# Games of Incomplete Information, Ergodic Theory, and the Measurability of Bayesian Equilibria

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**Abstract:** This paper discusses the difference between Harsanyi and Bayesian equilibria for games of incomplete information played on uncountable belief spaces. A conjecture belonging to ergodic theory is presented. If the conjecture were valid then there would exist a game played on an uncountable belief space with a common prior for which there are Bayesian equilibria but no Harsanyi equilibrium.

**Key words:** Bayesian Equilibria, Belief Spaces, Sequence Spaces, Nonmeasurable Sets

# 1 Introduction

An equilibrium of a game is a set of strategies, one for each player, such that no player does better by choosing a different strategy, given that the other players do not change their strategies. In a game of incomplete information, what does it mean to do no better by choosing a different strategy? Should one evaluate a player's actions according to the subjective and local belief of that player, or should one evaluate according to a common prior probability distribution determined objectively by the game?

When the subjective beliefs of the players are conditional probabilities of a common prior, a central question is whether there is an essential difference between the equilibria defined according to the players' subjective beliefs or those defined according to a global functional evaluation. J. Harsanyi (1967-8) showed that there is no essential difference between these two types of equilibria when the set of all possible situations in the game is finite. Is this claim correct when the possible situations are infinitely many?

If we mean by equivalence that Bayesian and Harsanyi equilibria should differ only in sets of measure zero, then the answer is no. In our conclusion we present an example of a payoff function resulting from a Bayesian equilibrium in a zero-sum game that cannot not measurable, meaning that the value of the game cannot be understood as the expected value of a random variable. The more interesting question is whether there is an example of a game with Bayesian equilibria but no Harsanyi equilibrium at all.

We present a conjecture and an example of a game played on a sequence space (with a shift operator) such that if the conjecture is true (and the axiom of choice is assume) then this game has Bayesian equilibria but none that are measurable with respect to the completion of the common prior (implying also that there can be no Harsanyi equilibrium).

The basic idea belongs to Ergodic Theory. One chooses a sequence space (Cantor set) with a Borel probability distribution that allows for finitely many non-commuting measure preserving involutions $\sigma_i$ whose orbits are almost everywhere dense in the space. ($\sigma_i \circ \sigma_i$ is the identity, and for almost every $x$ the subset of all the $\sigma_{i_1} \circ \sigma_{i_2} \circ \ldots \sigma_{i_n}(x)$ is dense in the space.) For each $i$ and $x$, at the point $x$ Player $i$ believes that only the points $x$ and $\sigma_i x$ are possible, with equal probability to both (and full probability if $x = \sigma_i x$). The entries in the payoff matrix are determined by the topological position in the sequence space. The conditions defining Bayesian equilibria pertain to the orbits of the involutions $\sigma_i$, and these orbits do not contruct the space in a measurable way. (With two players a set of representives, one for each orbit, is a classical non-measurable set.)

In the next section we define Bayesian and Harsanyi equilibria and prove that the countability of a belief structure implies the existence of Bayesian equilibria, regardless of whether there exists a common prior. In the third section we introduce the conjecture. In the fourth section we introduce an example of a zero-sum game of incomplete information with a common prior that would have a Bayesian equilibrium but no Harsanyi equilibrium, given that the conjecture is true. In conclusion, we explore the possible theoretical implications for zero-sum games.

# 2 Bayesian and Harsanyi Equilibria

Throughout the rest of this paper, we will assume the axiom of choice. By distance in a Euclidean space, we will mean the Euclidean distance. When $S$ is a topological space, $\Delta(S)$ will stand for the space of regular Borel probability distribution on $S$; $\Delta(S)$ will be given the weak topology induced from the topology on $S$.

## 2.1 Mertens-Zamir belief spaces

There is more than one definition of a belief space. Our example satisfies the exclusive Mertens-Zamir definition for a belief space, and so we will present this definition.

A Mertens-Zamir Belief Space (Mertens and Zamir, 1985) is a tuple $(\Omega, X, \psi, N, (t^j \mid j \in N))$, where $X$ is a compact parameter set, $\Omega$ is a compact set, $\psi$ is a continuous map from $\Omega$ to $X$, $N$ is a finite set of players, for every $j \in N$ $t^j : \Omega \to \Delta(\Omega)$ is a continous function, and for every player $j$ and every pair of points $s, s' \in \Omega$ if $s' \in$ support $(t^j(s))$ then $t^j(s) = t^j(s')$ (where support refers to the unique smallest compact set supporting the measure).

