, the condition can be checked for a single choice of x. Furthermore one can modify f by a homotopy so that f(e,z) = f(z,e) = z for all z, i.e. that f has a 2-sided identity.

H-spaces arise naturally in algebraic topology and have much in common with Lie groups. We shall make use of some of their theory in deducing corollaries such as the above from our main theorem. Our main theorem is the following:

Main Theorem: If X is a connected space of the homotopy type of a CW complex, then if X^k admits a weakly anonymous averaging operator, then X is an H-space whose homotopy groups (or equivalently) its homology groups are modules over the ring Z[1/k] (the rational numbers whose denominators only have prime factors that appear in k; equivalently, every element of these groups is uniquely divisible by k).

A complete analysis of which of the above X s have (weakly) anonymous averaging operators requires more of a discussion of symmetries: we shall delay H-spaces always have abelian fundamental groups and are examples of simple spaces, spaces whose fundamental groups act trivially on their higher homotopy groups[2]. Simplicity is enough to guarantee the equivalence of a Z[1/k]-module structure on homotopy and on homology. Such spaces are called 1/k-local. A space is **rational** if it is

[2] Recall that for any reasonable space V there is an action of $\pi_1(V)$ on $\pi_i(V)$ i>2 as follows. $\pi_i(V)$ is the same as that of its universal cover. $\pi_1(V)$ acts on the universal cover by covering translates.

1/k local for every k, i.e. if its reduced (integral!) homology groups are all rational vector spaces.

Without giving a complete analysis of all the symmetry issues we can state two corollaries:

Corollary 1: A connected space X homotopy equivalent to a CW complex admits an average for k=2 iff it is a 1/2-local homotopy commutative H-space.

Homotopy commutativity means that f(x,y) is homotopic to f(y,x); ordinarily this is rather stronger than being homotopic to a g with g(x,y) = g(y,x), but for 1/2 local H-spaces these are equivalent.

Corollary 2: A connected space X homotopy equivalent to a CW complex admits a weakly anonymous average for all k iff it admits a strongly anonymous average for all k which is iff it is a rational space and an H-space.

It is amusing to consider one of the spaces as in corollary 2. Then one has the following remark.

Remark: Suppose X is a non-contractible space as in corollary 2 given a proper metric[3]. Then there is a compact subset K such that for any real C, there are points $x_1.....x_k$ in K, such that $d(x_r, A(x_1....x_k)) > C$.

For these spaces, the fairness demands (continuity, (1) and (2)) of our method of compromising forces upon us a Wisdom of Solomon [4]. We must make compromises for some sets of agents that are worse for all of them than any of the terms they offer each other[5]. This is a strong failure of Pareto optimality.

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[3] A metric is proper if its closed balls are compact.

[4] Although we must really split the baby or abandon our model.

[5] I am imagining that each agent has a proper utility function defined on X, with their choice on X being their point of maximum utility. Since K is compact none of the choices of any of the other agents can be that bad, but since their aggregated A is so far away (by propriety) it will be viewed by all the agents as having very negative utility.

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Proof of the main theorem.

While some of the steps follow Chichilnisky and Heal, our discussion will be self-contained, aside from standard algebraic topological facts that can be found in [8]. Throughout X denotes a space satisfying the hypothesis of the main theorem.

Let D: $X \to X^k$ be defined by $D(x) = (x,x,...x)$.

Lemma 1: $\pi_1 X$ is abelian.

Proof: $A_* : \pi_1 X^k = (\pi_1 X)^k \to \pi_1 X$ is onto, since AD = identity. A_* can be written as a sum of factors $A_1+A_2+....A_k$ induced by the various inclusions of X in X^k (send x to (e,e,...,x,..e); there are k choices of where to insert the x). Symmetry by a transitive group implies that the A_i coincide. Consequently, the images of $\pi_1 X$ under these maps coincide and are all of $\pi_1 X$.

Note now that the first and second copies of $\pi_1 X$ commute with each other in the domain group; so their images commute in the range. Since they are $\pi_1 X$, that group must be commutative.

Lemma 2: Let R: $X \to X$ denote A(x,e,...e). R is a homotopy equivalence.

Proof: According to the Whitehead theorem (see [8]) all we must see is that R induces an isomorphism of homotopy groups. Since $A(x,x,...x) = x$, for any u in any homotopy group, $u = A(u,u,...) = A_1(ku) = kA_1(u) = kR(u)$.

R is called a k-th root operator. It induces an inverse to multiplication by k on homotopy groups. It follows, for instance, that X s homotopy is 1/k divisible, i.e. that X is 1/k local.

Lemma 3: Let P denote the homotopy inverse to R. Define f(x,y) = A(P(x),P(y),e,...e). Then f provides an H-space structure on X.

