The Dummy Paradox of the Bargaining Set^{*}

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Abstract

By means of an example of a superadditive 0-normalized game, we show that the maximum payoff to a dummy in the bargaining set may decrease when the marginal contribution of the dummy to the grand coalition becomes positive.

We consider the weighted majority game (N, v_0) which has the tuple (3; 1, 1, 1, 1, 1, 0) as a representation (see (3)). The maximum payoff to the dummy (the last player) in the bargaining set of (N, v_0) is shown to be 2/7 (see Remark 2). If we now increase $v_0(N)$ by δ , $0 < \delta < 2/3$, then the maximum payoff to the last player in the new game, in which this player is no longer a dummy and contributes δ to N, is smaller than 2/7 and strictly decreasing in δ (see Lemma 3).

We recall some definitions and introduce relevant notations. A (cooperative TU) game is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For any game (N, v) let

$$I(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x^i \ge v(\{i\}) \text{ for all } i \in N\}$$

denote the set of *imputations*. (We use $x(S) = \sum_{i \in S} x^i$ for every $S \subseteq N$.) Let (N, v) be a game, $x \in I(N, v)$, and $k, l \in N, k \neq l$. Let

$$\mathcal{T}_{kl} = \{ S \subseteq N \setminus \{l\} \mid k \in S \}.$$

An objection of k against l at x is a pair (P, y) satisfying

$$P \in \mathcal{T}_{kl}, \ y(P) = v(P), \ \text{and} \ y^i > x^i \ \text{for all} \ i \in P.$$
 (1)

We say that k can object against l via P, if there exists y such that (P, y) is an objection of k against l. Hence k can object against l via P, if and only if $P \in \mathcal{T}_{kl}$ and e(P, x, v) > 0,

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where e(S, x, v) = v(S) - x(S) is the excess of S at x for $S \subseteq N$. A counter objection to an objection (P, y) of k against l is a pair (Q, z) satisfying

$$Q \in \mathcal{T}_{lk}, \ z(Q) = v(Q), \ z^i \ge y^i \text{ for all } i \in Q \cap P \text{ and } z^j \ge x^j \text{ for all } j \in Q \setminus P.$$
 (2)

Aumann and Maschler (1964) introduced the concepts of objections and counter objections.

An imputation $x \in I(N, v)$ is stable if for every objection at x there exists a counter objection. The bargaining set $\mathcal{M}(N, v)$ is defined by $\mathcal{M}(N, v) = \{x \in I(N, v) \mid x \text{ is stable}\}$. The bargaining set was introduced by Davis and Maschler (1967).

Player $i \in N$ is a dummy of (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. The game (N, v) is superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S \subseteq N$ and $T \subseteq N \setminus S$.

Remark 1. Let (N, v) be a game. We recall that the core of (N, v) is the set $\mathcal{C}(N, v) = \{x \in I(N, v) \mid e(S, x, v) \leq 0 \text{ for all } S \subseteq N\}$. Also we remark (see [2]) that $\mathcal{C}(N, v) \subseteq \mathcal{M}(N, v)$.

In the sequel let $N = \{1, ..., 6\}$ and (N, v_0) be the weighted majority game mentioned above. That is, $v_0(S)$, $S \subseteq N$, satisfies the following equation:

$$v_0(S) = \begin{cases} 0 & , & \text{if } |S \setminus \{6\}| \le 2\\ 1 & , & \text{if } |S \setminus \{6\}| \ge 3 \end{cases}$$
(3)

Then (N, v_0) is a superadditive game and player 6 is a dummy. Also, for every $\delta \in \mathbb{R}, \delta > 0$, let (N, v_{δ}) be the game which differs from (N, v_0) only inasmuch as $v_{\delta}(N) = 1 + \delta$.

If $0 \leq \delta \leq 2/3$, then define $x_{\delta} \in \mathbb{R}^N$ by

$$x_{\delta}^{j} = \frac{1}{7} + \frac{2}{7}\delta \text{ for } j \in N \setminus \{6\} \text{ and } x_{\delta}^{6} = \frac{2}{7} - \frac{3}{7}\delta.$$

$$\tag{4}$$

If $\delta \geq 2/3$, then define $x_{\delta} \in \mathbb{R}^N$ by

$$x_{\delta}^{j} = \frac{1}{3} \text{ for } j \in N \setminus \{6\} \text{ and } x_{\delta}^{6} = \delta - \frac{2}{3}.$$
(5)

Remark 2. For every $\delta \geq 0$, $x_{\delta} \in \mathcal{M}(N, v_{\delta})$.

Proof: Clearly $x_{\delta} \in I(N, v_{\delta})$. If $\delta \geq 2/3$, then $x_{\delta} \in \mathcal{C}(N, v_{\delta})$, thus $x_{\delta} \in \mathcal{M}(N, v_{\delta})$ by Remark 1. Now we assume $0 \leq \delta < 2/3$. Then $\mathcal{C}(N, v_{\delta}) = \emptyset$. Let $k, l \in N, k \neq l$, and let (P, y)be an objection of k against l at x_{δ} . By (1), $|P \setminus \{6\}| \geq 3$. If $l \neq 6$ and $k \neq 6$, then let $Q = (P \setminus \{k\}) \cup \{l\}$. If k = 6, then there exists $i \in P$ and let $Q = (P \setminus \{k, i\}) \cup \{l\}$. If l = 6, then select $i \in P \setminus \{k\}$ satisfying $y^i \geq y^j$ for all $j \in P \setminus \{k\}$ and let $Q = N \setminus \{k, i\}$. Also, let $z \in \mathbb{R}^Q$ be given by

$$z^{j} = \begin{cases} y^{j} & , \text{ if } j \in Q \cap P \\ v(Q) - y(P \cap Q) - x_{\delta}(Q \setminus (P \cup \{l\})) & , \text{ if } j = l \\ x_{\delta}^{j} & , \text{ if } j \in Q \setminus (P \cup \{l\}). \end{cases}$$

$$(6)$$

Then (Q, z) is a counter objection to (P, y).

q.e.d.