Define a *cell* of a Mertens-Zamir belief space to a minimal set $C$ with the property that at every point $y$ in $C$ every player's support set for the point $y$ is contained in $C$ (without the requirement that $C$ must be compact).

Of special interest is the definition of mutual consistency for Mertens-Zamir belief spaces. For every player $j \in N$ and define $\mathcal{T}^j$ to be the smallest

Borel field of subsets of $\Omega$ such that the function $t^j$ is measurable. A probability distribution $\mu$ on $\Omega$ is defined to be *consistent* if for every Borel subset $A \subseteq \Omega$ we have that $\mu(A) = \int t^j(y)(A)d\mu(y)$. Mertens and Zamir (1985) showed that consistency is equivalent to the stronger statement that for every $B \in \mathcal{T}^j$ and Borel subset $A \subseteq \Omega$ we have $\mu(A \cap B) = \int_B t^j(y)(A)d\mu(y)$. A consistent $\mu$ is called a *common prior.*

If $X$ is a finite set, we define a game modeled on a Mertens-Zamir belief spaces $(\Omega, X, \psi, N, (t^j \mid j \in N))$ in the following way. For each player $j$, there is a finite action set $\mathcal{A}^j$ with $n^j := |\mathcal{A}^j|$. There are $|X|$ different $n^1 \times \ldots \times n^{|N|}$ matrices $(Q_x \mid x \in X)$ corresponding to the set $X$; every entry of every matrix is a vector payoff for the players in $\mathbf{R}^N$. The players choose actions in their respective $\mathcal{A}^j$ independently and according to mixed strategies in $\Delta(\mathcal{A}^j)$. After the choices are made the payoff to the players at $y \in \Omega$ is the vector entry corresponding to their actions in the matrix $Q_{\psi(y)}$. A Bayesian equilibrium for a point $z \in \Omega$ is an $|N|$-set of functions $(f^j \mid j \in N)$, each $f^j$ from the cell that contains $z$ to $\Delta(\mathcal{A}^j)$ with the following properties for every player $j \in N$

1) $f^j$ is constant within all support sets of Player $j$,

2) for all $j' \neq j$, within the support set of $t^{j'}(z)$ the function $f^j$ is $t^{j'}(z)$ measurable, and

3) within the support set of $t^j(z)$ Player $j$ can do no better than $f^j(z) \in \Delta(\mathcal{A}^j)$ in response to the other functions $f^{j'}, j' \neq j$, as evaluated by $t^j(z)$.

When the $|N|$-set of functions is an equilibrium for all points in a cell, then we call it a Bayesian equlibrium for that cell.

Let us assume that $\mu$ is a consistent common prior for the belief space $(\Omega, X, \psi, N, (t^j \mid j \in N))$. A Harsanyi equilibrium is a set of functions $(f^j : \Omega \to \Delta(\mathcal{A}^j) \mid j \in N)$, each $f^j$ *measurable* with respect to the Borel field $\mathcal{T}^j$, such that no player can attain a higher expected payoff as evaluated by $\mu$ by choosing another such measurable function, (given that the strategies of the other players do not change).

## 2.2 Countable structures

Now we will restrict ourselves to belief structures that are countable, and simplify the set-up.

As before, $N$ is the finite set of players and for each player $j \in N$ let $\mathcal{F}^j$ be a partition of $S$, a countably infinite set. For every $A \in \mathcal{F}^j$ let

$p^{j,A}$ be a probability distribution on the set $A$. As before, for each player $j \in N$ let $\mathcal{A}^j$ be the associated finite set of pure actions, with $n_j := |\mathcal{A}^j|$. For every $s \in S$ there will be an associated $n_1 \times \ldots n_{|N|}$ matrix $Q_s$ with entries in $\mathbf{R}^N$. The actions chosen by the players at $s \in S$ and the matrix $Q_s$ determine their payoffs. We define a Bayesian equlibrium to be a set of functions $(f^j : \mathcal{F}^j \to \Delta(\mathcal{A}^j) \mid j \in N)$, one for each player, so that for every $j \in N$ and $A \in \mathcal{F}^j$ Player $j$ cannot attain a higher payoff as evaluated by $p^{j,A}$ (given that the strategies of the other players remain constant).

