Singular Games in $bv'NA$

Abraham Neyman*

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Abstract

Every simple monotonic game in $bv'NA$ is a weighted majority game. Every game $v \in bv'NA$ has a representation $v = u + \sum_{i=1}^{\infty} f_i \circ \mu_i$ where $u \in pNA$, $\mu_i \in NA^1$ and f_i is a sequence of bv' functions with $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Moreover, the representation is unique if we require f_i to be singular and that for every $i \neq j$, $\mu_i \neq \mu_j$.

1 Introduction

Simple games with finitely many players were introduced by von Neumann and Morgenstern [9]. These games are appropriate to the study of political structures in which power is the fundamental driving force. A special class consists of the weighted majority games $[q; w_1, \dots, w_n]$, where n is the number of players, w_i is the number (weight) of votes of player i, and q is the quota of votes required for a winning coalition.

The seminal paper of Shapley and Shubik [8], interpreting the Shapley value as a measure of voting power in the context of simple games, initiated a long line of research. Special attention was given to weighted majority games with a large number of small voters (see [3], [4], [5], [6]); such games arise naturally in stockholder voting in corporations with one vote per share held.

*Institute of Mathematics, and Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904; Israel. E-mail: aneyman@math.huji.ac.il
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In these games, the voters are individually insignificant and wield influence only via coalitions. It is fruitful to model them as games with a continuum of players (see the definitive model of Aumann and Shapley [1]).

The underlying set of players is modeled by a measurable space $(I, \mathcal{C})$ admitting non-atomic (positive) measures. Here I is the set of *players*, and the σ-algebra $\mathcal{C}$ is the set of *coalitions*. A *game* is a real-valued function v on $\mathcal{C}$ with $v(\emptyset) = 0$.

Two spaces of games that play a central role in Aumann and Shapley [1] — pNA and the larger space $bv'NA$ — are generated by non-atomic scalar measure games, i.e., by games of the form $f \circ \mu$, where μ is in NA_1, the set of all non-atomic probability measures on $(I, \mathcal{C})$; and f is a real-valued function defined on $[0, 1]$ (the range of μ). The space pNA is obtained by considering all functions f in ac, where ac is the set of all absolutely continuous functions with $f(0) = 0$. The closure (in the bounded variation norm) of the linear span of all games $f \circ \mu$, for $f \in ac$ and $\mu \in NA^1$, is pNA. When we admit more functions $f \in bv'$, where bv' is the set of all functions of bounded variation on $[0, 1]$ which are continuous at $0 = \mu(\emptyset) = f(0)$ and at 1.

For $f \in bv'$, let $\|f\|$ denote the variation of f on $[0, 1]$. Whenever μ is a non-atomic probability measure, the bounded variation norm $\|f \circ \mu\|$ of $f \circ \mu$, equals $\|f\|$. Therefore, if $(f_i)_{i=1}^{\infty}$ is a sequence of functions in bv' with $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and μ_i is a sequence of non-atomic probability measures, the series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ converges to a game $v \in bv'NA$. If the functions f_i are in ac the series converges to a game in pNA.

A game of the form $f \circ \mu$ where μ is a non-atomic measure and f is an absolutely continuous function on the range of the scalar measure μ is called an *absolutely continuous scalar measure game*. If f is a polynomial, it is called a *polynomial scalar measure game*. Therefore any game in pNA is approximated in the bounded variation norm by a linear combination of absolutely continuous scalar measure games. Moreover, any absolutely continuous scalar measure game is approximated in the bounded variation norm by a polynomial scalar measure game and therefore any game in pNA is approximated by a linear combination of polynomial scalar measure games.

However, any polynomial of a vector of non-atomic measures that vanishes at the vector 0 is a linear combination of polynomial scalar measure games and therefore the space pNA contains many other games of interest. For example, if $\mu_1, \ldots, \mu_n$ are non-atomic probability measures and f is a continuously differentiable function on the range of the vector measure $(\mu_1, \ldots, \mu_n)$ with $f(0) = 0$ then $f \circ (\mu_1, \ldots, \mu_n)$ is in pNA. Such a vector measure smooth game need not have a representation as a convergent series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ where $f_i \in ac$ and $\mu_i \in NA^1$ where NA^1.