Lemma 3. Let $\delta \in \mathbb{R}_+$. If $x \in \mathcal{M}(N, v_{\delta})$, then $x^6 \leq x_{\delta}^6$.

Proof: Let $x \in I(N, v_{\delta})$ satisfy $x^6 > x^6_{\delta}$. It remains to show that $x \notin \mathcal{M}(N, v_{\delta})$. Without loss of generality we may assume

$$x^1 \le \dots \le x^5. \tag{7}$$

In what follows we shall construct a justified objection of 1 against 6 via the coalition $P = \{1, 2, 3\}$. We distinguish two cases:

- (1) $\delta \geq 2/3$: Then 1 can object against 6 via P by (7) and the assumption that $x^6 > x_{\delta}^6$. Also, $\{2,3,4,6\}$, $\{2,3,5,6\}$, and $\{2,4,5,6\}$ are the only coalitions in \mathcal{T}_{61} which **might** have a nonnegative excess at x. Now, player 2 is a member of all of them and $e(P, x, v_{\delta}) > e(Q, x, v_{\delta})$ for all $Q \in \mathcal{T}_{61}$, thus there exists $y \in \mathbb{R}^P$ such that y(P) = v(P), $y^i > x^i$ for all $i \in P$, and $y^2 x^2 > e(Q, x, v_{\delta})$ for all $Q \in \mathcal{T}_{61}$. We conclude that (P, y) is a justified objection of 1 against 6 at x_{δ} .
- (2) $0 \le \delta < 2/3$: Again, 1 can object against 6 via P, because $x^6 > x^6_{\delta}$. Let $Q_{\{i\}}, i = 2, 3$, and $Q_{\{2,3\}}$ be the members of \mathcal{T}_{61} defined by

$$Q_{\{i\}} = \{i, 4, 5, 6\}, \ i = 2, 3, \text{ and } Q_{\{2,3\}} = \{2, 3, 4, 6\}$$

Then

$$Q \in \mathcal{T}_{61}, \ e(Q, x, v_{\delta}) \ge 0 \Rightarrow u(Q) = 1, \tag{8}$$

because $x \ge 0$ and $x^6 > 0$. Also, we have

$$Q \in \mathcal{T}_{61}, \ u(Q) = 1 \Rightarrow e(Q, x, v_{\delta}) \le e(Q_{Q \cap \{2,3\}}, x, v_{\delta}).$$

$$\tag{9}$$

Indeed, every $Q \in \mathcal{T}_{61}$ satisfying u(Q) = 1, intersects $\{2, 3\}$, hence $Q_{Q \cap \{2,3\}}$ is defined. The inequality follows from (7). Also, $x \ge 0$, $x^6 > 0$, (7) - (9) imply that

$$e(P, x, v_{\delta}) > (e(Q, x, v_{\delta}))_{+} \text{ for all } Q \in \mathcal{T}_{61}.$$
(10)

We claim that

$$e(P, x, v_{\delta}) > (e(Q_{\{2\}}, x, v_{\delta}))_{+} + (e(Q_{\{3\}}, x, v_{\delta}))_{+}.$$
(11)

By (10) it suffices to show that

$$e(P, x, v_{\delta}) > e(Q_{\{2\}}, x, v_{\delta}) + e(Q_{\{3\}}, x, v_{\delta}),$$
(12)

which is equivalent to

$$1-x(P)>1-x(Q_{\{2\}})+1-x(Q_{\{3\}})$$

and, thus, to $-1 - x^1 + 2x(\{4, 5, 6\}) > 0$. By the observation that

$$-1 - x^{1} + 2x(\{4, 5, 6\}) = -1 + x(N) - 2x^{1} - x(\{2, 3\}) + x(\{4, 5, 6\}) \ge \delta + x^{6} - 2x^{1}$$

it suffices to show that $\delta + x^6 - 2x^1 > 0$. By (7), $5x^1 + x^6 \le 1 + \delta$, thus

$$\delta + x^6 - 2x^1 \ge \frac{3\delta + 7x^6 - 2}{5} > 0.$$

The last inequality is implied by the assumption that $x^6 > x_{\delta}^6 = 2/7 - (3/7)\delta$. Now the proof can be finished. By (10) and (11) there exists $t \in \mathbb{R}^P$ satisfying

$$t(P) = e(P, x, v_{\delta}), \ t(\{2, 3\}) > e(Q_{\{2, 3\}}, x, v_{\delta}),$$

$$t^{i} > (e(Q_{\{i\}}, x, v_{\delta}))_{+}, i \in \{2, 3\}, \text{ and } t^{1} > 0.$$
(13)

Let $y = x^P + t$. By (13), (P, y) is a justified objection of 1 against 6 at x. q.e.d.

Remark 4. The reactive bargaining set and the semi-reactive bargaining set, two variants of the bargaining set recently introduced by Granot and Maschler (1997) and Sudhölter and Potters (2001), do not show the dummy paradox. Indeed, in [4] it is shown, that both solutions, when restricted to superadditive games, satisfy the dummy property (that is, each member of the solution assigns $v(\{i\})$ to a dummy i).

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