**Proposition 1:** If there is a uniform bound $M$ on the absolute value of all payoffs to all players, then there exists a Bayesian equilibrium.

**Proof:** Let the set $S$ be enumerated by $S = \{s_1, s_2, s_3, \ldots\}$. For every $i = 1, 2, \ldots$ define the game $\Gamma_i$ to be that played on the set $S_i := \{s_1, \ldots, s_i\}$ with the corresponding partitions $\mathcal{F}^j_i := \{A \cap S_i \mid A \in \mathcal{F}^j, A \cap S_i \neq \emptyset\}$ and probability distributions $p^{j,A}_i$ on $S_i$ defined by $p^{j,A}_i(B) := p^{j,A}(B)/p^{j,A}(S_i)$ for all $B \subseteq S_i$, with any choice for the $p^{j,A}_i(B)$ if $p^{j,A}(S_i) = 0$. Since each $S_i$ is finite, for every $i$ there exists a Nash equilibrium $\sigma_i := (\sigma^{j,A}_i \in \Delta(\mathcal{A}^j) \mid j \in N, A \in \mathcal{F}^j_i)$ to the game $\Gamma_i$. Assigning any member of $\Delta(\mathcal{A}^j)$ to $\sigma^{j,A}_i$ when $A \cap S_i = \emptyset$ and $A \in \mathcal{F}^j$, we have a sequence $\sigma_i$ in the set

$$\tilde{A} := \prod_{j \in N} \prod_{A \in \mathcal{F}^j} \Delta(\mathcal{A}^j).$$

We give $\tilde{A}$ the product topology. Due to Tychonoff's Theorem, $\tilde{A}$ is compact, and we can assume that there exists a convergent subsequence $(\sigma_{i_n} \mid n = 1, 2, \ldots)$ of the $\sigma_i$ converging to some $\sigma \in \tilde{A}$.

We aim to show that $\sigma$ defines a Bayesian equilibrium of the original game. Fix an $\epsilon > 0$. We will show that $\sigma$ is an $\epsilon$-perfect Bayesian equilibrium, meaning that at every $A \in \mathcal{F}^j$ Player $j$ can gain no more than $\epsilon$ by choosing a different strategy.

Let Player $j$ and $A \in \mathcal{F}^j$ be given. Let $\hat{i}$ be so large that $p^{j,A}(S_{\hat{i}}) \geq 1 - \epsilon/10M$. Let $\delta$ be the smallest positive probability by which Player $j$ chooses some pure action with the strategy $\sigma^{j,A}$. Now choose an $n$ so large so that $m \geq n$ implies that $\sigma^{k,B}_{i_m}$ is within $\min(\epsilon/10M|N|, \delta/3)$ of $\sigma^{k,B}$ for all the (finitely many) $B \in \mathcal{F}^k$, $k \in N$, intersecting the set $S_{\hat{i}}$ (including those with $k = j$). Given that all other players $k$ stay with the $(\sigma^k \mid k \neq j)$, we need to show that the differences in expected payoffs for Player $j$ by choosing different pure actions in support $(\sigma^{j,A})$ do not exceed $\epsilon/2$, and furthermore that Player $j$ cannot improve his payoff by at least $\epsilon/2$ compared to the best actions in support $(\sigma^{j,A})$ by choosing an action outside of support $(\sigma^{j,A})$. Because support $(\sigma^{j,A}) \subseteq$ support $(\sigma^{j,A}_{i_n})$, both claims are easy to confirm using the equilibrium properties of the $\sigma_{i_n}$ in the game $\Gamma_{i_n}$. □