Proof: f(x,e) = A(P(x),e,...e) = R(P(x)) which is homotopic to the identity by the definition of a homotopy inverse. f(e,x) = f(x,e) by symmetry.

Remark: A 1/k local H-space always has a k-th root operator R. It is the inverse of the k-th power map, defined using the multiplication k-1 times.

Proofs of the corollaries.

To prove theorem 1 it suffices to see that there are no finite CW complexes that are H-spaces with 1/k local homology. This follows immediately from a theorem of [1] that asserts that the top homology group of any connected finite H complex is **Z**, which is not at all divisible.

Corollary 1 follows immediately, because if A is symmetric, then one gets a commutative H-space from the proof of the main theorem. Conversely if X is a homotopy commutative 1/2 local H-space then one can actually assume that it is symmetric (this is a straightforward consequence of obstruction theory, the Grothendieck lemma above, and locality). We claim that C has a square root operator: R: X -> X. It is the homotopy inverse to f(x,x). Define first A: $XxX/\mathbf{Z}_2$ -> X, by A(x,y) = Rf(x,y). It is symmetric, but does not satisfy unanimity. However AD is homotopic to the identity (by construction), so by the homotopy extension principle, we can define A : $XxX/\mathbf{Z}_2$ -> X homotopic to the identity so that A is the identity on the diagonal. Thinking of A as a function on XxX, it is continuous, symmetric and respectful of unanimity.

Corollary 2 requires the characterization of rational H-spaces (see [9]; it follows directly from rational homotopy theory and the classification of Hopf algebras in characteristic 0).

Definition: An Eilenberg MacLane space of type (R,n) is a space K(R,n) whose homotopy is nonzero only in dimension n and for which $\pi_n = R$. These exist for all abelian groups R and all n, and are unique up to homotopy type. Note that K(R, n) x K(S,n) = K(RxS, n). For any Z, the homotopy classes of maps from Z into K(R,n) are in a 1-1 correspondence with the group $H^n(Z; R)$.

Proposition: $H^n(K(Q,n)^k/S_k ; Q) = Q$ and the restriction map to the diagonal is an isomorphism.

Here S_k is the symmetric group. This is well known and follows, for example, directly from the Kunneth theorem [9] and Grothendieck s lemma from the first chapter of [6].

Corollary: For any rational vector space V, and any k, there is a completely symmetric averaging operator respecting unanimity. Hence any product of rational Eilenberg-MacLane spaces has such an operator.

(This can also be seen directly with appropriate explicit models for Eilenberg-MacLane spaces; that is what one must do to verify the example involving the infinite dimensional lens space.) The second statement follows from the first just by averaging in each coordinate separately.

Corollary 2 follows immediately from the fact that any rational H-space is homotopy equivalent to a product of Eilenberg-MacLane spaces where the R s are rational vector spaces. Thus for the spaces satisfying the necessary condition for a weakly anonymous average, one can construct strongly symmetric ones.

Note: Even for K(Q,2) x K(Q,4) the averaging operator is not at all unique. This is due to the existence of many maps from K(Q,2) x K(Q,2) into K(Q,4). These correspond to quadratic polynomials in two variables: one for each factor. For k=4 one can then construct $\mathbf{Z}_4$ symmetric averages (e.g. using the quadratic polynomial $x_1x_2+x_2x_3+x_3x_4+x_4x_1$) that are not homotopic to an S_4-symmetric average.

The first basic question that should be addressed is whether a $\mathbf{Z}[1/3]$ local H-space which has an operator $T(x,y,z)$ of 3-variables which restricts to an H-structure on the first two variables, necessarily has a function T which is symmetric in the first two variables. There are certainly obstructions to deforming a given H-space structure to having this property, but the question is whether there exists any other. In any case, there certainly are more weakly symmetric averages than strongly symmetric ones (up to homotopy) in general.

In terms of functions T as above, one can extend corollary 1 to characterize (for instance) exactly which X support cyclically symmetric averages. The 1/k locality allows one to modify a homotopy cyclic symmetry to be genuinely symmetric with respect to the cyclic group and then the argument is no different than the k=2 case.

Finally the Solomonic nature of the averaging. Take an X as in corollary 2 and let K be the image of a sphere that represents a nontrivial element of homotopy (assuming X is not contractible[6]). If the k-averages of all elements of K lie within a subset P of X, then (by the proof of the main theorem) one can do the division of this homotopy class within P; It is no loss of generality to assume that P is simple, because in the standard telescope models of X, it has an exhaustion by compact simple spaces, and therefore that it has finitely generated homotopy (using the mod C theory of Serre s thesis, see [9]); this cycle cannot be arbitrarily divided, so the averages must, for large k, leave an arbitrary compact set.

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