Values of Games with Infinitely Many Players

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1 Introduction

The Shapley value is one of the basic solution concepts of cooperative game theory. It can be viewed as a sort of average or expected outcome, or as an *a priori* evaluation of the players' expected payoffs.

The value has a very wide range of applications, particularly in economics and political science (see chapters 32, 33 and 34 in this Handbook). In many of these applications it is necessary to consider games that involve a large number of players. Often most of the players are individually insignificant, and are effective in the game only via coalitions. At the same time there may exist big players who retain the power to wield single-handed influence. A typical example is provided by voting among stockholders of a corporation, with a few major stockholders and an "ocean" of minor stockholders. In economics, one considers an oligopolistic sector of firms embedded in a large population of "perfectly competitive" consumers. In all these cases, it is fruitful to model the game as one with a continuum of players. In general, the continuum consists of a non-atomic part (the "ocean"), along with (at most countably many) atoms. The continuum provides a convenient framework for mathematical analysis, and approximates the results for large finite games well. Also, it enables a unified view of games with finite, countable, or oceanic player-sets, or indeed any mixture of these.

1.1 An Outline of the Chapter

In Section 2 we highlight a sample of a few asymptotic results on the Shapley value of finite games. These results motivate the definition of games with infinitely many players and the corresponding value theory; in particular, Proposition 1 introduces a formula, called the diagonal formula of the value, that expresses the value (or an approximation thereof) by means of an integral along a specific path.

The Shapley value for finite games is defined as a linear map from the linear space of games to payoffs that satisfies a list of axioms: symmetry, efficiency and the null player axiom. Section 3 introduces the definitions of the space of games with a continuum of players, and defines the symmetry, positivity and efficiency of a map from games to payoffs (represented by finitely additive games) which enables us to define the value on a space of games as a linear, symmetric, positive and efficient map.

The value does not exist on the space of all games with a continuum of players. Therefore the theory studies spaces of games on which a value does exist and moreover looks for spaces for which there is a unique value. Section 4 introduces several spaces of games with a continuum of players including the two spaces of games, pNA and $bv'NA$, which played an important role in the development of value theory for non-atomic games. Theorem 2 asserts that there exists a unique value on each of the spaces pNA and $bv'NA$ as well as on pNA_∞. In addition there exists a (unique) value of norm 1 on the space $'NA$. Section 4 also includes a detailed outline of the proofs.

A game with infinitely many players can be considered as a limit of finite games with a large number of players. Section 5 studies limiting values which are defined by means of the limits of the Shapley value of finite games that approximate the given game with infinitely many players. Section 5 also introduces the asymptotic value. The asymptotic value of a game v is defined whenever all the sequences of Shapley values of finite games that"approximate" v have the same limit. The space of all games of bounded variation for which the asymptotic value exist is denoted $ASYMP$. Theorem 4 asserts that $ASYMP$ is a closed linear space of games that contains $bv'M$ and the map associating an asymptotic value with each game in $ASYMP$ is a value on $ASYMP$.

Section 6 introduces the mixing value which is defined on all games of bounded variation for which an analogous formula of the random order one for finite games is well defined. The space of all games having a mixing value is denoted MIX. Theorem 5 asserts that MIX is a closed linear space which contains pNA, and the map that associates a mixing value with each game v in MIX is a value on MIX.

Section 7 introduces value formulas. These value formulas capture the idea that the value of a player is his expected marginal contribution to a perfect sample of size t of the set of all players where the size t is uniformly distributed on [0,1]. A value formula can be expressed as an integral of directional derivative whenever the game is a smooth function of finitely many non-atomic measures. More generally, by extending the game to be defined over all ideal coalitions – measurable [0,1]-valued functions on the (measurable) space of players – this value formula provides a formula for the value of any game in pNA. Changing the order of derivation and integration results in a formula that is applicable to a wider class of games: all those games for which the formula yields a finitely additive game. In particular, this modified formula defines a value on $bv'NA$ or even on $bv'M$. When this formula does not yield a finitely additive game, applying the directional derivative to a carefully crafted small perturbation of perfect samples of the set of players yields a value formula on a much wider space of games (Theorem 7). These include games which are nondifferentiable functions (e.g., a piecewise linear function) of finitely many non-atomic probability measures.

Section 8 describes two spaces of games that are spanned by nondifferentiable functions of finitely many mutually singular non-atomic probability measures that have a unique value (Theorem 8). One is the space spanned by all piecewise linear functions of finitely many mutually singular non-atomic probability measures, and the other is the space spanned by all concave Lipschitz functions with increasing returns to scale of finitely many mutually singular non-atomic probability measures.

The value is defined as a map from a space of games to payoffs that satisfies a short list of plausible conditions: linearity, symmetry, positivity and efficiency. The Shapley value of finite games satisfies many additional desirable properties. It is continuous (of norm 1 with respect to the bounded variation norm), obeys the null player and the dummy axioms, and it is strongly positive. In addition, any value that is obtained from values of finite approximations obeys an additional property called diagonality. Section 9 introduces such additional desirable value properties as strong positivity and diagonality. Theorem 9 asserts that strong positivity can replace positivity and linearity in the characterization of the value on pNA_∞, and thus strong positivity and continuity can replace positivity and linearity in the characterization of a value on pNA. A striking property of the value on pNA, the asymptotic value, the mixing value, or any of the values obtained by the formulas described in Section 7, is the diagonal property: if two games coincide on all coalitions that are "almost" a perfect sample (within ε with respect to finitely many non-atomic probability measures) of the set of players, their values coincide. Theorem 11 asserts that any continuous value is diagonal.

Section 10 studies semivalues which are generalizations of the value that do not necessarily obey the efficiency property. Theorem 13 provides a characterization of all semivalues on pNA. Theorems 14 and 15 provide results on asymptotic semivalues.

Section 11 studies partially symmetric values which are generalizations of the value that do not necessarily obey the symmetry axiom. Theorems 15 and 16 characterize in particular all the partially symmetric values on pNA and pM that are symmetric with respect to those symmetries that preserve a fixed finite partition of the set of players. A corollary of this characterization on pM is the characterization of all values on pM. A partially symmetric value, called a μ-value, is symmetric with respect to all symmetries that preserve a fixed non-atomic population measure μ and (in addition to linearity, positivity and efficiency), obeys the dummy axiom. Such a partially symmetric value is called a μ-value. Theorem 18 asserts the uniqueness of a μ-value on $pNA(\mu)$. Theorem 19 asserts that an important class of nondifferentiable markets have an asymptotic μ-value.

Games that arise from the study of non-atomic markets, called market games, obey two specific conditions: concavity and homogeneity of degree 1. The core of such games is nonempty. Section 12 provides results on the relation between the value and the core of market games. Theorem 20 asserts the coincidence of the core and the value in the differentiable case. Theorem 21 relates the existence of an asymptotic value of nondifferentiable market games to the existence of a center of symmetry of its core. Theorems 22 and 23 describe the asymptotic μ-value and the value of market games with a finite-dimensional core as an average of the extreme points of the core.

2 Prelude: Values of Large Finite Games

This section introduces several asymptotic results on the Shapley value ψ of finite games. These results can serve as an introduction to the theory of values of games with infinitely many players, by motivating the definitions of games with a continuum of players and the corresponding value theory.

We start with the introduction of a representation of games with finitely many players. Let ℓ be a fixed positive integer and $f : \mathbb{R}^\ell \to \mathbb{R}$ a function with $f(0) = 0$. Given $w_1, \ldots, w_\ell \in \mathbb{R}^\ell_+$, define the n-person game $v = [f; w_1, \ldots, w_\ell]$ by

$$v(S) = f(\sum_{i \in S} w_i).$$

A special subclass of these games is weighted majority games where $\ell = 1$, $0 < q < \sum_{i=1}^n w_i$ and $f(x) = 1$ if $x \geq q$ and 0 otherwise. The asymptotic results relate to sequences of finite games where the function f is fixed and the vector of weights varies. If the function f is differentiable at x, the directional derivative at x of the function f in the direction $y \in \mathbb{R}^\ell$ is $f_y(x) := \lim_{\varepsilon \to 0+}(f(x + \varepsilon y) - f(x))/\varepsilon$.

The following result follows from the proofs of results on the asymptotic value of non-atomic games. It can be derived from the proof in Kannai (1966) or from the proof in Dubey (1980).

Proposition 1 *Assume that f is a continuously differentiable function on $[0,1]^\ell$. For every $\varepsilon > 0$ there is $\delta > 0$ such that if $w_1, \ldots, w_n \in I\!R^\ell_+$ with (the normalization) $\sum_{i=1}^n w_i = (1, \ldots, 1)$ and $\max_{i=1}^n \|w_i\| < \delta$, then*

$$|\psi v(i) - \int_0^1 f_{w_i}(t, \ldots, t)\ dt| \leq \varepsilon \|w_i\|,$$

where $v = [f; w_1, \ldots, w_n]$. Thus, for every coalition S of players,

$$|\psi v(S) - \int_0^1 f_{w(S)}(t, \ldots, t)\ dt| < \varepsilon,$$

where $\psi v(S) = \sum_{i \in S} \psi v(i)$ and $w(S) = \sum_{i \in S} w_i$.

The limit of a sequence of games of the form $[f; w_1^k, \ldots, w_{n_k}^k]$, where f is a fixed function and $w_1^k, \ldots, w_{n_k}^k$ is a sequence of vectors in $I\!R^\ell_+$ with $\sum_{i=1}^{n_k} w_i^k = (1, \ldots, 1)$ and $\max_{i=1}^{n_k} \|w_i\| \to_{k \to \infty} 0$ can be modeled by a 'game' of the form $f \circ (\mu_1, \ldots, \mu_\ell)$ where $\mu = (\mu_1, \ldots, \mu_\ell)$ is a vector of non-atomic probability measures defined on a measurable space $(I, \mathcal{C})$. If f is continuously differentiable, the limiting value formula thus corresponds to:

$$\varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t, \ldots, t)\ dt.$$

The next results address the asymptotics of the value of weighted majority games. The n-person weighted majority game $v = [q; w_1, \ldots, w_n]$ with $w_1, \ldots, w_n \in I\!R_+$ and $0 < q < \sum_{i=1}^m w_i$, is defined by $v(S) = 1$ if $\sum_{i \in S} w_i \geq q$ and 0 otherwise.

Proposition 2 *[Neyman 1981a]. For every $\varepsilon > 0$ there is $K > 0$ such that if $v = [q; w_1, \ldots, w_n]$ is a weighted majority game with $w_1, \ldots, w_n \in I\!R^+$, $\sum_{i=1}^n w_i = 1$ and $K \max_{i=1}^n w_i < q < 1 - K \max_{i=1}^n w_i$,*

$$|\psi v(S) - \sum_{i \in S} w_i| < \varepsilon$$

for every coalition $S \subset \{1, \ldots, n\}$.

In particular, if $v_k = [q_k; w_1^k, \ldots, w_{n_k}^k]$ is a sequence of weighted majority games with $w_i^k \geq 0$, $\sum_{i=1}^{n_k} w_i^k \rightarrow_{k\rightarrow\infty} 1$, and $q_k \rightarrow_{k\rightarrow\infty} q$ with $0 < q < 1$, then for any sequence of coalitions $S_k \subset \{1, \ldots, n_k\}$, $\psi v_k(S_k) - \sum_{i\in S_k} w_i^k \rightarrow_{k\rightarrow\infty} 0$. The limit of such a sequence of weighted majority games is modeled as a game of the form $v = f_q \circ \mu$ where μ is a non-atomic probability measure and $f_q(x) = 1$ if $x \geq q$ and 0 otherwise. The non-atomic probability measure μ describes the value of this limiting game:

$$\varphi v(S) = \mu(S).$$

The next result follows easily from the result of Berbee (1981).