# 3 The Conjecture

Let $\Delta$ be an $n$-dimensional simplex (generated by $n+1$ extremal points), and let $A_0, ..., A_n$ be polytope subsets of $\Delta$ of dimension less than $n$ (not necessarily convex finite unions of simplices of dimensions $n-1$ or less) such that the affine hull of $\cup_{i=0}^{n} A_n$ contains all of $\Delta$ and the intersection $\cap_{i=0}^{n} A_i$ is empty. Define $A := \{A_0, ..., A_n\}$, a finite set. Let $\Omega := A^{\mathbf{Z}}$, where $\mathbf{Z}$ is the (doubly infinite) set of integers. Let $P$ be the canonical Borel probability measure on $\Omega$, induced from the probabilities on the cylinder sets by giving each of the members of $A$ in each coordinate position independently the probability of $\frac{1}{n+1}$. Define a function on $\Omega$ to be measurable if it is measurable with respect to the completion of $P$ (meaning that all sets of outer measure zero are measurable). Let $T$ be the canonical shift operator on $\Omega$, such that $(Tx)^i = x^{i-1}$, where for every $y \in \Omega$ $y^k$ is the $k$ coordinate of $y$. Let $f : \Omega \to \Delta$ be a function with the property that for all $x \in \Omega$ we have $\frac{1}{2}(f(x) + f(Tx)) \in x^0$, (where $x^0 \in A$ is now perceived as a subset of $\Delta$).

**Conclusion of the conjecture:** The function $f$ cannot be measurable.

Let us consider the following example. $\Delta := [0,2]$, $A_1 := \{1/3, 4/3\}$, $A_2 := \{2/3, 5/3\}$, and $\Omega := \{A_1, A_2\}^{\mathbf{Z}}$. Due to determination relative to modulo one, any start in defining the function $f$ for any $x \in \Omega$ will define $f$ for the rest of the $T, T^{-1}$ orbit of $x$. By the axiom of choice, there exists such a function $f$ defined on all of $\Omega$.

For the sake of contradiction, let us assume that there exists such a function $f$ that is measurable. By Luzin's Theorem, we should be able to approximate our function $f$ in probability with continuous functions, meaning that for every $\epsilon > 0$ there exists a continuous function $g : \{A_1, A_2\}^{\mathbf{Z}} \to [0,2]$ such that $f$ and $g$ differ only on a set of measure $\epsilon$. Let $\epsilon$ be $1/100$. By the continuity of $g$ there is a cylinder set $C$ and a value $r \in [0,1]$ such that within the set $C$ the probability (conditioned on membership in $C$) that the value of $f$ is further than $\epsilon$ from $r$ (modulo 1) is less than $\epsilon$. Without loss of generality, we can assume that our cylinder set $C$ is defined by the choice of coordinates $y^{-k}, \dots, y^0, \dots, y^{k-1}$ for some positive integer $k$. Now let us define the value $q$ (modulo 1) by it being $f(T^{-k}(x))$ (modulo 1) for any $x \in C$ with $f(x) = r$ and define the value $s$ to be the corresponding value $f(T^k(x))$ (modulo 1). As long as $x \in C$ is mapped by $f$ to within $\epsilon$ of $r$ (modulo 1), $T^{-k}(x)$ and $T^k(x)$ will be mapped to within $\epsilon$ of $q$ and $s$, respectively (modulo 1). The occurances of points in the cylinder set $C$ in a typical orbit (by the transformation $T$) will be separated by many different sequences of intermediate values for the 0-coordinate. By our assumption, these seperating sequences must, with near certain probability, connect values within $\epsilon$ of $s$ to values within $\epsilon$ of $q$. This is an absurdity, given that $\epsilon$ is small compared to the distance between $A_1$ and $A_2$.

The difficulty with the conjecture is the degree of freedom in allowing the function $f$ to be defined for $Tx$ once it is defined for $x$. We do not know if we can drop the dimension conditions and require for the $A_i$ only that $\cap_i A_i = \emptyset$, with the conjecture remaining valid. If so, we could prove stronger statements concerning the non-existence of approximate Harsanyi equilibria.

# 4 The Example

The following is an example of a belief space with a common prior and corresponding actions and payoff matrices for which, if the conjecture is valid, there would exist Bayesian equilibria but no Harsanyi equilibria.

There are three states of nature, $w_1$, $w_2$, and $w_3$, and $W = \{w_1, w_2, w_3\}$. The space is $\Omega := W^{\mathbf{Z}}$, where $\mathbf{Z}$ is the set of integers, including both the positive and the negative integers. The 0-coordinate determines the state of nature, so that if $y \in \Omega$ and $y^0 = w_i$ then the payoff matrix is determined by the state $w_i$. The Borel probability distribution $\mu$ on $\Omega$ will be that generated canonically, giving 1/3 probability to each state independently in each position.