Any function $f \in bv'$ has a unique representation as a sum $f = f^{ac} + f^s$ where $f^{ac} \in ac$ and f^s is a singular function in bv', i.e., a function whose variation is on a set of measure zero. The subspace of all singular functions in bv' is denoted s'. The closed linear space generated by all games of the form $f \circ \mu$ where $f \in s'$ and $\mu \in NA^1$ is denoted $s'NA$.

Aumann and Shapley [1] show that $bv'NA$ is the algebraic sum of the two spaces pNA and $s'NA$ and that for any two games, $u \in pNA$ and $v \in s'NA$, $\|u + v\| = \|u\| + \|v\|$.

The present paper demonstrates two results. The first asserts that the only simple monotonic games in $bv'NA$ are the weighted majority games. The second result shows that any game $v \in s'NA$ is the sum of a convergent series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ where $f_i \in s'$ and $\sum_{i=1}^{\infty} \|f_i\| < \infty$.

2 Simple Monotonic Games in $bv'NA$

The set of all positive and finitely additive games is denoted FA^+. A weighted majority game is a game of the form

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) \geq q \\ 0 & \text{if} \quad \mu(S) < q \end{cases}$$

or

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) > q \\ 0 & \text{if} \quad \mu(S) \leq q \end{cases}$$

where $\mu \in FA^+$ and $0 < q < \mu(I)$. The finitely additive measure μ is called the *weight measure* and q is called the *quota*. Every weighted majority game has a representation with a weighted measure in FA^1 and thus we assume further the normalization $\mu \in FA^1$.

Theorem 1 *Every simple monotonic game in $bv'NA$ is a weighted majority game, and the weighted measure is a non-atomic probability measure.*

Proof: Assume that $v \in bv'NA$ is a simple monotonic game. As $bv'NA$ is the closed linear span of all games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in bv'$, v can be approximated by linear combinations of the form $\sum_{i=1}^n f_i \circ \mu_i$ where $f_i \in bv'$ and $\mu_i \in NA^1$. We can assume w.l.o.g. that $\mu_i \neq \mu_j$ for every $1 \leq i < j \leq n$. As any function $f \in bv'$ is the sum of an absolutely continuous function $f^{ac} \in bv'$ and a singular function $f^s \in s'$, any such linear combination is the sum of a game $u \in pNA$ and a linear combination $\sum_{i=1}^n f_i \circ \mu_i$ where each one of the functions f_i is singular, i.e., in s'. Fix $\varepsilon > 0$, and let $S_0 \subseteq \ldots \subseteq S_m$ be an increasing chain of coalitions so that the variation of u over it, $\sum_{0 \leq j < m} |u(S_{j+1}) - u(S_j)|$, is $\geq \|u\| - \varepsilon$. Let $S(t)$, $0 \leq t \leq 1$, be the increasing path of ideal coalitions defined by $S(t) = S_j + (tm - j)(S_{j+1} - S_j)$ if $j/m \leq t \leq j + 1/m$. The function $G : [0,1] \to I\!R$, defined by $G(t) = (v - u - \sum_{i=1}^n f_i \circ \mu_i)(S(t))$, is a sum of an absolutely continuous function, $t \mapsto -u(S(t))$, and a singular function of bounded variation $t \mapsto (v - \sum_{i=1}^n f_i \circ \mu_i)(S(t))$. Therefore the variation of the game $v - u - \sum_{i=1}^n f_i \circ \mu_i$ over the increasing path $(S(t))_{0 \leq t \leq 1}$ is greater than or equal to the variation of u over this increasing path which is $\geq \|u\| - \varepsilon$. Therefore