Proposition 3 *Let $w_1, \ldots, w_n, \ldots$ be a sequence of positive numbers with $\sum_{i=1}^{\infty} w_i = 1$ and $0 < q < 1$. There exists a sequence of nonnegative numbers ψ_i, $i \geq 1$, with $\sum_{i=1}^{\infty} \psi_i = 1$, such that for every $\varepsilon > 0$ there is a positive constant $\delta > 0$ and a positive integer n_0 such that if $v = [q'; u_1, \ldots, u_n]$ is a weighted majority game with $n \geq n_0$, $|q' - q| < \delta$ and $\sum_{i=1}^{n} |w_i - u_i| < \delta$, then*

$$|\psi v(i) - \psi_i| < \varepsilon.$$

Equivalently, let $v_k = [q_k; w_1^k, \ldots, w_{n_k}^k]$ be a sequence of weighted majority games with $\sum_{i=1}^{n_k} w_i^k \rightarrow_{k\rightarrow\infty} 1$ and $w_j^k \rightarrow_{k\rightarrow\infty} w_j$ and $q_k \rightarrow_{k\rightarrow\infty} q$. Then the Shapley value of player i in the game v_k converges as $k \rightarrow \infty$ to a limit ψ_i and $\sum_{i=1}^{\infty} \psi_i = 1$. The limit of such a sequence of weighted majority games can be modeled as a game of the form $v = f_q \circ \mu$ where μ is a purely atomic measure with atoms $i = 1, 2, \ldots$ and $\mu(i) = w_i$. The sequence ψ_i describes the value of this limiting game:

$$\varphi v(S) = \sum_{i\in S} \psi_i.$$

3 Definitions

The space of players is represented by a measurable space $(I, \mathcal{C})$. The members of set I are called *players*, those of $\mathcal{C}$ – *coalitions*. A game in coalitional form is a real-valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. For each coalition S in $\mathcal{C}$, the number $v(S)$ is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members; it is called the *worth* of S. The set of all games forms a linear space over the field of real numbers, which is denoted

G. We assume that $(I, \mathcal{C})$ is isomorphic to $([0,1], \mathcal{B})$, where $\mathcal{B}$ stands for the Borel subsets of $[0,1]$. A game v is *finitely additive* if $v(S \cup T) = v(S) + v(T)$ whenever S and T are two disjoint coalitions. A distribution of payoffs is represented by a finitely additive game. A value is a mapping from games to distributions of payoffs, i.e., to finitely additive games that satisfy several plausible conditions: linearity, symmetry, positivity and efficiency. There are several additional desirable conditions: e.g., continuity, strong positivity, a null player axiom, a dummy player axiom, and so on. It is remarkable, however, that a short list of plausible conditions that define the value suffices to determine the value in many cases of interest, and that in these cases the unique value satisfies many additional desirable properties. We start with the notations and definitions needed formally to define the value, and set forth four spaces of games on which there is a unique value.

A game v is *monotonic* if $v(S) \geq v(T)$ whenever $S \supset T$; it is of *bounded variation* if it is the difference of two monotonic games. The set of all games of bounded variation (over a fixed measurable space $(I, \mathcal{C})$) is a vector space. The *variation* of a game $v \in G$, $\|v\|$, is the supremum of the variation of v over all increasing chains $S_1 \subset S_2 \subset \ldots \subset S_n$ in $\mathcal{C}$. Equivalently, $\|v\| = \inf\{u(I) + w(I) : u, w \text{ are monotonic games with } v = u - w\}$ $(\inf \emptyset = \infty)$. The variation defines a topology on G; a basis for the open neighborhood of a game v in G consists of the sets $\{u \in G : \|u - v\| < \varepsilon\}$ where ε varies over all positive numbers. Addition and multiplication of two games are continuous and so is the multiplication of a game and a real number. A game v has bounded variation if $\|v\| < \infty$. The space of all games of bounded variation, BV, is a Banach algebra with respect to the variation norm $\| \ \|$, i.e., it is complete with respect to the distance defined by the norm (and thus it is a Banach Space) and it is closed under multiplication which is continuous. Let $\mathcal{G}$ denote the group of automorphisms (i.e., one-to-one measurable mappings Θ from I onto I with Θ^{-1} measurable) of the underlying space $(I, \mathcal{C})$. Each Θ in $\mathcal{G}$ induces a linear mapping Θ_* of BV (and of the space of all games) onto itself, defined by $(\Theta_* v)(S) = v(\Theta S)$. A set of games Q is called *symmetric* if $\Theta_* Q = Q$ for all Θ in $\mathcal{G}$. The space of all finitely additive and bounded games is denoted FA; the subspace of all measures (i.e., countably additive games) is denoted M and its subspace consisting of all non-atomic measures is denoted NA. The space of all non-atomic elements of FA is denoted AN. Obviously, NA $\subset$ M $\subset$ FA $\subset$ BV, and each of the spaces NA, AN, M, FA and BV is a symmetric space. Given a set of games Q, we denote by Q^+

all monotonic games in Q, and by Q^1 the set of all games v in Q^+ with $v(I) = 1$. A map $\varphi : Q \to BV$ is called *positive* if $\varphi(Q^+) \subset BV^+$; *symmetric* if for every $\Theta \in \mathcal{G}$ and v in Q, $\Theta_* v \in Q$ implies that $\varphi(\Theta_* v) = \Theta_*(\varphi v)$; and *efficient* if for every v in Q, $(\varphi v)(I) = v(I)$.

Definition 1 *[Aumann and Shapley 1974]. Let Q be a symmetric linear subspace of BV. A* value *on Q is a linear map $\varphi : Q \to FA$ that is symmetric, positive and efficient.*

The definition of a value is naturally extended also to include spaces of games that are not necessarily included in BV. Thus let Q be a linear and symmetric space of games.

Definition 2 *A* value *on Q is a linear map from Q into the space of finitely additive games that is symmetric, positive (i.e., $\varphi(Q^+) \subset BV^+$), efficient, and $\varphi(Q \cap BV) \subset FA$.*

There are several spaces of games that have a unique value. One of them is the space of all games with a finite support, where a subset T of I is called a *support* of a game v if $v(S) = v(S \cap T)$ for every S in $\mathcal{C}$. The space of all games with finite support is denoted FG.

Theorem 1 *[Shapley 1953a]. There is a unique value on FG.*

The unique value ψ on FG is given by the following formula. Let T be a finite support of the game v (in FG). Then, $\psi v(S) = 0$ for every S in $\mathcal{C}$ with $S \cap T = \emptyset$, and for $t \in T$,

$$\psi v(\{t\}) = \frac{1}{n!} \sum_{\mathcal{R}} \left[v(\mathcal{P}_t^{\mathcal{R}} \cup \{t\}) - v(\mathcal{P}_t^{\mathcal{R}}) \right],$$

where n is the number of elements of the support T, the sum runs over all $n!$ orders of T, and $\mathcal{P}_t^{\mathcal{R}}$ is the set of all elements of T preceding t in the order $\mathcal{R}$.

Finite games can also be identified with a function v defined on a finite subfield π of $\mathcal{C}$: given a real-valued function v defined on a finite subfield π of $\mathcal{C}$ with $v(\emptyset) = 0$, the Shapley value of the game v is defined as the additive function $\psi : \pi \to I\!R$ defined by its values on the atoms of π; for every atom a of π,

$$\psi v(a) = \frac{1}{n!} \sum \mathcal{R} \left[v\left(\mathcal{P}_a^{\mathcal{R}} \cup a\right) - v\left(\mathcal{P}_a^{\mathcal{R}}\right) \right],$$

where n is the number of atoms of π, the sum runs over all $n!$ orders of the atoms of π, and $\mathcal{P}_a^{\mathcal{R}}$ is the union of all atoms of π preceding a in the order $\mathcal{R}$.

4 The Value on pNA and $bv'NA$

Let pNA (pM, pFA) be the closed (in the bounded variation norm) algebra generated by NA (M, FA, respectively). Equivalently, pNA is the closed linear subspace that is generated by powers of measure in NA^1 (Aumann and Shapley 1974, Lemma 7.2). Similarly, pM (pFA) is the closed algebra that is generated by powers of elements in M (FA). Let $\| \ \|_\infty$ be defined on the space of all games G by $\|v\|_\infty = \inf\{\mu(I) : \mu \in NA^+,\ \mu - v \in BV^+, \mu + v \in BV^+\} + \sup\{|v(S)| \ : S \in \mathcal{C}\}$. In the definition of $\|v\|_\infty$, we use the common convention that $\inf \emptyset = \infty$. The definition of $\| \ \|_\infty$ is an analog of the C^1 norm of continuously differentiable functions; e.g., if f is continuously differentiable on $[0,1]$ and μ is a non-atomic probability measure, $\|f \circ \mu\|_\infty = \max\{|f'(t)| : 0 \le t \le 1\} + \max\{|f(t)| : 0 \le t \le 1\}$. Note that $\|uv\|_\infty \le \|u\|_\infty \|v\|_\infty$ for two games u and v. The function $v \mapsto \|v\|_\infty$ defines a topology on G; a basis for the open neighborhood of a game v in G consists of the sets $\{u \in G : \|u - v\|_\infty < \varepsilon\}$ where ε varies over all positive numbers. The space of games pNA_∞ is defined as the $\| \ \|_\infty$ closed linear subspace that is generated by powers of measures in NA^1. Obviously, $pNA_\infty \subset pNA$ ($\| \ \|_\infty \ge \| \ \|$). Both spaces contain many games of interest. If μ is a measure in NA^1 and $f : [0,1] \to I\!R$ is a function with $f(0) = 0$, then $f \circ \mu \in pNA$ if and only if f is absolutely continuous (Aumann and Shapley 1974, Theorem C) and in that case $\|f \circ \mu\| = \int_0^1 |f'(t)| dt$ where f' is the (Radon-Nikodym) derivative of the function f, and $f \circ \mu \in pNA_\infty$ if and only if f is continuously differentiable on $[0,1]$. Also, if $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of NA measures with range $\{\mu(S) : S \in \mathcal{C}\} =: \mathcal{R}(\mu)$, and f is a C^1 function on $\mathcal{R}(\mu)$ with $f(\mu(\emptyset)) = 0$, then $f \circ \mu \in pNA_\infty$ (Aumann and Shapley 1974, Prop. 7.1).

Let $bv'NA$ ($bv'M$, $bv'FA$) be the closed linear space generated by games of the form $f \circ \mu$ where $f \in bv' =: \{f : [0,1] \to I\!R \mid f$ is of bounded variation and continuous at 0 and at 1 with $f(0) = 0\}$, and $\mu \in NA^1$ ($\mu \in M^1, \mu \in FA^1$). Obviously, $pNA_\infty \subset pNA \subset bv'NA \subset bv'M \subset bv'FA$.

Theorem 2 *[Aumann and Shapley 1974, Theorems A, B; Monderer 1990, Theorem 2.1]. Each of the spaces pNA, $bv'NA$, pNA_∞ has a unique value.*

The proof of the uniqueness of the value on these spaces relies on two basic arguments. Let $\mu \in NA^1$ and let $\mathcal{G}(\mu)$ be the subgroup of $\mathcal{G}$ of all automorphisms Θ that are μ-measure-preserving, i.e., $\mu(\Theta S) = \mu(S)$ for every coalition S in $\mathcal{C}$. Then the set of all finitely additive games u that are fixed points of the action of $\mathcal{G}(\mu)$ (i.e., $\Theta_* u = u$ for every Θ in $\mathcal{G}(\mu)$) are games of the form $\alpha\mu$, $\alpha \in I\!R$ (here we use the standardness assumption that $(I, \mathcal{C})$ is isomorphic to $([0,1], \mathcal{B})$). Therefore, if Q is a symmetric space of games and $\varphi : Q \to FA$ is symmetric, $\mu \in NA^1$, and $f : [0,1] \to I\!R$ is such that $f \circ \mu \in Q$, then $\Theta_*(f \circ \mu) = f \circ \mu$ for every $\Theta \in \mathcal{G}(\mu)$. It follows that the finitely additive game $\varphi(f \circ \mu)$ is a fixed point of $\mathcal{G}(\mu)$, which implies that $\varphi(f \circ \mu) = \alpha\mu$. If, moreover, φ is also efficient, then $\varphi(f \circ \mu) = f(1)\mu$. The spaces pNA and $bv'NA$ are closed linear spans of games of the form $f \circ \mu$. The space pNA_∞ is in the closed linear span of games of the form $f \circ \mu$ with $f \circ \mu$ in pNA_∞. Therefore, each of the spaces pNA_∞, pNA and $bv'NA$ has a dense subspace on which there is at most one value.

To complete the uniqueness proof, it suffices to show that any value on these spaces is indeed continuous. For that we apply the concept of internality. A linear subspace Q of BV is said to be *reproducing* if $Q = Q^+ - Q^+$. It is said to be *internal* if for all v in Q, $\|v\| = \inf\{u(I) + w(I) : u, w \in Q^+ \text{ and } v = u - w\}$. Clearly, every internal space is reproducing. Also any linear positive mapping φ from a closed reproducing space Q into BV is continuous, and any linear positive and efficient mapping φ from an internal space Q into BV has norm 1 (Aumann and Shapley 1974, Propositions 4.15 and 4.7). Therefore, to complete the proof of uniqueness, it is sufficient to show that each of the spaces pNA, $bv'NA$ and pNA_∞ is internal. The closure of an internal space is internal (Aumann and Shapley 1974, Proposition 4.12) and any linear space of games that is the union of internal spaces is internal. Therefore one demonstrates that each of these spaces is the union of spaces that contain a dense internal subspace. There are several dense internal subspaces of pNA (Aumann and Shapley 1974, Lemma 7.18; Reichert and Tauman 1985; Monderer and Neyman 1988, Section 5). The proof of the internality of $bv'NA$ is more involved (Aumann and Shapley 1974, Section 8).

Note that if Q and Q' are two linear symmetric spaces of games and $Q \supset Q'$, the existence of a value on Q implies the existence of a value on Q'. However, the uniqueness of a value on one of them does not imply uniqueness on the other. There are, for instance, subspaces of pNA which have more than one value (Hart and Neyman 1988, Example 4). There are however results that provide sufficient conditions for subspaces of pNA to have a unique value. For a coalition S we define the *restriction* of v to S, v_S, by $v_S(T) = v(S \cap T)$ for every $T \in \mathcal{C}$. A set of games M is *restrictable* if $v_S \in M$ for every $v \in M$ and for every $S \in \mathcal{C}$. Hart and Monderer (1997, Theorem 3.6) prove the uniqueness of the value on restrictable subspaces of pNA_∞ that contain NA. This is the non-atomic version of the analogous result for finite games proved in Hart and Mas-Colell 1989a and Neyman 1989.