There are two players, and at each state each player has three pure actions. It is a zero-sum game, so that the payoffs will be those of the first player and the payoffs for the second player will be their negations. The following payoffs are for the state $w_i$ (modulo 3). The pure actions are also represented modulo 3.

If Player One chooses the pure action $i$, then the payoff is
100 if Player Two chooses $i+1$ or $i-1$, and
$-100$ if Player Two chooses $i$.

If Player One chooses the pure action $i+1$, then the payoff is
99 if Player Two chooses $i$,
100 if Player Two chooses $i+1$, and
$-100$ if Player Two chooses $i-1$.

If Player One chooses the pure action $i-1$, then the payoff is
99 if Player Two chooses $i$,
$-100$ if Player Two chooses $i+1$, and
100 if Player Two chooses $i-1$.

Next, consider the measure preserving involution $\sigma_1 : \Omega \to \Omega$ defined by $\sigma_1(y)^i := y^{-i}$, where $x^i$ is the $i$th coordinate of $x$. $\sigma_1$ is the reflection of the doubly infinite sequence about the position zero. The measure preserving involution $\sigma_2 : \Omega \to \Omega$ is defined by $\sigma_2(y)^i := y^{1-i}$. It follows that $T := \sigma_2 \circ \sigma_1$ is the shift operator $(T(y))^i = y^{i-1}$. The subjective beliefs of the players are determined as follows: at any point $y \in \Omega$ Player $j$ considers only $y$ and $\sigma_j(y)$ to be possible, and with equal probability. For $j = 1,2$ let $\mathcal{T}^j := \{B \mid B$ is Borel and $y \in B \Leftrightarrow \sigma_j y \in B\}$, the Borel subsets that are invarient with respect to $\sigma_j$. For both $j = 1,2$ we define $t^j : \Omega \to \Delta(\Omega)$ by $t^j(x)(A) := 1$ if both $x$ and $\sigma_j x$ is in $A$, $t^j(x)(A) = 1/2$ if one but not both of $x$ and $\sigma_j x$ is in $A$, and $t^j(x)(A) = 0$ if neither $x$ nor $\sigma_j x$ is in $A$. Because the $\sigma_j$ are measure preserving, it follows that the $t^j$ are regular conditional probabilities of $\mu$ with respect to the Borel fields $\mathcal{T}^j$; (see Simon, 2000).

Notice that Player One always knows the state of nature, but not necessary what Player Two believes.

**Lemma 1:** With regard to the example, any Harsanyi equilibrium will generate a Bayesian equilibrium that is measurable with respect to the completion of the common prior $\mu$.

**Proof:** For both $j = 1,2$ assume that $f^j : \Omega \to \Delta(\mathcal{A}^j)$ are Harsanyi equilibria, meaning also that they are Borel measurable functions. For both $j = 1,2$, for all $i = 1,2,3$, and all positive integers $m$ define $U_0^j(i,m)$ to be the subset of $\Omega$ such that Player $j$ can improve upon his payoff by at least

$1/m$ through deviation by the pure action $i$, as evaluated by $t^j$. Since $f^k$ for $k \neq j$ is Borel measurable, implying also that the $t^j$ evaluation of the pure action $i$ is measurable with respect to $\mathcal{T}^j$, the set $U_0^j(i,m)$ is also in $\mathcal{T}^j$. If $\mu(U_0^j(i,m))$ were positive for some $j$, $i$, and $m$, then the pair $(f^1, f^2)$ would not have been a Harsanyi equilibrium (since we could alter $f^j$ on $U_0^j(i,m)$ to choose the pure action $i$ with full probability, and Player $j$ would gain in expected payoff by at least $\frac{1}{m}\mu(U_0^j(i,m))$). We define $U_0^j$ to be $\cup_{i,m} U_0^j(i,m)$ and $U_0$ to be $\cup_j U_0^j$. Because these unions are countable, the sets $U_0^j$ and $U_0$ are also Borel and of measure zero, with $U_0^j \in \mathcal{T}^j$. Now for all $i \geq 1$ and $j = 1, 2$ define inductively the sets $U_l^j := \{x \in \Omega \mid t^j(U_{l-1}) > 0\}$ and $U_l := U_l^1 \cup U_l^2$. We claim for all $l$ and $j$ that $U_l^j$ is in $\mathcal{T}^j$, $U_l$ is Borel, and $\mu(U_l) = 0$. We proceed by induction, assuming the claim for $l-1$. That $U_{l-1}$ is Borel implies that $U_l^j$ is in $\mathcal{T}^j$, and this implies that $U_l = U_l^1 \cup U_l^2$ is Borel. Due to the formula $\mu(U_{l-1} \cap U_l^j) = \int_{U_l^j} t^j(y)(U_{l-1}) d\mu(y)$ and the fact that $t^j(y)(U_{l-1}) \geq 1/2$ for all $y \in U_l^j$, $\mu(U_l^j) > 0$ would imply that $\mu(U_{l-1}) \geq \mu(U_{l-1} \cap U_l^j) = \int_{U_l^j} t^j(y)(U_{l-1}) d\mu(y) \geq \frac{1}{2}\mu(U_l^j) > 0$, a contradiction. Since $U_l = U_l^1 \cup U_l^2$ we conclude that $\mu(U_l) = 0$.