$$\|v - u - \sum_{i=1}^n f_i \circ \mu_i\| \geq \|u\|$$

Therefore, if v is a simple monotonic game in $bv'NA$ it is approximated by games of the form $\sum_{i=1}^n f_i \circ \mu_i$ where $f_i \in s'$ and $\mu_i \in NA^1$. Assume $\|v - \sum_{i=1}^n f_i \circ \mu_i\| < \varepsilon$, $f_i \in s'$ and w.l.o.g. $\mu_i \neq \mu_j \in NA^1$ for $i \neq j$. Let $T \in \mathcal{C}$ with $\alpha_i := \mu_i(T) \neq \alpha_j := \mu_j(T)$ for every $i \neq j$. For each fixed $0 < r < 1$, the functions $f_i^r : [0,1] \to I\!R$ which are defined by $f_i^r(t) = f_i((1-r)t + r\alpha_i)$ are in s'. By Corollary 8.10 of Aumann and Shapley [1] there is r sufficiently small such that for every $j \neq i$ the functions f_i^r and f_j^r are mutually singular when $i \neq j$. If r is sufficiently small so that the variation of f_i on each of the intervals $[0, r]$ and $[1-r, 1]$ is at most $\varepsilon/(2n)$, then the variation of f_i^r is at least $\|f_i\| - \varepsilon/n$. Let $\chi(t) = (1-r)t + rT$, $0 \leq t \leq 1$, and consider the function $F : [0,1] \to I\!R$ defined by $F(t) = (v - \sum_{i=1}^n f_i \circ \mu_i)(\chi(t))$. Note that $F(t) = v(\chi(t)) - \sum_{i=1}^n f_i^r(t)$. As $\chi(t)$, $0 \leq t \leq 1$, is an increasing path of ideal coalitions, the variation of F over $[0,1]$ is bounded by $\|v - \sum_{i=1}^n f_i \circ \mu_i\| < \varepsilon$. As v is a simple monotonic game, the function $t \mapsto v(\chi(t))$ is singular to all but possibly one, say f_1^r, of the functions f_i^r. Therefore $\|F\| \geq \sum_{i=2}^n \|f_i^r\| \geq \sum_{i=2}^n \|f_i\| - \varepsilon$. Therefore $\|v - f_1 \circ \mu_1\| < 2\varepsilon$. Therefore v is a limit in the bounded variation norm of games of the form $f \circ \mu$ where $f \in s'$ and $\mu \in NA^1$.

Consider the function $g(t) = v(tI)$. It follows that $\|f \circ \mu - g \circ \mu\| = \|f - g\| \leq \|f \circ \mu - v\|$ and therefore $\|g \circ \mu - v\| \leq 2\|f \circ \mu - v\|$ and thus v is a limit in the bounded variation norm of weighted majority games. As the bounded variation distance of any two distinct weighted majority games is at least 1, the result follows. ■

3 Singular Games in $bv'NA$

Aumann and Shapley [1] prove that the space $bv'NA$ is the algebraic sum of two spaces: the space pNA and the space $s'NA$. The space $s'NA$ is the closed linear space spanned by all games of the form $f \circ \mu$ where $f \in s'$ and μ is a non-atomic probability measure.

Theorem 2 *Every game $v \in s'NA$ is a countable (possibly finite) sum*

$$v = \sum_{i=1}^{\infty} f_i \circ \mu_i$$

where $f_i \in s'$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and $\mu_i \in NA^1$.

Proof Let $(\mu_i)_{i=1}^{\infty}$ be a fixed sequence of distinct non-atomic probability measure. Consider the set $X(\mu_1, \mu_2, \ldots)$ of all games in $s'NA$ which are a countable sum $\sum_{i=1}^{\infty} f_i \circ \mu_i$ with $f_i \in s'$ and $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Assume that $v = \sum_{i=1}^{\infty} f_i \circ \mu_i \in X$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$. Then $\sum_{i=1}^{n} f_i \circ \mu_i \to v$ as $n \to \infty$. By Proposition 8.17 of Aumann and Shapley [1], $\|\sum_{i=1}^{n} f_i \circ \mu_i\| = \sum_{i=1}^{n} \|f_i\|$. Therefore $\|v\| = \sum_{i=1}^{\infty} \|f_i\|$.

We prove next that $X = X(\mu_1, \mu_2, \ldots)$ is a closed subspace of $s'NA$. If $v_k = \sum_{i=1}^{\infty} f_{k,i} \circ \mu_i$ is a Cauchy sequence of games in X, $\|v_k - v_m\| \geq \|f_{k,i} - f_{m,i}\|$ for each fixed $1 \leq i < \infty$, and thus $(f_{k,i})_{k=1}^{\infty}$ is a Cauchy sequence in s' which is a Banach space and therefore has a limit f_i in s'. Moreover, for every n, $\sum_{i=1}^{n} \|f_i\| = \lim_{k \to \infty} \sum_{i=1}^{n} \|f_{k,i}\| \leq \sup_{k \geq 1} \|v_k\|$ and therefore $v := \sum_{i=1}^{\infty} f_i \circ \mu_i \in X$. As $\|v_k - v\| = \sum_{i=1}^{\infty} \|f_{k,i} - f_i\|$ and $\|v_k - v_m\| = \sum_{i=1}^{\infty} \|f_{k,i} - f_{m,i}\|$ we deduce that $\|v_k - v\| \leq \sup_{m \geq k} \|v_k - v_m\| \to_{k \to \infty} 0$ and therefore $v_k \to v$ as $k \to \infty$.