Existence of a value on pNA and $bv'NA$ is proved in Aumann and Shapley, Chapter I. We outline below a proof that is based on the proof of Mertens and Neyman (2001). Assume that $v = \sum_{i=1}^n f_i \circ \mu_i$ with $f_i \in bv'$ and $\mu_i \in NA^1$. Define $\varphi v(S) = \sum_{i=1}^n f_i(1)\mu_i(S)$. It follows from the continuity at 0 and 1 of each of the functions f_i that $\sum_{i=1}^n f_i(1)\mu_i(S) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^{1-\varepsilon} [\sum_{i=1}^n f_i(t + \varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t)]\, dt$. By Lyapunov's Theorem, applied to the vector measure $(\mu_1, \ldots, \mu_n)$, for every $0 \leq t \leq 1 - \varepsilon$ there are coalitions $T \subset R$ with $\mu_i(T) = t$ and $\mu_i(R) = t + \varepsilon\mu_i(S)$. Therefore, if v is monotonic, $v(R) - v(T) = \sum_{i=1}^n f_i(t + \varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t) \geq 0$, and therefore $\varphi v(S) \geq 0$. In particular, if $0 = \sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g_j \circ \nu_j$ where $f_i, g_j \in bv'$ and $\mu_i, \nu_j \in NA^1$, $\sum_{i=1}^n f_i(1)\mu_i = \sum_{j=1}^k g_j(1)\nu_j$, and thus also φv is well defined, i.e., it is independent of the presentation. Obviously, $\varphi v \in NA \subset FA$ and $\varphi v(I) = v(I)$ and φ is symmetric and linear. Therefore φ defines a value on the linear and symmetric subspace of $bv'NA$ of all games of the form $\sum_{i=1}^n f_i \circ \mu_i$ with n a positive integer, $f_i \in bv'$ and $\mu_i \in NA^1$. We next show that φ is of norm 1 and thus has a (unique) extension to a continuous value (of norm 1) on $bv'NA$. For each $u \in FA$, $\|u\| = \sup_{S \in \mathcal{C}} |u(S)| + |u(S^c)|$. Therefore, it is sufficient to prove that for every coalition $S \in \mathcal{C}$, $|\varphi v(S)| + |\varphi v(S^c)| \leq \|v\|$. For each positive integer m let $S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_m$ and $S_0^c \subset S_1^c \subset \ldots \subset S_m^c$ be measurable subsets of S and $S^c = I \setminus S$ respectively with $\mu_i(S_j) = \frac{j}{m+1}\mu_i(S)$ and $\mu_i(S_j^c) = \frac{j}{m+1}\mu_i(S^c)$. For every $0 \leq t \leq \frac{1}{m+1}$ let I_t be a measurable subset of $I \setminus (S_m \cup S_m^c)$ with $\mu_i(I_t) = t$.

Define the increasing sequence of coalitions $T_0 \subset T_1 \subset \ldots T_{2m}$ by $T_0 = I_t$, $T_{2j-1} = I_t \cup S_j \cup S_{j-1}^c$ and $T_{2j} = I_t \cup S_j \cup S_j^c$, $j = 1, \ldots, m$. Obviously, $\|v\| \geq \sum_{j=1}^{2m} |v(T_j) - v(T_{j-1})| \geq |\sum_{j=0}^{m-1} v(T_{2j+1}) - v(T_{2j})| + |\sum_{j=1}^{m} v(T_{2j}) - v(T_{2j-1})|$. Set $\varepsilon = \frac{1}{m+1}$. Note that $\frac{1}{\varepsilon}\int_0^\varepsilon \sum_{j=0}^{m-1}[\sum_{i=1}^n f_i(t + \varepsilon j + \varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t+j\varepsilon)]\,dt = \frac{1}{\varepsilon}\int_0^{1-\varepsilon}[\sum_{i=1}^n f_i(t+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t)]\,dt \to_{m\to\infty} \varphi v(S)$, and similarly $\frac{1}{\varepsilon}\int_0^\varepsilon \sum_{j=1}^{m}[\sum_{i=1}^n f_i(t + \varepsilon j) - \sum_{i=1}^n f_i(t + j\varepsilon - \varepsilon\mu_i(S^c))]\,dt = \frac{1}{\varepsilon}\int_\varepsilon^1[\sum_{i=1}^n f_i(t) - \sum_{i=1}^n f_i(t - \varepsilon\mu_i(S^c))]\,dt \to_{m\to\infty} \varphi v(S^c)$. As $v(T_{2j+1}) - v(T_{2j}) = \sum_{i=1}^n f_i(t + \varepsilon j + \varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t + j\varepsilon)$, and $v(T_{2j}) - v(T_{2j-1}) = \sum_{i=1}^n f_i(t + 2j\varepsilon) - \sum_{i=1}^n f_i(t + 2j\varepsilon - \varepsilon\mu_i(S^c))$, we deduce that for each fixed $0 \leq t \leq \varepsilon$, $|\sum_{j=0}^{m-1}[\sum_{i=1}^n f_i(t+\varepsilon j+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t+j\varepsilon)]| + |\sum_{j=1}^{m}[\sum_{i=1}^n f_i(t+\varepsilon j) - \sum_{i=1}^n f_i(t+j\varepsilon-\varepsilon\mu_i(S^c))]| \leq \|v\|$ and therefore $|\varphi v(S)| + |\varphi v(S^c)| \leq \|v\|$. It follows that the map φ, which is defined on a dense subspace of $bv'NA$ and with values in NA, is linear, symmetric, efficient, and of norm 1, and therefore φ has a unique extension to a continuous value on $bv'NA$.

The bounded variation of the functions f_i is used in the above proof to show that the integrals $\int_a^b f(x)\,dx$, $0 \leq a < b \leq 1$, are well defined. Therefore, the proof also demonstrates the result of Tauman (1979) – that there is a unique value of norm 1 on $In'NA$: the closed linear space of games that is generated by games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in In' = \{f : [0,1] \to I\!R \mid f(0) = 0, f \text{ integrable and continuous at 0 and 1}\}$. The integrability requirement can also be dropped. Mertens and Neyman (2000) prove the existence of a value (of norm 1) on the spaces $'AN \supset' NA$: the closure in the variation distance of the linear space spanned by all games $f \circ \mu$ where $\mu \in AN^1 \supset NA^1$ and f is a real-valued function defined on $[0,1]$ with $f(0) = 0$ and continuous at 0 and 1. Moreover, the continuity of the function f at 0 and 1 can be further weakened by requiring the upper and lower averages of f on the intervals $[0,\varepsilon]$ and $[1-\varepsilon, 1]$ to converge to $f(0) = 0$ and $f(1)$, respectively.

The existence of a value (of norm 1) on pNA and $bv'NA$ also follows from various constructive approaches to the value, such as the limiting approach discussed immediately below and the value formulas approach (Section 7). Other constructive approaches include the mixing approach (Section 6) that provides a value on pNA. Each constructive approach describes a way of "computing" the value for a specific game.

The space of all games that are in the (feasible) domain of each approach will form a subspace and the associated map will be a value. Each of the approaches has its advantages and each sheds additional interpretive light on the value obtained. We start with limiting value approaches.

5 Limiting Values

Limiting values are defined by means of the limits of the Shapley value of the finite games that approximate the given game.

Given a game v on $(I, \mathcal{C})$ and a finite measurable field π, we denote by v_π the restriction of the game v to the finite field π. The real-valued function v_π (defined on π) can be viewed as a finite game. The Shapley value for finite games ψv_π on π is given by

$$(\psi v_\pi)(a) = \frac{1}{n!} \sum_{\mathcal{R}} \left[v\left(\mathcal{P}_a^{\mathcal{R}} \cup a\right) - v\left(\mathcal{P}_a^{\mathcal{R}}\right)\right],$$

where n is the number of atoms of π, the sum runs over all orders $\mathcal{R}$ of the atoms of π, and $\mathcal{P}_a^{\mathcal{R}}$ is the union of all atoms preceding a in the order $\mathcal{R}$.

5.1 The Weak Asymptotic Value

The family of measurable finite fields (that contain a given coalition T) is ordered by inclusion, and given two measurable finite fields π and π' there is a measurable finite field π'' that contains both π and π'. This enables us to define limits of functions of finite measurable fields. The weak asymptotic value of the game v is defined as $\lim_\pi \psi v_\pi$ whenever the limit of $\psi v_\pi(T)$ exists where π ranges over the family of measurable finite fields that include T. Equivalently,

Definition 3 *Let v be a game. A game φv is said to be the* weak asymptotic value *of v if for every $S \in \mathcal{C}$ and every $\varepsilon > 0$, there is a finite subfield π of $\mathcal{C}$ with $S \in \pi$ such that for any finite subfield π' with $\pi' \supset \pi$, $|\psi v_{\pi'}(S) - \varphi v(S)| < \varepsilon$.*

Remarks: (1) Let v be a game. The game v has a weak asymptotic value if and only if for every S in $\mathcal{C}$ and $\varepsilon > 0$ there exists a finite subfield π with $S \in \pi$ such that for any finite subfield π' with $\pi' \supset \pi$, $|\psi v_{\pi'}(S) - \psi v_\pi(S)| < \varepsilon$.

(2) A game v has at most one weak asymptotic value.

(3) The weak asymptotic value φv of a game v is a finitely additive game, and $\|\varphi v\| \leq \|v\|$ whenever $v \in BV$.

Theorem 3 *[Neyman 1988, p.559]. The set of all games having a weak asymptotic value is a linear symmetric space of games, and the operator mapping each game to its weak asymptotic value is a value on that space. If $ASYMP^*$ denotes all games with bounded variation having a weak asymptotic value, then $ASYMP^*$ is a closed subspace of BV with $bv'FA \subset ASYMP^*$.*

Remarks: The linearity, symmetry, positivity and efficiency of ψ (the Shapley value for games with finitely many players) and of the map $v \mapsto v_\pi$ imply the linearity, symmetry, positivity and efficiency of the weak asymptotic value map and also imply the linearity and symmetry of the space of games having a weak asymptotic value. The closedness of $ASYMP^*$ follows from the linearity of ψ and from the inequalities $\|\psi v_\pi\| \leq \|v_\pi\| \leq \|v\|$. The difficult part of the theorem is the inclusion $bv'FA \subset ASYMP^*$, on which we will comment later. The inclusion $bv'FA \subset ASYMP^*$ implies in particular the existence of a value on $bv'FA$.

5.2 The Asymptotic Value

The asymptotic value of a game v is defined whenever all the sequences of the Shapley values of finite games that "approximate" v have the same limit. Formally: given T in $\mathcal{C}$, a T-admissible sequence is an increasing sequence $(\pi_1, \pi_2, \ldots)$ of finite subfields of $\mathcal{C}$ such that $T \in \pi_1$ and $\cup_i \pi_i$ generates $\mathcal{C}$.

Definition 4 *A game φv is said to be the* asymptotic value *of v, if $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists and $= \varphi v(T)$ for every $T \in \mathcal{C}$ and every T-admissible sequence $(\pi_i)_{i=1}^\infty$.*

Remarks: (1) A game v has an asymptotic value if and only if $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists for every T in $\mathcal{C}$ and every T-admissible sequence $\mathcal{P} = (\pi_k)_{k=1}^\infty$, and the limit is independent of the choice of $\mathcal{P}$.

(2) For any given v, the asymptotic value, if it exists, is clearly unique.

(3) The asymptotic value φv of a game v is finitely additive, and $\|\varphi v\| \leq \|v\|$ whenever $v \in BV$.

(4) If v has an asymptotic value φv then v has a weak asymptotic value, which $= \varphi v$.

(5) The *dual* of a game v is defined as the game v^* given by $v^*(S) = v(I) - v(I \backslash S)$. Then $\psi v^*_\pi = \psi v_\pi = \psi \left(\frac{v+v^*}{2}\right)_\pi$, and therefore v has a (weak) asymptotic value if and only if v^* has a (weak) asymptotic value if and only if $\left(\frac{v+v^*}{2}\right)$ has a (weak) asymptotic value, and the values coincide.

The space of all games of bounded variation that have an asymptotic value is denoted $ASYMP$.

Theorem 4 *[Aumann and Shapley 1974, Theorem F; Neyman 1981a; and Neyman 1988, Theorem A]. The set of all games having an asymptotic value is a linear symmetric space of games, and the operator mapping each game to its asymptotic value is a value on that space. $ASYMP$ is a closed linear symmetric subspace of BV which contains $bv'M$.*

Remarks: (1) The linearity and symmetry of ψ (the Shapley value for games with finitely many players), and of the map $v \to v_\pi$, imply the linearity and symmetry of the asymptotic value map and its domain. The efficiency and positivity of the asymptotic value follows from the efficiency and positivity of the maps $v \to v_\pi$ and $v_\pi \to \psi v_\pi$. The closedness of $ASYMP$ follows from both the linearity of ψ and the inequalities $\|\psi v_\pi\| \leq \|v\|_\pi \leq \|v\|$.