Define $U := \cup_{l=0}^{\infty} U_l$. We have two important properties, that $U$ is Borel with $\mu(U) = 0$, and also from the structure of the example that $U$ is the union of cells (orbits of the $\sigma_j$ for $j = 1, 2$). We can alter our Harsanyi equilibrium. We keep the original functions $f^j$ on $\Omega \setminus U$, and for all the cells in the set $U$ we introduce any Bayesian equilibria obtained from Proposition 1. The result is a Bayesian equilibrium that we seek. □

**Lemma 2:** In a Bayesian equilibrium, at every pair of points $x$ and $\sigma_1 x$ considered possible by Player One, he receives an expected payoff of at least $9,900/299$ (which is approximately 33.11).

**Proof:** We consider the game of complete information played on any one of the three symmetric payoff matrices. Without loss of generality, look at that payoff matrix corresponding to State $w_1$. The result follows by considering the optimal mixed strategies of $(100/299, 199/598, 199/598)$ for Player Two and $(99/299, 100/299, 100/299)$ for Player One. □

**Lemma 3:** In a Bayesian equilibrium, at every pair of points $x$ and $\sigma_2 x$ considered possible by Player Two, Player One receives an expected payoff of no more than $19,900/599$ (which is approximately 33.222).

**Proof:** If $x$ and $\sigma_2 x$ share the same state of nature, then from the strategies given in the proof of Lemma 2 we know that Player Two can obtain at least $-9,900/299$ for himself. If there are two states of nature, due to symmetry we will consider only the case that these two states are States $w_1$ and $w_2$. Consider the following three strategies of Player One:
(i) at either state choose Action 1,
(ii) at State $w_1$ choose Action 1 and at State $w_2$ choose the Action 2,
(iii) at either state choose Action 2.
The result follows by consider the mixed strategies $(200/599, 200/599, 199/599)$ for Player Two and $(200/599, 199/599, 200/599)$ for Player One, where for Player One these are the three pure strategies defined above. Against the above mixed strategy of Player Two, none of Player One's six other pure strategies are superior. $\square$

The difference of $19,900/599 - 9,900/299$ (approximately .112) is the maximal quantity of exploitation that can be obtained by Player One from his knowing the state of nature.

**Lemma 4:** In a Bayesian equilibrium, at no pair of points $x$ and $\sigma_1 x$ does Player One choose only one pure action.

**Proof:** The proof of Lemma 4 rests on the following claim.

**Claim:** If Player One chooses any pure action at the pair $x$ and $\sigma_1 x$ with at least a probability of $1 - \delta$ with $0 \leq \delta \leq 3/10$, then at the pair $x$ and $\sigma_2 x$ and at the pair $\sigma_1 x$ and $\sigma_2 \circ \sigma_1 x$ Player Two obtains a payoff of at least $-100\delta$ (meaning that Player One gets no more than $100\delta$) and Player One chooses some pure action at $\sigma_2 x$ and $\sigma_2 \circ \sigma_1 x$ with probability of at least $1 - 2\delta - 1/200$.