As any sequence v_k of finite linear combinations of singular scalar measure games is in $X(\mu_1, \mu_2, \ldots)$ where $\mu_1, \mu_2, \ldots$ is a sequence of pairwise different non-atomic probability measure containing all non-atomic probability measure appearing in the representations of the v_ks, the result follows. ■

4 Simple Monotonic Games in $'AN$

The space $'AN$ is introduced in [2]. It is the linear closed span of scalar measure games $f \circ \mu$ where μ is in AN_1, the set of finitely additive non-atomic positive measure, and f obeys a weakened continuity at $0 = \mu(\emptyset)$ and at $\mu(I)$.

Theorem 3 *Every simple monotonic game in $'AN$ is a weighted majority game, and the weighted measure is a non-atomic finitely additive probability measure.*

Proof: Assume that $v \in\ 'NA$ is a simple monotonic game. As $'NA$ is the closed linear span of all games of the form $f \circ \mu$ where $\mu \in AN^1$ and $f \in\ '$, v can be approximated by linear combinations of the form $\sum_{i=1}^n f_i \circ \mu_i$ where $f_i \in\ '$ and $\mu_i \in AN^1$ and w.l.o.g. for every $1 \leq i < j \leq n$, $\mu_i \neq \mu_j$. As v has bounded variation, the game $\sum_{i=1}^n f_i \circ \mu_i$ has bounded variation. Lemma 3 of [2] implies that each function f_i is a sum of a function $g_i \in s'$ and a function h_i which is continuous on $[0, 1]$ and absolutely continuous on its interior. Set $u = \sum_{i=1}^n h_i \circ \mu_i$. Let $\chi_0 \leq \chi_2 \leq \ldots \leq \chi_m$ be an increasing chain of ideal coalitions so that the variation of u over this chain is $\geq \|u\| - \varepsilon$. We can assume w.l.o.g that for some $\delta > 0$ we have $\delta < \chi_0 \leq \chi_m < 1 - \delta$. The proof proceeds as in the proof of Theorem 1; the only modification required is replacing the increasing path $S(t)$ with the increasing path $\chi(t)$ where $\chi(t) = \chi_j + (tm - j)(\chi_{j+1} - \chi_j)$ if $j/m \leq t \leq (j + 1)/m$ and replacing the non-atomic probability measures μ_i with finitely additive non-atomic probability measures. This shows that v can be approximated by the sum $\sum_{i=1}^n f_i \circ \mu_i$, where $f_i \in s'$ and $\mu_i \in AN_1$. ■

References

[1] Aumann, R.J. and Shapley, L.S. (1974), *Values of Non-Atomic Games*, Princeton University Press, Princeton, NJ.

[2] Mertens, J.F. and Neyman, A. (2001), A Value on $'AN$.

[3] Milnor, J.W. and Shapley, L.S. (1978), Values of Large Games II: Oceanic Games, *Mathematics of Operations Research* **3**: 290-307.

[4] Neyman, A. (1981), Singular Games Have Asymptotic Values, *Mathematics of Operations Research* **6**: 205-212.

[5] _______ (1988), Weighted Majority Games Have Asymptotic Value, *Mathematics of Operations Research* **13**: 556-580.

[6] Shapiro, N. Z. and Shapley, L.S. (1978), Values of Large Games I: A Limit Theorem, *Mathematics of Operations Research* **3**: 1-9.

[7] Shapley, L. S. (1962), Simple Games: An Outline of the Descriptive Theory, *Behavioral Science* **7**: 59-66.

[8] Shapley, L. S. and Shubik, M. (1954), A Method for Evaluating the Distribution of Power in a Committee System, American Political Science Review **48**: 787-792.

[9] von Neuman, J. and Morgenstern, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.