(2) Several authors have contributed to proving the inclusion $bv'M \subset ASYMP$. Let FL denote the set of measures with at most finitely many atoms and M_a all purely atomic measures. Each of the spaces, $bv'NA$, $bv'FL$ and $bv'M_a$ is defined as the closed subspace of BV that is generated by games of the form $f \circ \mu$ where $f \in bv'$ and μ is a probability measure in NA^1, FL^1, or M^1_a respectively.

Kannai (1966) and Aumann and Shapley (1974) show that $pNA \subset ASYMP$. Artstein (1971) proves essentially that $pM_a \subset ASYMP$. Fogelman and Quinzii (1980) show that $pFL \subset ASYMP$. Neyman (1981a, 1979) shows that $bv'NA \subset ASYMP$ and $bv'FL \subset ASYMP$ respectively. Berbee (1981), together with Shapiro and Shapley (1971, Theorem 14) and the proof of Neyman (1981a, Lemma 8 and Theorem A), imply that $bv'M_a \subset ASYMP$. It was further announced in Neyman (1979) that Berbee's result implies the existence of a partition value on $bv'M$. Neyman (1988) shows that $bv'M \subset ASYMP$.

(3) The space of games, $bv'M$, is generated by scalar measure games, i.e., by games that are real-valued functions of scalar measures. Therefore, the essential part of the inclusion $bv'M \subset ASYMP$ is that whenever $f \in bv'$ and μ is a probability measure, $f \circ \mu$ has an asymptotic value. The tightness of the assumptions on f and μ for $f \circ \mu$ to have an asymptotic value follows from the next three remarks.

(4) $pFA \notin ASYMP$ (Neyman 1988, p. 578). For example, $\mu^3 \notin ASYMP$ whenever μ is a positive non-atomic finitely additive measure which is not countably additive. This illustrates the essentiality of the countable additivity of μ for $f \circ \mu$ to have an asymptotic value when f is a polynomial.

(5) For $0 < q < 1$, we denote by f_q the function given by $f_q(x) = 1$ if $x \geq q$ and $f_q(x) = 0$ otherwise. Let μ be a non-atomic measure with total mass 1. Then $f_q \circ \mu$ has an asymptotic value if and only if μ is positive (Neyman 1988, Theorem 5.1). There are games of the form $f_q \circ \mu$, where μ is a purely atomic signed measure with $\mu(I) = 1$, for which $f_{\frac{1}{2}} \circ \mu$ does not have an asymptotic value (Neyman 1988, Example 5.2). These two comments illustrate the essentiality of the positivity of the measure μ.

(6) There are games of the form $f \circ \mu$, where $\mu \in NA^1$ and f is continuous at 0 and 1 and vanishes outside a countable set of points, which do not have an asymptotic value (Neyman 1981a, p. 210). Thus, the bounded variation of the function f is essential for $f \circ \mu$ to have an asymptotic value.

(7) The set of games $DIAG$ is defined (Aumann and Shapley 1974, p. 252) as the set of all games v in BV for which there is a positive integer k, measures $\mu_1, \ldots, \mu_k \in NA^1$, and $\varepsilon > 0$, such that for any coalition $S \in \mathcal{C}$ with $\mu_i(S) - \mu_j(S) \leq \varepsilon$ for all $1 \leq i, j \leq k$, $v(S) = 0$. $DIAG$ is a symmetric linear subspace of $ASYMP$ (Aumann and Shapley 1974).

(8) If μ_1, μ_2, μ_3, are three non-atomic probabilities that are linearly independent, then the game v defined by $v(S) = \min[\mu_1(S), \mu_2(S), \mu_3(S)]$ is not in $ASYMP$ (Aumann and Shapley 1974, Example 19.2).

(9) Games of the form $f_q \circ \mu$ where $\mu \in M^1$ and $0 < q < 1$ are called *weighted majority games*; a coalition S is winning, i.e., $v(S) = 1$, if and only if its total μ-weight is at least q. The measure μ is called the *weight measure* and the number q is called the *quota*. The result $bv'M \subset ASYMP$ implies in particular that weighted majority games have an asymptotic value.

(10) A *two-house weighted majority game* v is the product of two weighted majority games, i.e., $v = (f_q \circ \mu)(f_c \circ \nu)$, where $\mu, \nu \in M^1$, $0 < c < 1$, and $0 < q < 1$. If $\mu, \nu \in NA^1$ with $\mu \neq \nu$ and $q \neq 1/2$, v has an asymptotic value if and only if $q \neq c$. Indeed, if $q < c$, $(f_q \circ \mu)(f_c \circ \nu) - (f_c \circ \nu) \in DIAG$ and therefore $(f_q \circ \mu)(f_c \circ \nu) \in bv'NA + DIAG \subset ASYMP$, and if $q = c \neq 1/2$, $(f_q \circ \mu)(f_c \circ \nu) \notin ASYMP$ (Neyman and Tauman 1979, proof of Proposition 5.42). If μ has infinitely many atoms and the set of atoms of ν is disjoint to the set of atoms of μ, $(f_q \circ \mu)(f_c \circ \nu)$ has an asymptotic value (Neyman and Smorodinski 1993).

(11) Let $\mathcal{A}$ denote the closed algebra that is generated by games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in bv'$ is continuous. It follows from the proof of Proposition 5.5 in Neyman and Tauman (1979) together with Neyman (1981a) that $\mathcal{A} \subset ASYMP$ and that for every $u \in \mathcal{A}$ and $v \in bv'NA$, $uv \in ASYMP$.

The following should shed light on the proofs regarding asymptotic and limiting values.

Let π be a finite subfield of $\mathcal{C}$ and let v be a game. We will recall formulas for the Shapley value of finite games by applying these formulas to the game v_π. This will enable us to illustrate proofs regarding the limiting properties of the Shapley value ψv_π of the finite game v_π. Let $A(\pi)$, or A for short, denote the set of atoms of π. The Shapley value of the game v_π to an atom a in A is given by

$$\psi v_\pi(a) = \frac{1}{|A|!} \sum_{\mathcal{R}} \left[v\left(P_a^{\mathcal{R}} \cup a\right) - v\left(P_a^{\mathcal{R}}\right)\right],$$

where $|A|$ stands for the number of elements of A, the summation ranges over all orders $\mathcal{R}$ of the atoms in A, and $P_a^{\mathcal{R}}$ is the union of all atoms preceding a in the order R. An alternative formula for ψv_π could be given by means of a family X_a, $a \in A(\pi)$ of i.i.d. real-valued random variables that are uniformly distributed on $(0,1)$. The values of X_a, $a \in A(\pi)$, induce with probability 1 an order $\mathcal{R}$ on A; a precedes b if and only if $X_a < X_b$. As X_a, $a \in A$, are i.i.d., all orders are equally likely. Thus, identifying sets with their indicator functions, and letting $I(X_a \leq X_b)$ denote the indicator of the event $X_a \leq X_b$, the value formula is

$$\psi v_\pi(a) = E\left[v\left(\sum_{b \in A} bI(X_b \leq X_a)\right) - v\left(\sum_{b \in A} bI(X_b < X_a)\right)\right].$$

Thus, by conditioning on the value of X_a, and letting $S(t)$, $0 \leq t \leq 1$, denote the union of all atoms b of π for which $X_b \leq t$, we find that

$$\psi v_\pi(a) = \int_0^1 E\left(v\left(S(t) \cup a\right) - v\left(S(t) \setminus a\right)\right) dt.$$

Assume now that the game v is a continuously differentiable function of finitely many non-atomic probability measures; i.e., $v = f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n) \in (NA^1)^n$ and $f : \mathcal{R}(\mu) \to \mathcal{R}$ is continuously differentiable. Then $v(S(t) \cup a) - v(S(t) \setminus a) = \sum_{i=1}^n \frac{\partial f}{\partial \xi_i}(y)\mu_i(a)$, where y is a point in the interval $[\mu(S(t) \setminus a), \mu(S(t) \cup a)]$. Thus, in particular, $\sum_{a \in \pi} |v(S(t) \cup a) - v(S(t) \setminus a)|$ is bounded.

For any scalar measure ν, $\nu(S(t))$ is the sum $\sum_{a \in A(\pi)} \nu(a) I(X_a \leq t)$, i.e., it is a sum of $|A(\pi)|$ independent random variables with expectation $t\nu(a)$ and variance $t(1-t)\nu^2(a)$. Therefore, $E(\nu(S(t))) = t\nu(I)$ and $\text{Var}\ (S(t)) = t(1-t)\sum_{a \in A(\pi)} \nu^2(a) \leq t(1-t)\nu(I) \max\{\nu(a) : a \in A(\pi)\}$. Therefore, if $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of measures, and $(\pi_k)_{k=1}^\infty$ is a sequence of finite fields with $\max\{|\mu_j(a)| : 1 \leq j \leq n,\ a \in A(\pi_k)\} \to_{k \to \infty} 0$, $\mu(S(t))$ converges in distribution to $t\mu(I)$; so if T is a coalition with $T \in \pi_k$ for every k, then

$$\sum_{\substack{a \subset T \\ a \in A(\pi_k)}} \left[v\left(S_k(t) \cup a\right) - v\left(S_k(t) \setminus a\right)\right] \to_{k \to \infty} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t\mu(I))\mu_i(T),$$

and therefore $\lim_{k \to \infty} \psi v_{\pi_k}(T)$ exists and

$$\lim_{k \to \infty} \psi v_{\pi_k}(T) = \int_0^1 f_{\mu(T)}(t\mu(I)) dt.$$

For any T-admissible sequence $(\pi_k)_{k=1}^\infty$ and non-atomic measure ν, $\max\{\nu(a) : a \in A(\pi_k)\} \to_{k \to \infty} 0$. Therefore any game $v = f \circ \mu$, where f is a continuously differentiable function defined on the range of μ, $\{\mu(S) : S \in \mathcal{C}\}$, has an asymptotic value φv whenever $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of non-atomic measures.

Also, given any $\varepsilon > 0$ and a non-atomic element ν of FA^+, there exists a partition π such that $\max\{\nu(a) : a \in A(\pi)\} < \varepsilon$. Therefore $v = f \circ \mu$ has a weak asymptotic value φv whenever $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of non-atomic bounded finitely additive measures. In both cases the value is given by

$$\varphi v(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt.$$

The following tool which is of independent interest will be used later on to show how to reduce the proof of $bv'NA \subset ASYMP$ (or $bv'M \subset ASYMP$) to the proof that $f_q \circ \mu \in ASYMP$ where $f_q(x) = I(x \geq q)$, $0 < q < 1$, and $\mu \in NA^1$ (or $\mu \in M^1$).

Let v be a monotonic game. For every $\alpha > 0$, let v^α denote the indicator of $v \geq \alpha$, i.e., $v^\alpha(S) = I(v(S) \geq \alpha)$, i.e., $v^\alpha(S) = 1$ if $v(S) \geq \alpha$ and 0 otherwise. Note that for every coalition S, $v(S) = \int_0^\infty v^\alpha(S)d\alpha$. Then for every finite field π, $\psi v_\pi = \int_0^\infty \psi(v^\alpha)_\pi d\alpha = \int_0^{v(I)} \psi\left(v^\alpha\right)_\pi d\alpha$. Therefore if $(\pi_k)_{k=1}^\infty$ is a T-admissible sequence such that for every $\alpha > 0$, $\lim_{k\to\infty} \psi\left(v^\alpha\right)_{\pi_k}(T)$ exists, then from the Lebesgue's bounded convergence theorem it follows that $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists and

$$\lim_{k\to\infty} \psi v_{\pi_k}(T) = \int_0^{v(I)} \left(\lim_{k\to\infty} \psi v^\alpha_{\pi_k}(T)\right) d\alpha.$$

Thus, if for each $\alpha > 0$, $v^\alpha = I\left(v \geq \alpha\right)$ has an asymptotic value φv^α, then v also has an asymptotic value φv, which is given by $\int_0^\infty \varphi v^\alpha(T)d\alpha = \varphi v(T)$.

The spaces $ASYMP$ and $ASYMP^*$ are closed and each of the spaces $bv'NA$, $bv'M$ and $bv'FA$ is the closed linear space that is generated by scalar measure games of the form $f \circ \mu$ with $f \in bv'$ monotonic and $\mu \in NA^1$, $\mu \in M^1$, $\mu \in FA^1$ respectively. Also, each monotonic f in bv' is the sum of two monotonic functions in bv', one right continuous and the other left continuous. If $f \in bv'$, with f monotonic and left continuous, then $(f \circ \mu)^* = g \circ \mu$, with $g \in bv'$ monotonic and right continuous.