**Proof of the claim:** Without loss of generality, we consider only the points $x$ and $\sigma_2 x$. That Player Two can obtain $-100\delta$ follows from the fact that he can always choose the appropriate action for which at $x$ he obtains for himself at least $100(1-\delta) - 100\delta = 100 - 200\delta$, (and the worst that can happen to him at the other point is $-100$).

We assume that Player One has chosen Action $i$ at $x$ with probability of at least $1 - \delta$. Due to $\delta \leq 3/10$ and Lemma 2, we must assume that Player Two has some reason to choose some other action than the one which, when paired with Player One's Action $i$, the payoff for Player One at the point $x$ is $-100$. Since Player Two must obtain $-100\delta$ from this alternative action when averaging between the point $x$ and $\sigma_2 x$, the only way to offer Player Two at least an equally desirable payoff is if Player One chooses some action at $\sigma_2 x$ with a probability of $p$, where $p$ satisfies $(1-\delta)99-\delta 100+100(1-p)-100p \leq 200\delta$. This equation deliver the claim.

Now that the claim is proven, let us assume that Player One chooses some pure action with certainty. By induction, we can assume that there is a sequence $x, Tx, \ldots, T^8 x$ where at all the points $Tx$ to $T^7 x$ Player One chooses some action with probability at least $1 - 7/200 = .965$ (and some action is chosen with certainty at $T^4 x$). Assigning the value of $v$ to the payoff of Player One at the point $x$, because Player One can obtain at least 33 at the pair $x$ and $\sigma_1 x$, we know that the payoff at $\sigma_1 x$ must be at least $33 - v$. But since at the pair $\sigma_1 x$ and $\sigma_2 \circ \sigma_1 x = Tx$ Player Two can hold down the payoff to $7/2$, we must conclude that the payoff at $\sigma_2 \sigma_1 x = Tx$ cannot exceed $v - 26$. By induction we conclude that the payoff to Player One at the point $T^8 x$ cannot exceed $v - 208$. But this is impossible, since no difference of payoffs can exceed 200. □

**Lemma 5:** At no pair of points $x$ and $\sigma_2 x$ can the probability that Player Two chooses some pure action exceed $3/4$.

**Proof:** Due to Lemma 2, at neither $x$ nor $\sigma_2 x$ can Player One choose the action that gives $-100$ when combined with this action of Player Two (since Player One would get no more than $\frac{5}{8}100 - \frac{3}{8}100 = 25$). We conclude that Player Two obtains a payoff for himself at $x$ and $\sigma_2 x$ no better than $-99$, a contradiction to Lemma 3. □

Now we will consider the behavior strategy of Player Two.

From Lemma 4 we know that Player Two must act so that at every pair $x$ and $\sigma_1 x$ Player One is indifferent between at least two pure actions. Let us consider the case of $x^0 = w_1$ first, and the following four possibilities for the pair $x$ and $\sigma_1 x$:

1) Player One is indifferent between the actions 1 and 2 and does not prefer Action 3,

2) Player One is indifferent between the actions 1 and 3 and does not prefer Action 2,

3) Player One is indifferent between the actions 2 and 3 and does not prefer Action 1,

4) Player One is indifferent between all three actions.

The last case is the easiest to consider. We know from Lemma 2 that the average distribution of Player Two's mixed strategies between the pair $x$ and $\sigma_1 x$ must be the probability vector $p_1 := (100/299, 199/598, 199/598)$. But with Case 4 understood, the other cases are also easy to understand. For Case 1, we obtain the line segment between $(0,1,0)$ and $p_1$. For Case 2, we obtain the line segment between $(0,0,1)$ and $p_1$. For Case 3, we obtain the line segment between $(1,0,0)$ and $p_1$. Note, however, that by Lemma 5 we can restrict ourselves to the subset where no pure action of Player Two receives probability greater than 3/4. Define the subset $C_1 \subseteq \Delta(\{1,2,3\})$ by $C_1$ being the union of these three line segments intersected with $\{p \in \Delta(\{1,2,3\}) \mid \forall k = 1,2,3 \; p^k \leq 3/4\}$.

Define the points $p_2$ and $p_3$ and the sets $C_2$ and $C_3$ symmetrically, with respect to the states $w_2$ and $w_3$. Notice that $C_1 \cap C_2 \cap C_3$ is empty, (now that no pure action of Player Two is used with any probability above 3/4).