Therefore, to show that $bv'NA \subset ASYMP$ ($bv'M \subset ASYMP$ or $bv'FA \subset ASYMP^*$), it suffices to show that $f \circ \mu \in ASYMP$ (or $\in ASYMP^*$) for any monotonic and right continuous function f in bv' and $\mu \in NA^1$ ($\mu \in M^1$ or $\mu \in FA^1$). Note that $(f \circ \mu)^\alpha(S) = I(f(\mu(S)) \geq \alpha) = I(\mu(S) \geq \inf\{x : f(x) \geq \alpha\}) = f_q(\mu(S))$, where $q = \inf\{x : f(x) \geq \alpha\}$. Thus, in view of the above remarks, to show that $bv'NA \subset ASYMP$ (or $bv'M \subset ASYMP$), it suffices to prove that $f_q \circ \mu \in ASYMP$ for any $0 < q < 1$ and $\mu \in NA^1$ (or $\mu \in M^1$), where $f_q(x) = 1$ if $x \geq q$ and $= 0$ otherwise.

The proofs of the relations $f_q \circ \mu \in ASYMP$ for $0 < q < 1$ and $\mu \in NA^1$ (or $\mu \in M_a^1$ or $\mu \in M^1$) rely on delicate probabilistic results, which are of independent mathematical interest. These results can be formulated in various equivalent forms. We introduce them as properties of Poisson bridges. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables that are uniformly distributed on the open unit interval $(0,1)$. With any finite or infinite sequence $w = (w_i)_{i=1}^n$ or $w = (w_i)_{i=1}^\infty$ of positive numbers, we associate a stochastic process, $S_w(\cdot)$, or $S(\cdot)$ for short, given by $S(t) = \sum_i w_i I(X_i \leq t)$ for $0 \leq t \leq 1$ (such a process is called a pure jump Poisson bridge). The proof of $f_q \circ \mu \in ASYMP$ for $\mu \in NA^1$ uses the following result, which is closely related to renewal theory.

Proposition 4 *[Neyman 1981a]. For every $\epsilon > 0$ there exists a constant $K > 0$, such that if $w_1, \ldots, w_n$ are positive numbers with $\sum_{i=1}^n w_i = 1$, and $K \max_{i=1}^n w_i < q < 1 - K \max_{i=1}^n w_i$ then*

$$\sum_{i=1}^{n} \mid w_i - \ \textit{Prob}\ (S(X_i) \in [q, q + w_i)) \mid < \epsilon.$$

The proof of $f_q \circ \mu \in ASYMP$ for $\mu \in M_a$ uses the following result, due to Berbee (1981).

Proposition 5 *[Berbee 1981]. For every sequence $(w_i)_{i=1}^\infty$ with $w_i > 0$, and every $0 < q < \sum_{i=1}^\infty w_i$, the probability that there exists $0 \leq t \leq 1$ with $S(t) = q$ is 0, i.e.,*

$$\textit{Prob}\ (\exists t \ \ , \ \ S(t) = q) = 0.$$

6 The Mixing Value

An *order* on the underlying space I is a relation $\mathcal{R}$ on I that is transitive, irreflexive, and complete. For each order $\mathcal{R}$ and each s in I, define the initial segment $I(s; \mathcal{R})$ by $I(s; \mathcal{R}) = \{t \in I : s\mathcal{R}t\}$. An order $\mathcal{R}$ is measurable if the σ-field generated by all the initial segments $I(s; \mathcal{R})$ equals $\mathcal{C}$. Let ORD

be the linear space of all games v in BV such that for each measurable $\mathcal{R}$, there exists a unique measure $\varphi(v;\mathcal{R})$ satisfying

$$\varphi(v;\mathcal{R})(I(s;\mathcal{R})) = v(I(s;\mathcal{R})) \text{ for all } s \in I.$$

Let μ be a probability measure on the underlying space $(I,\mathcal{C})$. A μ-mixing sequence is a sequence $(\Theta_1, \Theta_2, \ldots)$ of μ-measure-preserving automorphism of $(I,\mathcal{C})$, such that for all S, T in $\mathcal{C}$, $\lim_{k\to\infty} \mu(S \cap \Theta_k T) = \mu(S)\mu(T)$. For each order $\mathcal{R}$ on I and automorphism Θ in $\mathcal{G}$ we denote by $\Theta\mathcal{R}$ the order on I defined by $\Theta s \Theta\mathcal{R} \Theta t \iff s\mathcal{R}t$. If ν is absolutely continuous with respect to the measure μ we write $\nu << \mu$.

Definition 5 *Let $v \in ORD$. A game φv is said to be the* mixing value *of v if there is a measure μ_v in NA^1 such that for all μ in NA^1 with $\mu_v << \mu$ and all μ-mixing sequences $(\Theta_1, \Theta_2, \ldots)$, for all measurable orders $\mathcal{R}$, and all coalitions S, $\lim_{k\to\infty} \varphi(v;\Theta_k\mathcal{R})(S)$ exists and $= (\varphi v)(S)$.*

The set of all games in ORD that have a mixing value is denoted MIX.

Theorem 5 *[Aumann and Shapley 1974, Theorem E]. MIX is a closed symmetric linear subspace of BV which contains pNA, and the map φ that associates a mixing value φv to each v is a value on MIX with $\|\varphi\| \leq 1$.*

7 Formulas for Values

Let v be a vector measure game, i.e., a game that is a function of finitely many measures. Such games have a representation of the form $v = f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of (probability) measures, and f is a real-valued function defined on the range $\mathcal{R}(\mu)$ of the vector measure μ with $f(0) = 0$.

When f is continuously differentiable, and $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of non-atomic measures with $\mu_i(I) \neq 0$, then $v = f \circ \mu$ is in $pNA_\infty (\subset pNA \subset$

$ASYMP$) and the value (asymptotic, mixing, or the unique value on pNA) is given by (Aumann and Shapley 1974, Theorem B),

$$\varphi v(S) \equiv \varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

(where $f_{\mu(S)}$ is the derivative of f in the direction of $\mu(S)$); i.e.,

$$\varphi(f \circ \mu)(S) = \sum_{i=1}^{n} \mu_i(S) \int_0^1 \frac{\partial f}{\partial x_i} (t\mu_1(I), \ldots, t\mu_n(I))\, dt.$$

The above formula expresses the value as an average of derivatives, that is, of marginal contributions. The derivatives are taken along the interval $[0, \mu(I)]$, i.e., the line segment connecting the vector 0 and the vector $\mu(I)$. This interval, which is contained in the range of the non-atomic vector measure μ, is called the *diagonal* of the range of μ, and the above formula is called the *diagonal formula*. The formula depends on the differentiability of f along the diagonal and on the game being a non-atomic vector measure game.

Extensions of the diagonal formula (due to Mertens) enable one to apply (variations of) the diagonal formula to games with discontinuities and with essential nondifferentiabilities. In particular, the generalized diagonal formula defines a value of norm 1 on a closed subspace of BV that includes all finite games, $bv'FA$, the closed algebra generated by $bv'NA$, all games generated by a finite number of algebraic and lattice operations[1] from a finite number of measures, and all market games that are functions of finitely many measures.

The value is defined as a composition of positive linear symmetric mappings of norm 1. One of these mappings is an extension operator which is an operator from a space of games into the space of ideal games, i.e., functions that are defined on ideal coalitions, which are measurable functions from $(I, \mathcal{C})$ into $([0,1], \mathcal{B})$.

The second one is a "derivative" operator, which is essentially the diagonal formula obtained by changing the order of integration and differentiation. The third one is an averaged derivative; it replaces the derivatives on this

[1] max and min

diagonal with an average of derivatives evaluated in a small invariant perturbation of the diagonal. The invariance of the perturbed distribution is with respect to all automorphisms of $(I, \mathcal{C})$.

First we illustrate the action and basic properties of the "derivative" and the "averaged derivative" mappings in the context of non-atomic vector measure games.

The derivative operator (Mertens 1980) provides a diagonal formula for the value of vector measure games in $bv'NA$; let $\mu = (\mu_1, \ldots, \mu_n) \in (NA^+)^n$ and let $f : \mathcal{R}(\mu) \to I\!R$ with $f(0) = 0$ and $f \circ \mu$ in $bv'NA$. Then f is continuous at $\mu(I)$ and $\mu(\emptyset) = 0$, and the value of $f \circ \mu$ is given by

$$\begin{aligned} \varphi\left(f \circ \mu\right)(S) &= \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^{1-\varepsilon} [f\left(t\mu(I) + \varepsilon\mu(S)\right) - f\left(t\mu(I)\right)]dt \\ &= \lim_{\varepsilon \to 0+} \frac{1}{2\epsilon} \int_\varepsilon^{1-\varepsilon} [f\left(t\mu(I) + \varepsilon\mu(S)\right) - f\left(t\mu(I) - \varepsilon\mu(S)\right)]dt. \end{aligned}$$

The limits of integration in these formulas, $1 - \varepsilon$ and ε, could be replaced by 1 and 0 respectively whenever f is extended to $I\!R^n$ in such a way that f remains continuous at $\mu(I)$ and $\mu(\emptyset)$.

Let $\mu = (\mu_1, \ldots, \mu_n) \in (NA^+)^n$ be a vector of measures. Consider the class $\mathcal{F}(\mu)$ of all functions $f : \mathcal{R}(\mu) \to I\!R$ with $f(0) = 0$, f continuous at 0 and $\mu(I)$, $f \circ \mu$ of bounded variation, and for which the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_\varepsilon^{1-\varepsilon} [f\left(t\mu(I) + \varepsilon\mu(S)\right) - f\left(t\mu(I) - \varepsilon\mu(S)\right)]dt$$

exists for all S in $\mathcal{C}$. Note that $f \in \mathcal{F}(\mu)$ whenever f is continuously differentiable with $f(0) = 0$. An example of a Lipschitz function that is not in $\mathcal{F}(\mu)$ where μ is a vector of 2 linearly independent probability measures is $f(x_1, x_2) = (x_1 - x_2) \sin \log |x_1 - x_2|$. Denote this limit by $Dv(S)$ and note that Dv is a function of the measures $\mu_1, \ldots, \mu_n$; i.e., $Dv = \bar{f} \circ \mu$ for some function $\bar{f}$ that is defined on the range of μ. The derivative operator D acts on all games of the form $f \circ \mu$, $\mu \in (NA^+)^n$, and $f \in \mathcal{F}(\mu)$; it is a value of norm 1 on the space of all these games for which $D(f \circ \mu) \in FA$.

Let $v = f \circ \mu$, where $\mu \in (NA^+)^n$ and $f \in \mathcal{F}(\mu)$. Then $D(f \circ \mu)$ is of the form $\bar{f} \circ \mu$, where $\bar{f}$ is linear on every plane containing the diagonal $[0, \mu(I)]$,

$\bar{f}(\mu(I)) = f(\mu(I))$ and $\|\bar{f} \circ \mu\| \leq \|f \circ \mu\|$. There is no loss of generality in assuming that the measures $\mu_1, \ldots, \mu_n$ are independent (i.e., that the range of the vector measure $\mathcal{R}(\mu)$ is full-dimensional), and then we identify $\bar{f}$ with its unique extension to a function on $I\!R^n$ that is linear on every plane containing the interval $[0, \mu(I)]$.

The averaging operator expresses the value of the game $\bar{f} \circ \mu$ $(= D(f \circ \mu)$ for f in $\mathcal{F}(\mu))$ as an average, $E(\sum_{i=1}^n \frac{\partial \bar{f}}{\partial x_i}(z)\mu_i)$, where z is an n-dimensional random variable whose distribution, P_μ, is strictly stable of index 1 and has a Fourier transform ξ to be described below. When $\mu_1, \ldots, \mu_n$ are mutually singular, $z = (z_1, \ldots, z_n)$ is a vector of independent "centered" Cauchy distributions, and then $\xi(y) = \exp\left(-\sum_{i=1}^n \mu_i(I)\right) = \exp\left(-\sum_{i=1}^n \|\mu_i\|\right)$. Otherwise, let $\nu \in NA^+$ be such that $\mu_1, \ldots, \mu_n$ are absolutely continuous with respect to ν; e.g., $\nu = \sum \mu_i$. Define $N_\mu : I\!R^n \to I\!R^n$ by

$$N_\mu(y) = \int_I |\sum_{i=1}^n (d\mu_i/d\nu)\, y_i| d\nu.$$

Then z is an n-dimensional random variable whose distribution has a Fourier transform $\xi_z(y) = \exp\left(-N_\mu(y)\right)$.

Let Q be the linear space of games generated by games of the form $v = f \circ \mu$ where $\mu = (\mu_1, \ldots, \mu_n) \in (NA^+)^n$ is a vector of linearly independent non-atomic measures and $f \in \mathcal{F}(\mu)$. The space Q is symmetric and the map $\varphi : Q \to FA$ given by

$$\varphi(f \circ \mu) = E_{P_\mu}\left(\bar{f}_{\mu(S)}(z)\right) = E_{P_\mu}\left(\sum_{i=1}^n \frac{\partial \bar{f}_i}{\partial x_i}(z)\mu_i(S)\right)$$

is a value of norm 1 on Q and therefore has an extension to a value of norm 1 on the closure $\bar{Q}$ of Q.

The space $\bar{Q}$ contains $bv'NA$, the closed algebra generated by $bv'NA$, all games generated by a finite number of algebraic and lattice operations from a finite number of non-atomic measures and all games of the form $f \circ \mu$ where $\mu \in (NA^+)^n$ and f is a continuous concave function on the range of μ.

To show that $E_{P_\mu}(\bar{f}_{\mu(S)}(z))$ is well defined for any f in $\mathcal{F}(\mu)$, one shows that $\bar{f}$ is Lipschitz and therefore on the one hand $[\bar{f}\,(z + \delta\mu(S)) - \bar{f}(z)]/\delta$ is bounded and continuous, and on the other hand, given S in $\mathcal{C}$, for almost every z (with respect to the Lebesgue measure on $I\!R^n$) $\bar{f}_{\mu(S)}(z)$ exists and is bounded (as a function of z). The distribution P_μ is absolutely continuous with respect to Lebesgue measure and therefore $E_{P_\mu}\left(\bar{f}_{\mu(S)}(z)\right)$ is well defined and equals $\lim_{\delta\to 0} E_{P_\mu}\left([\bar{f}(z+\delta\mu(S)\right) - \bar{f}(z)]/\delta)$ (using the Lebesgue bounded convergence theorem).

To show that $E_{P_\mu}\left(\bar{f}_{\mu(S)}(z)\right)$ is finitely additive in S, assume that S, T are two disjoints coalitions. For any $\delta > 0$, let P_μ^δ be the translation of the measure P_μ by $-\delta\mu(S)$, i.e., $P_\mu^\delta(B) = P_\mu\left(B + \delta\mu(S)\right)$. Since P_μ^δ converges in norm to P_μ as $\delta \to 0$ (i.e., $\|P_\mu^\delta - P_\mu\| \longrightarrow_{\delta\to 0} 0$), and $[\bar{f}\left(z+\delta\mu(T)\right) - \bar{f}(z)]/\delta$ is bounded, it follows that

$$\lim_{\delta\to 0} E_{P_\mu}\left([\bar{f}\left(z+\delta\mu(S)+\delta\mu(T)\right) - \bar{f}\left(z+\delta\mu(S)\right)]/\delta\right) =$$

$$\lim_{\delta\to 0} E_{P_\mu^\delta}\left([\bar{f}(z+\delta\mu(T)) - \bar{f}(z)]/\delta\right) = \lim_{\delta\to 0} E_{P_\mu}\left([\bar{f}\left(z+\delta\mu(T)\right) - \bar{f}(z)]/\delta\right)$$
$$= E_{P_\mu}\left(\bar{f}_{\mu(T)}(z)\right).$$

Therefore

$$\begin{aligned} E_{P_\mu}\left(\bar{f}_{\mu(S\cup T)}(z)\right) &= \lim_{\delta\to 0} E_{P_\mu}([\bar{f}\left(z+\delta\mu(S)+\delta\mu(T)\right) \\ &\quad -\bar{f}\left(z+\delta\mu(S)\right) + \bar{f}\left(z+\delta\mu(S)\right) - \bar{f}(z)]/\delta) \\ &= E_{P_\mu}\left(\bar{f}_{\mu(S)}(z)\right) + E_{P_\mu}\left(\bar{f}_{\mu(T)}(z)\right). \end{aligned}$$

We now turn to a detailed description of the mappings used in the construction of a value due to Mertens on a large space of games.

Let $B(I, \mathcal{C})$ be the space of bounded measurable real-valued functions on $(I, \mathcal{C})$, and $B_1^+(I, \mathcal{C}) = \{f \in B(I, \mathcal{C}) : 0 \le f \le 1\}$ the space of "ideal coalitions" (measurable fuzzy sets). A function $\bar{v}$ on $B_1^+(I, \mathcal{C})$ is *constant sum* if $\bar{v}(f) + \bar{v}(1-f) = \bar{v}(1)$ for every $f \in B_1^+(I, \mathcal{C})$. It is *monotonic* if for every $f,\, g \in B_1^+(I, \mathcal{C})$ with $f \ge g$, $\bar{v}(f) \ge \bar{v}(g)$. It is *finitely additive* if $\bar{v}(f+g) = \bar{v}(f) + \bar{v}(g)$ whenever $f,\, g \in B_1^+(I, \mathcal{C})$ with $f + g \le 1$. It is *of bounded variation* if it is the difference of two monotonic functions, and the variation norm $\|\bar{v}\|_{IBV}$ (or $\|\bar{v}\|$ for short) is the supremum of the variation of $\bar{v}$ over all increasing sequences $0 \le f_1 \le f_2 \le \ldots \le 1$ in $B_1^+(I, \mathcal{C})$. The symmetry group $\mathcal{G}$ acts on $B_1^+(I, \mathcal{C})$ by $(\Theta_* f)(s) = f(\Theta s)$.

Definition 6 *An* extension operator *is a linear and symmetric map ψ from a linear symmetric set of games to real-valued functions on $B_1^+(I,\mathcal{C})$, with $(\psi v)(0) = 0$, so that $(\psi v)(1) = v(I)$, $\|\psi v\|_{IBV} \leq \|v\|$, and ψv is finitely additive whenever v is finitely additive, and ψv is constant sum whenever v is constant sum.*

It follows that an extension operator is necessarily positive (i.e., if v is monotonic so is ψv) and that every extension operator ψ on a linear subspace Q of BV has a unique extension to an extension operator $\bar{\psi}$ on $\bar{Q}$ (the closure of Q).

One example of an extension operator is Owen's (1972) multilinear extension for finite games. For another example, consider all games that are functions of finitely many non-atomic probability measures, i.e., $Q = \{f \circ (\mu_1 \ldots, \mu_n) : n$ a positive integer, $\mu_i \in NA^1$ and $f : \mathcal{R}(\mu_1, \ldots, \mu_n) \to \mathbb{R}$ with $f(0) = 0\}$, where $\mathcal{R}(\mu_1, \ldots, \mu_n) = \{(\mu_1(S), \ldots, \mu_n(S)) : S \in \mathcal{C}\}$ is the range of the vector measure $(\mu_1, \ldots, \mu_n)$. Then Q is a linear and symmetric space of games. An extension operator on Q is defined as follows: for $\mu = (\mu_1, \ldots, \mu_n) \in (NA^1)^n$ and $g \in B_1^+(I,\mathcal{C})$, let $\mu(g) = (\int_I g d\mu_1, \ldots, \int_I g d\mu_n)$ and define $\psi(f \circ \mu)(g) = f(\mu(g))$. Then it is easy to verify that ψ is well defined, linear, and symmetric, that $\psi v(1) = v(I)$, ψv is finitely additive whenever v is, and ψv is constant sum whenever v is. The inequality $\|\psi v\|_{IBV} \leq \|v\|$ follows from the result of Dvoretzky, Wald, and Wolfowitz (1951, p.66, Theorem 4), which asserts that for every $\mu \in (NA^1)^n$ and every sequence $0 \leq f_1 \leq f_2 \leq \ldots \leq f_k \leq 1$ in $B_1^+(I,\mathcal{C})$, there exists a sequence $\emptyset \subset S_1 \subset S_2 \subset \ldots \subset S_k \subset I$ with $\mu(f_i) = \mu(S_i)$. This extension operator satisfies additional properties: $(\psi v)(S) = v(S)$, $\psi(vu) = (\psi v)(\psi u)$, $\psi(f \circ v) = f \circ \psi v$ whenever $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, $\psi(v \wedge u) = (\psi v) \wedge (\psi u)$ and $\psi(v \vee u) = (\psi v) \vee (\psi u)$. The NA-topology on $B(I,\mathcal{C})$ is the minimal topology for which the functions $f \mapsto \mu(f)$ are continuous for each $\mu \in NA$. For every game $v \in pNA$, ψv is NA-continuous. Moreover, as the characteristic functions $I_S \in B_+^1(I,\mathcal{C})$, $S \in \mathcal{C}$, are NA-dense in $B_+^1(I,\mathcal{C})$, ψv is the unique NA-continuous function on $B_+^1(I,\mathcal{C})$ with $\psi v(S) = v(S)$. Similarly, for every game v in pNA' – the closure in the supremum norm of pNA – there is a unique function $\bar{v} : B_+^1(I,\mathcal{C}) \to \mathbb{R}$ which is NA-continuous and $\bar{v}(S) = v(S)$. Moreover, the map $v \mapsto \bar{v}$ is an extension operator on pNA' and it satisfies: $\overline{vu} = \bar{v}\bar{u}$, $\bar{v} \wedge \bar{u} = \overline{v \wedge u}$ and $\bar{v} \vee \bar{u} = \overline{v \vee u}$ for all $v, u \in pNA'$ and $(f \bar{\circ} v) = f \circ \bar{v}$ whenever $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$.

Another example is the following natural extension map defined on $bv'M$. Assume that $\mu \in M^1$ with a set of atoms $A = A(\mu)$. Let $v = f \circ \mu$ with $f \in bv'$ and $\mu \in NA^1$. For $g \in B^1_+(I, \mathcal{C})$ define $\psi v(g) = E\left(f(\mu(gA^c + \sum_{a \in A} X_a a))\right)$ where $A^c = I \backslash A$ and $(X_a)_{a \in A}$ are independent $\{0,1\}$-valued random variables with $E(X_a) = g(a)$. If $v = \sum_{i=1}^k v_i$, $v_i = f_i \circ \mu_i$ with $\mu_i \in M^1$ and $f_i \in bv'$ we define $\psi v(g) = \sum_{i=1}^k \bar{v}_i(g)$. It is possible to show that $\|\psi v\|_{IBV} \leq \|v\|$ and thus in particular $\bar{v}$ is well defined (i.e., it is independent of the representation of v) and can be extended to the closure of all these finite sums, i.e., it defines an extension map on $bv'M$.

The space $bv'NA$ is a subset of both $bv'M$ and $\bar{Q}$ and the above two extensions that were defined on $\bar{Q}$ and on $bv'M$ coincide on $bv'NA$, and thus induce (by restriction) an extension operator ψ on $bv'NA$ and on pNA. The extension on $bv'NA$ can be used to provide formulas for the values on $bv'NA$ and on pNA.

Given v in $\bar{Q}$, we denote by $\bar{v}$ the extension (ideal game) $\bar{\psi} v$. We also identify S with the indicator of the set S, χ_S, and the scalar t also stands for the constant function in $B^+_1(I, \mathcal{C})$ with value t. Then, if $v \in pNA$, for almost all $0 < t < 1$, for every coalition S, the derivative $\partial \bar{v}(t, S) =: \lim_{\delta \to 0}[\bar{v}(t) + \delta_S) - \bar{v}(t)]/\delta$ exists (Aumann and Shapley 1974, Proposition 24.1) and the value of the game is given by

$$\varphi v(S) = \int_0^1 \partial \bar{v}(t, S) dt.$$

For any game v in $bv'NA$, the value of v is given by

$$\begin{aligned} \varphi v(S) &= \lim_{\delta \to 0+} \frac{1}{\delta} \int_0^{1-\delta} [\bar{v}\,(t + \delta S) - \bar{v}(t)] dt \\ &= \lim_{\delta \to 0} \frac{1}{2\delta} \int_{|\delta|}^{1-|\delta|} [\bar{v}\,(t + \delta S) - \bar{v}\,(t - \delta S)] dt. \end{aligned}$$

More generally, given an extension operator ψ on a linear symmetric space of games Q, we extend $\bar{v} = \psi v$, for v in Q, to be defined on all of $B(I, \mathcal{C})$ by $\bar{v}(f) = \bar{v}\,(\max[0, \min(1, f)])$ (and then we can replace the limits $|\delta|$ and $1 - |\delta|$ in the above integrals with 0 and 1 respectively, and define the derivative operator ψ_D by

$$\psi_D \bar{v}(\chi) = \lim_{\delta \to 0} \int_0^1 \frac{\bar{v}(t + \delta\chi) - \bar{v}(t - \delta\chi)}{2\delta} dt$$

whenever the integral and the limit exists for all χ in $B(I, \mathcal{C})$. The integral exists whenever $\bar{v}$ has bounded variation. The derivative operator is positive, linear and symmetric. Assuming continuity in the NA-topology of $\bar{v}$ at 1 and 0, ψ_D also satisfies $\psi_D \bar{v}(\alpha + \beta\chi) = \alpha\bar{v}(1) + \beta[\psi_D \bar{v}](\chi)$ for every α, $\beta \in I\!R$, so that $\psi_D \bar{v}$ is linear on every plan in $B(I, \mathcal{C})$ containing the constants and ψ_D is efficient, i.e., $\psi_D \bar{v}(1) = \bar{v}(1)$.

The following (due to Mertens 1988) will (essentially) extend the previously mentioned extension operators to a large class of games that includes $bv'FA$ and all games of the form $f \circ \mu$ where $\mu = (\mu_1, \ldots, \mu_n) \in NA$. Intuitively we would like to define $\psi_E v(f)$ for f in $B_1^+(I, \mathcal{C})$ as the "limit" of the expectation $v(S)$ with respect to a distribution P of a random set that is very similar to the ideal set f.