**Proposition 2:** If the conjecture is valid, then the above is an example of a game with a Bayesian equilibrium but no Harsanyi equilibrium.

**Proof:** By Proposition 1 (and the axiom of choice) there exists a Bayesian equilibrium.

For the sake of contradiction suppose that $f : \Omega \to \Delta(\{1,2,3\})$ is the behavior strategy of Player Two in a Harsanyi equilibrium. By Lemma 1 we can assume without loss of generality that $f$ is the behavior strategy of Player Two in a Bayesian equilibrium that is measurable with respect to the completion of the common prior $\mu$.

By Lemmatta 4 and 5, we know for every $x \in \Omega$ that $f$ must satisfy $\frac{1}{2}(f(x) + T(f(x)) \in C_k$ with $x^0 = w_k$. But the conjecture claims that this is not compatible with the measurability of $f$. □

# 5 Conclusion: The Value of Zero-Sum Games

We are interested in zero-sum games on uncountably large belief spaces because of the possibilities for violating conventional game theory. In finitely defined zero-sum games, if an equilibrium exists we say that the game has a value, namely the payoff to the first player in equilibrium. Equilibrium strategies are also called optimal strategies, because if one exchanges the strategies in two pairs of equilibria, one obtains again an equilibrium and the same value as an expected payoff.

The lack of a common prior on a cell prevents the usual kind of payoff evaluation that would justify the exchange property. A point by point evaluation to support an exchange property would be useless, because two different equilibria could give different expected payoffs to a given point (even with a finite belief space). This is also the problem behind assigning a value to such a game. If the only equilibria are non-measurable, then the payoff function of all equilibria may fail to be measurable, so that a value as the expected value of a random variable would be meaningless.

One does not need a game with the complication of our above example to demonstrate that a zero-sum game may have a Bayesian equlibrium without a random variable as the payoff function. Let $\Omega$ be $\{0,1\}^{\mathbf{Z}}$ with the same belief structure as before, defined by the measure preserving involutions $\sigma_1$ and $\sigma_2$, and $T = \sigma_2 \circ \sigma_1$. We assume that there is only one payoff matrix, that corresponding to the well known game of *matching pennies*. Both players have only two pure actions, 0 and 1, and if the sum of their pure actions is even then Player One receives 1 and if this sum is odd than Player One receives $-1$. Let $f^j : \Omega \to [0,1]$ represent the probability that Player $j$ chooses the pure action 0, and of course we must assume that for both $j = 1,2$ and every $x \in \Omega$ we have $f^j(x) = f^j(\sigma_j(x))$.

Consider the following behavior of the players. At some $x \in \Omega$ we have $f^1(x) = f^2(x) = 1$, but also we have $f^1(\sigma_2(x)) = f^1(\sigma_1 \circ \sigma_2(x)) = 0$ and $f^2(\sigma_1(x)) = f^2(\sigma_2 \circ \sigma_1(x)) = 0$. Continue defining the strategies $f^1$ and $f^2$ on the cell containing $x$ so that at all $y$ in the cell we have $f^1(y) + f^1(\sigma_2(y)) = 1$ and $f^2(y) + f^2(\sigma_1(y)) = 1$, with the values for $f^1$ and $f^2$ always either 0 or 1. This defines a Bayesian equilibrium on the cell. To create a Bayesian equilibrium for all of $\Omega$, choose such strategies for all the cells of $\Omega$ (assuming the axiom of choice). Notice that the subset of $\Omega$ where Player One receives a payoff of 1 is a $T$-invarient set, and the same holds for the subset where Player One receives $-1$. If the payoff function of this Bayesian equilibrium were measurable, then the subset where Player One receives $-1$ must be a set of measure 0 or 1 (by the mixing property of the transformation $T$). We would conclude that the global expected payoff for Player One is either $-1$ or 1, an absurdity.

Of course there does exist a Harsanyi equilibrium for this game, namely both players at all points choose the probability of $1/2$ for the pure action 0.

In spite of the inability to define the value of a zero-sum game with a Bayesian equilibrium in the conventional way, one could define a value on a cell as the limit of average payoffs radiating from any given point in the cell (should such a value exist), and then define the global value as a common (or expected) value for the cells containing almost all points (should it exist). With games defined on sequence spaces (as modeled above) there should be hope for proving the existence of such a value.

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# 6 References

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