Let $\mathcal{F}$ be the set of all triplets $\alpha = (\pi, \nu, \varepsilon)$ where π is a finite subfield of C, ν is a finite set of non-atomic elements of FA and $\varepsilon > 0$. $\mathcal{F}$ is ordered by $\alpha = (\pi, \nu, \varepsilon) \leq \alpha' = (\pi', \nu', \varepsilon')$ if $\pi' \supset \pi$, $\nu' \supset \nu$ and $\varepsilon' \leq \varepsilon$. $(\mathcal{F}, \leq)$ is filtering-increasing with respect to this order, i.e., given $\alpha = (\pi, \nu, \varepsilon)$ and $\alpha' = (\pi', \nu', \varepsilon')$ in $\mathcal{F}$ there is $\alpha'' = (\pi'', \nu'', \varepsilon'')$ with $\alpha'' \geq \alpha$ and $\alpha'' \geq \alpha'$.

For any $\alpha = (\pi, \nu, \varepsilon)$ in $\mathcal{F}$ and f in $B_1^+(I, \mathcal{C})$, $P_{\alpha,f}$ is the set of all probabilities P with finite support on $\mathcal{C}$ such that $|\sum_{S \in C} SP(S) - f| \equiv |E_P(S) - f| < \varepsilon$ uniformly on I (i.e., for every $t \in I$, $|\sum_{S \in C} P(S) I(t \in S) - f(t)| < \varepsilon$), and such that for any T_1, T_2 in π with $T_1 \cap T_2 = \emptyset$, $T_1 \cap S$ is independent of $T_2 \cap S$, and for all μ in ν $\mu(S \cap T_1) = \int_{T_1} f \, d\mu$. That $P_{\alpha,f} \neq \emptyset$ follows from Lyapunov's convexity theorem on the range of a vector of non-atomic elements of FA.

For any game v, and any $f \in B_1^+(I, \mathcal{C})$, let $\bar{v}(f) = \lim_{\alpha \in \mathcal{F}} \sup_{P \in P_{\alpha,f}} E_P(v(S)) \equiv \lim_{\alpha \in \mathcal{F}} \sup_{P \in P_{\alpha,f}} \sum_{S \in C} v(S) P(S)$ and $\underline{v}(f) = \lim_{\alpha \in \mathcal{F}} \inf_{P \in P_{\alpha,f}} E_P(v(S))$. The definition of $\bar{v}$ is extended to all of $B(I, \mathcal{C})$ by $\bar{v}(f) = \bar{v}(\max(0, \min(1, f)))$.

Note that if v is a game with finite support, then for any f in $B_1^+(I, \mathcal{C})$, $\bar{v}(f) = \underline{v}(f)$ coincides with Owen's multilinear extension, i.e., letting T be the finite support of v,

$$\bar{v}(f) = \sum_{S \subset T} [\Pi_{s \in S} f(s) \Pi_{s \in T \setminus S} (1 - f(s))] v(S)$$

and whenever v is a function of finitely many non-atomic elements of FA,

$\nu_1, \ldots, \nu_n$, i.e., $v = g \circ (\nu_1, \ldots, \nu_n)$ where g is a real-valued function defined on the range of $(\nu_1, \ldots, \nu_n)$, then for any $f \in B_1^+(I, \mathcal{C})$

$$\bar{v}(f) = \underline{v}(f) = g(\int f \, d\nu_1, \ldots, \int f \, d\nu_n).$$

Let $V_\varepsilon = \{\chi \in B_1^+(I, \mathcal{C}) : \sup \chi - \inf \chi \leq \varepsilon\}$. Let D be the space of all games v in BV for which $\frac{1}{\varepsilon} \sup\{\bar{v}(\chi) - \underline{v}(\chi) : \chi \in V_\varepsilon\} \to 0$ as $\varepsilon \to 0+$. Obviously, $D \supset D_\varepsilon$ where D_ε is the set of all games v for which $\bar{v}(\chi) = \underline{v}(\chi)$ for any $\chi \in V_\varepsilon$. Next we define the derivative operators φ_D; its domain is the set of all games v in D for which the integrals (for sufficiently small $\varepsilon > 0$) and the limit

$$[\varphi_D v](\chi) = \lim_{\tau \to 0} \int_0^1 \frac{\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)}{2\tau} dt$$

exist for every χ in $B(I, \mathcal{C})$.

The integrals always exist for games in BV. The limit, on the other hand, may not exist even for games that are Lipschitz functions of a non-atomic signed measure. However, the limit exists for many games of interest, including the concave functions of finitely many non-atomic measures, and the algebra generated by $bv'FA$. Assuming, in addition, some continuity assumption on $\bar{v}$ at 1 and 0 (e.g., $\bar{v}(f) \to \bar{v}(1)$ as $\inf(f) \to 1$ and $\bar{v}(f) \to \bar{v}(0)$ as $\sup(f) \to 0$) we obtain that $\varphi_D v$ obeys the following additional properties: $\varphi_D v(\chi) + \varphi_D v(1 - \chi) = v(1)$ whenever v is a constant sum; $[\varphi_D v](a + bx) = av(1) + b[\varphi_D v](x)$ for every a, $b \in I\!R$.

Theorem 6 *[Mertens 1988, Section 1]. Let $Q = \{v \in$ Dom $\varphi_D : \varphi_D v \in FA\}$. Then Q is a closed symmetric space that contains $DIFF$, $DIAG$ and $bv'FA$ and $\varphi_D : Q \to FA$ is a value on Q.*

For every function $w : B(I, \mathcal{C}) \to I\!R$ in the range of φ_D, and every two elements χ, h in $B(I, \mathcal{C})$, we define $w_h(\chi)$ to be the (two-sided) directional derivative of w at χ in the direction h, i.e.,

$$w_h(\chi) = \lim_{\delta \to 0} [w(\chi + \delta h) - w(\chi - \delta h)]/(2\delta)$$

whenever the limit exists. Obviously, when w is finitely additive then $w_h(\chi) = w(h)$. For a function w in the range of φ_D and of the form $w = f \circ \mu$ where $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of measures in NA, f is Lipschitz, satisfying

$f(a\mu(I)+bx) = af(\mu(I))+bf(x)$ and for every y in $L(\mathcal{R}(\mu))$—the linear span of the range of the vector measure μ—for almost every x in $L(\mathcal{R}(\mu))$ the directional derivative of f at x in the direction y, $f_y(x)$ exists, and then obviously $w_h(\chi) = f_{\mu(h)}(\mu(\chi))$.

The following theorem will provide an existence of a value on a large space of games as well as a formula for the value as an average of the derivatives $(\varphi_D \circ \varphi_E v)_h(\chi) \equiv w_h(\chi)$ where χ is distributed according to a cylinder probability measure on $B(I,\mathcal{C})$ which is invariant under all automorphisms of $(I,\mathcal{C})$. We recall now the concept of a cylinder measure on $B(I,\mathcal{C})$.

The algebra of cylinder sets in $B(I,\mathcal{C})$ is the algebra generated by the sets of the firm $\mu^{-1}(B)$ where B is any Borel subset of $I\!R^n$ and $\mu=(\mu_1,\ldots,\mu_n)$ is any vector of measures. A cylinder probability is a finitely additive probability P on the algebra of cylinder sets such that for every vector measure $\mu=(\mu_1,\ldots,\mu_n)$, $P\circ\mu^{-1}$ is a countably additive probability measure on the Borel subsets of $I\!R^n$. Any cylinder probability P on $B(I,\mathcal{C})$ is uniquely characterized by its Fourier transform, a function on the dual that is defined by $F(\mu) = E_P(\exp(i\mu(\chi))$. Let P be the cylinder probability measure on $B(I,\mathcal{C})$ whose Fourier transform F_P is given by $F_P(\mu) = \exp(-\|\mu\|)$. This cylinder measure is invariant under all automorphisms of $(I,\mathcal{C})$. Recall that $(I,\mathcal{C})$ is isomorphic to $[0,1]$ with the Borel sets, and for a vector of measures $\mu=(\mu_1,\ldots,\mu_n)$, the Fourier transform of $P\circ\mu^{-1}$, $F_{P\circ\mu^{-1}}$ is given by $F_{P\circ\mu^{-1}}(y) = \exp(-N_\mu(y))$, and $P\circ\mu^{-1}$ is absolutely continuous with respect to the Lebesgue measure, with moreover, continuous Radon-Nikodym derivatives.

Let Q_M be the closed symmetric space generated by all games v in the domain of φ_D such that either $\varphi_D v \in FA$, or $\varphi_D(v)$ is a function of finitely many non-atomic measures.

Theorem 7 *[Mertens 1988, Section 2]. Let $v \in Q_M$. Then for every h in $B(I,\mathcal{C})$, $(\varphi_D(v))_h(\chi)$ exists for P almost every χ and is P integrable in χ, and the mapping of each game v in Q_M to the game φv given by*

$$\varphi v(S) = \int (\varphi_D(v))_S(\chi)dP(\chi)$$

is a value of norm 1 on Q_M.

Remarks: The extreme points of the set of invariant cylinder probabilities on $B(I,\mathcal{C})$ have a Fourier transform $F_{m,\sigma}(\mu) = \exp(im\mu(1) - \sigma\|\mu\|)$ where

$m \in I\!R$, $\sigma \geq 0$. More precisely, there is a one-to-one and onto map between countably additive measures Q on $I\!R \times I\!R_+$ and invariant cylinder measures on $B(I, \mathcal{C})$ where every measure Q on $I\!R \times I\!R_+$ is associated with the cylinder measure whose Fourier transform is given by

$$F_Q(\mu) = \int_{I\!R \times I\!R_+} F_{m,\sigma}(\mu) dQ(m, \sigma).$$

The associated cylinder measure is nondegenerate if $Q(r = 0) = 0$, and in the above value formula and theorem, P can be replaced with any invariant nondegenerate cylinder measure of total mass 1 on $B(I, \mathcal{C})$.

Neyman (2001) provides an alternative approach to define a value on the closed space spanned by all bounded variation games of the form $f \circ \mu$ where μ is a vector of non-atomic probability measures and f is continuous at $\mu(\emptyset)$ and $\mu(I)$, and Q_M. This alternative approach stems from the ideas in Mertens (1988) and expresses the value as a limit of averaged marginal contributions where the average is with respect to a distribution which is strictly stable of index 1.

For any $I\!R^n$-valued non-atomic vector measure μ we define a map φ^δ_μ from $Q(\mu)$ — the space of all games of bounded variation that are functions of the vector measure μ and are continuous at $\mu(\emptyset)$ and at $\mu(I)$ — to BV. The map φ^δ_μ depends on a small positive constant $\delta > 0$ and the vector measure $\mu = (\mu_1, \ldots, \mu_n)$.

For $\delta > 0$ let $I_\delta(t) = I(3\delta \leq t < 1 - 3\delta)$ where I stands for the indicator function. The essential role of the function I_δ is to make the integrands that appear in the integrals used in the definition of the value well defined.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be a vector of non-atomic probability measures and $f : \mathcal{R}(\mu) \to I\!R$ be continuous at $\mu(\emptyset)$ and $\mu(I)$ and with $f \circ \mu$ of bounded variation.

It follows that for every $x \in 2\mathcal{R}(\mu) - \mu(I)$, $S \in \mathcal{C}$, and t with $I_\delta(t) = 1$, $t\mu(I) + \delta x$ and $t\mu(I) + \delta x + \delta\mu(S)$ are in $\mathcal{R}(\mu)$ and therefore the functions $t \mapsto I_\delta(t) f(t\mu(I) + \delta x)$ and $t \mapsto I_\delta(t) f(t\mu(I) + \delta x + \delta\mu(S))$ are of bounded variation on $[0, 1]$ and thus in particular they are integrable functions. Therefore, given a game $f \circ \mu \in Q(\mu)$, the function $F_{f,\mu}$, defined on all triples (δ, x, S) with $\delta > 0$ sufficiently small (e.g., $\delta < 1/9$), $x \in I\!R^n$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$, and $S \in \mathcal{C}$ by

$$F_{f,\mu}(\delta, x, S) = \int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - f(t\mu(I) + \delta^2 x)}{\delta^3} dt,$$

is well defined.

Let P_μ^δ be the restriction of P_μ to the set of all points in $\{x \in I\!\!R^n \ : \ \delta x \in 2\mathcal{R}(\mu) - \mu(I)\}$. The function $x \mapsto F_{f,\mu}(\delta, x, S)$ is continuous and bounded and therefore the function φ_μ^δ defined on $Q(\mu) \times \mathcal{C}$ by

$$\varphi_\mu^\delta(f \circ \mu, S) = \int_{AF(\mu)} F_{f,\mu}(\delta, x, S) dP_\mu^\delta(x),$$

where $AF(\mu)$ stands for the affine space spanned by $\mathcal{R}(\mu)$, is well defined.

The linear space of games $Q(\mu)$ is not a symmetric space. Moreover, the map φ_μ^δ violates all value axioms. It does not map $Q(\mu)$ into FA, it is not efficient, and it is not symmetric. In addition, given two non-atomic vector measures, μ and ν, the operators φ_μ^δ and φ_ν^δ differ on the intersection $Q(\mu) \cap Q(\nu)$. However, it turns out that the violation of the value axioms by φ_μ^δ diminishes as δ goes to 0, and the difference $\varphi_\mu^\delta(f \circ \mu) - \varphi_\nu^\delta(g \circ \nu)$ goes to 0 as $\delta \to 0$ whenever $f \circ \mu = g \circ \nu$. Therefore, an appropriate limiting argument enables us to generate a value on the union of all the spaces $Q(\mu)$.

Consider the partially ordered linear space $\mathcal{L}$ of all bounded functions defined on the open interval $(0, 1/9)$ with the partial order $h \succ g$ if and only if $h(\delta) \geq g(\delta)$ for all sufficiently small values of $\delta > 0$. Let $L : \mathcal{L} \to I\!\!R$ be a monotonic (i.e., $L(h) \geq L(g)$ whenever $h \succ g$) linear functional with $L(\mathbf{1}) = 1$.

Let Q denote the union of all spaces $Q(\mu)$ where μ ranges over all vectors of finitely many non-atomic probability measures. Define the map $\varphi : Q \to I\!\!R^{\mathcal{C}}$ by

$$\varphi v(S) = L(\varphi_\mu^\delta(v, S))$$

whenever $v \in Q(\mu)$. It turns out that φ is well defined, φv is in FA (whenever $v \in Q$) and φ is a value of norm 1 on Q. As φ is a value of norm 1 on Q and the continuous extension of any value of norm 1 defines a value (of norm 1) on the closure, we have:

Proposition 6 *φ is a value of norm 1 on $\bar{Q}$.*

8 Uniqueness of the Value of Nondifferentiable Games

The symmetry axiom of a value implies (see, e.g., Aumann and Shapley 1974, p.139) that if $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of mutually singular measures in

NA^1, $f : [0,1]^n \to I\!R$, and $v = f \circ \mu$ is a game which is a member of a space Q on which a value φ is defined, then

$$\varphi v = \sum_{i=1}^{n} a_i(f)\mu_i.$$

The efficiency of φ implies that $\sum_{i=1}^{n} a_i(f) = f(1,\ldots,1)$, and if in addition f is symmetric in the n coordinates all the coefficients $a_i(f)$ are the same, i.e., $a_i(f) = \frac{1}{n} f(1,\ldots,1)$. The results below will specify spaces of games that are generated by vector measure games $v = f \circ \mu$ where μ is a vector of mutually singular non-atomic probability measures and $f : [0,1]^n \to I\!R$ and on which a unique value exists.

Denote by $\bar{M}_+^k$ the linear space of function on $[0,1]^k$ spanned by all concave Lipschitz functions $f : [0,1]^k \to I\!R$ with $f(0) = 0$ and $f_y(x) \le f_y(tx)$ whenever x is in the interior of $[0,1]^k$, $y \in I\!R_+^k$ and $0 < t \le 1$. $\bar{M}_\ell^k$ is the linear space of all continuous piecewise linear functions f on $[0,1]^k$ with $f(0) = 0$.

Definition 7 *[Haimanko 2000d]. CON_s (respectively, LIN_s) is the linear span of games of the form $f \circ \mu$ where for some positive integer k, $f \in \bar{M}_+^k$ (respectively, $f \in \bar{M}_+^k$) and $\mu = (\mu_1, \ldots, \mu_k)$ is a vector of mutually singular measures in NA^1.*

The spaces CON_s and LIN_s are in the domain of the Mertens value; therefore there is a value on CON_s and on LIN_s. If $f \in \bar{M}_+^k$ or $f \in \bar{M}_\ell^k$, the directional derivative $f_y(x)$ exists whenever x is in the interior of $[0,1]^k$ and $y \in I\!R^k$, and for each fixed x, the directional derivative of the function $y \mapsto f_y(x)$ in the direction $z \in I\!R^k$, $\lim_{\varepsilon \to 0+} \frac{f_{y+\varepsilon z}(x) - f_y(x)}{\varepsilon}$, exists and is denoted $\partial f(x; y; z)$. If $\mu = (\mu_1, \ldots, \mu_k)$ is a vector of mutually singular measures in NA^1 and φ_M is the Mertens value,

$$\varphi_M(f \circ \mu)(S) = \int_0^1 \partial f(t\mathbf{1}_k; X; \mu(S))\, dt$$

where $\mathbf{1}_k = (1, \ldots, 1) \in I\!R^k$ and $X = (X_1, \ldots, X_k)$ is a vector of independent random variables, each with the standard Cauchy distribution.

Theorem 8 *[Haimanko 2000d and 2001b]. The Mertens value is the unique value on CON_s and on LIN_s.*

9 Desired Properties of Values

The short list of conditions that define the value, linearity, positivity, symmetry and efficiency, suffices to determine the value on many spaces of games, like pNA_∞, pNA and $bv'NA$. Values were defined in other cases by means of constructive approaches. All these values satisfy many additional desirable properties like continuity, the projection axiom, i.e., $\varphi v = v$ whenever $v \in FA$ is in the domain of φ, the dummy axiom, i.e., $\varphi v(S^c) = 0$ whenever S is a carrier of v, and stronger versions of positivity.

9.1 Strong Positivity

One of the stronger versions of positivity is called strong positivity: Let $Q \subseteq BV$. A map $\varphi : Q \to FA$ is *strongly positive* if $\varphi v(S) \geq \varphi w(S)$ whenever $v, w \in Q$ and $v(T \cup S') - v(T) \geq w(T \cup S') - w(T)$ for all $S' \subseteq S$ and T in $\mathcal{C}$.

It turns out that strong positivity can replace positivity and linearity in the characterization of the value on spaces of "smooth" games.

Theorem 9 *[Monderer and Neyman 1988, Theorems 8 and 5].*

(a) *Any symmetric, efficient, strongly positive map from pNA_∞ to FA is the Aumann-Shapley value.*

(b) *Any symmetric, efficient, strongly positive and continuous map from pNA to FA is the Aumann-Shapley value.*

9.2 The Partition Value

Given a value φ on a space Q, it is of interest to specify whether it could be interpreted as a limiting value. This leads to the definition of a partition value.

A value φ on a space Q is a *partition value* (Neyman and Tauman 1979) if for every coalition S there exists a T-admissible sequence $(\pi_k)_{k=1}^\infty$ such that $\varphi v(T) = \lim_{k\to\infty} \psi v_{\pi_k}(T)$.

It follows that any partition value is continuous (with norm ≤ 1) and coincides with the asymptotic value on all games that have an asymptotic value (Neyman and Tauman 1979). There are however spaces of games that include ASYMP and on which a partition value exists. Given two sets of games, Q and R, we denote by $Q * R$ the closed linear and symmetric space generated by all games of the form uv where $u \in Q$ and $v \in R$.

Theorem 10 *[Neyman and Tauman 1979]. There exists a partition value on the closed linear space spanned by* $ASYMP$, $bv'NA * bv'NA$ *and* $\mathcal{A} * bv'NA * bv'NA$.

9.3 The Diagonal Property

A remarkable implication of the diagonal formula is that the value of a vector measure game $f \circ \mu$ in pNA or in $bv'NA$ is completely determined by the behavior of f near the diagonal of the range of μ. We proceed to define a diagonal value as a value that depends only on the behavior of the game in a diagonal neighborhood. Formally, given a vector of non-atomic probability measures $\mu = (\mu_1, \ldots, \mu_n)$ and $\varepsilon > 0$, the $\varepsilon - \mu$ diagonal neighborhood, $\mathcal{D}(\mu, \varepsilon)$, is defined as the set of all coalitions S with $|\mu_i(S) - \mu_j(S)| < \varepsilon$ for all $1 \leq i,\ j \leq n$. A map φ from a set Q of games is diagonal if for every $\varepsilon - \mu$ diagonal neighborhood $\mathcal{D}(\mu, \varepsilon)$ and every two games v, w in $Q \cap BV$ that coincide on $\mathcal{D}(\mu, \varepsilon)$ (i.e., $v(S) = w(S)$ for any S in $\mathcal{D}(\mu, \varepsilon)$), $\varphi v = \varphi w$.

There are examples of non-diagonal values (Neyman and Tauman 1976) even on reproducing spaces (Tauman 1977). However,

Theorem 11 *[Neyman 1977a]. Any continuous value (in the variation norm) is diagonal.*

Thus, in particular, the values on (pNA and) $bv'NA$ (Aumann and Shapley 1974, Proposition 43.1), the weak asymptotic value, the asymptotic value (Aumann and Shapley 1974, Proposition 43.11), the mixing value (Aumann and Shapley 1974, Proposition 43.2), any partition value (Neyman and Tauman 1979) and any value on a closed reproducing space, are all diagonal.

Let $DIAG$ denote the linear subspace of BV that is generated by the games that vanish on some diagonal neighborhood. Then $DIAG \subset ASYMP$ and any continuous (and in particular any asymptotic or partition) value vanishes on DIAG.

10 Semivalues

Semivalues are generalizations and/or analogies of the Shapley value that do not (necessarily) obey the efficiency, or Pareto optimality, property. This has stemmed partly from the search for value function that describes the prospects of playing different roles in a game.

Definition 8 *[Dubey, Neyman and Weber 1981]. A* semivalue *on a linear and symmetric space of games Q is a linear, symmetric and positive map $\psi : Q \to FA$ that satisfies the projection axiom (i.e., $\psi v = v$ for every v in $Q \cap FA$).*

The following result characterizes all semivalues on the space FG of all games having a finite support.

Theorem 12 *[Dubey, Neyman and Weber 1981, Theorem 1].*

(a) *For each probability measure ξ on [0,1], the map ψ_ξ that is given by*

$$\psi_\xi v(S) = \int_0^1 \partial \bar{v}(t, S) d\xi(t),$$

where $\bar{v}$ is Owen's multilinear extension of the game v, defines a semivalue on FG. Moreover, every semivalue on FG is of this form and the mapping $\xi \to \psi_\xi$ is $1-1$.

(b) *ψ_ξ is continuous (in the variation norm) on G if and only if ξ is absolutely continuous w.r.t the Lebesgue measure λ on $[0,1]$ and $d\xi/d\lambda \in L_\infty[0,1]$. In that case, $\|\psi_\xi\|_{BV} = \|d\xi/d\lambda\|_\infty$.*

Let W be the subset of all elements g in $L_\infty[0,1]$ such that $\int_0^1 g(t)dt = 1$.

Theorem 13 *[Dubey, Neyman and Weber 1981, Theorem 2].*

For every g in W, the map $\psi_g : pNA \to FA$ defined by

$$\psi_g v(S) = \int_0^1 \partial \bar{v}(t, S) g(t) dt$$

is a semivalue on pNA. Moreover, every semivalue on pNA is of this form and the map $g \to \psi_g$ maps W onto the family of semivalues on pNA and is a linear isometry.

Let W_c be the subset of all elements g in W which are (represented by) continuous functions on $[0,1]$. We identify W (W_c) with the set of all probability measures ξ on $[0,1]$ which are absolutely continuous $w.r.t$ the Lebesgue measure λ on $[0,1]$, and their Radon-Nikodym derivative $d\xi/d\lambda \in W$ ($\in W_c$).

Definition 9 *Let v be a game and let ξ be a probability measure on $[0,1]$. A game φv is said to be the* weak asymptotic ξ-semivalue *of v if, for every $S \in \mathcal{C}$ and every $\varepsilon > 0$, there is a finite subfield π of $\mathcal{C}$ with $S \in \pi$ such that for any finite subfield $\pi\prime$ with $\pi\prime \supset \pi$,*

$$|\psi_\xi v_{\pi\prime}(S) - \varphi v(S)| < \varepsilon.$$

Remarks: (1) A game v has a weak asymptotic ξ-semivalue if and only if, for every S in C and every $\varepsilon > 0$, there is a finite subfield π of $\mathcal{C}$ with $S \in \pi$ such that for any subfield $\pi\prime$ with $\pi\prime \supset \pi$, $|\psi_\xi v_{\pi\prime}(S) - \psi_\xi v_\pi(S)| < \varepsilon$.

(2) A game v has at most one weak asymptotic ξ-semivalue.

(3) The weak asymptotic ξ-semivalue φv of a game v is a finitely additive game and $\|\varphi v\| \leq \|v\| \|d\xi/d\lambda\|_\infty$ whenever $\xi \in W$.

Let ξ be a probability measure on [0,1]. The set of all games of bounded variation and having a weak asymptotic ξ-semivalue is denoted $ASYMP^*(\xi)$.

Theorem 14 *Let ξ be a probability measure on $[0,1]$.*

(a) *The set of all games having a weak asymptotic ξ-semivalue is a linear symmetric space of games and the operator that maps each game to its weak asymptotic ξ-semivalue is a semivalue on that space.*

(b) *$ASYMP^*(\xi)$ is closed $\iff \xi \in W \iff pFA \subset ASYMP^*(\xi) \iff pNA \subset ASYMP^*(\xi)$.*

(c) $\xi \in W_c \Longrightarrow bv'NA \subset bv'FA \subset ASYMP^*(\xi)$.

The asymptotic ξ-semivalue of a game v is defined whenever all the sequences of the ξ-semivalues of finite games that "approximate" v have the same limit.

Definition 10 *Let ξ be a probability measure on $[0,1]$. A game φv is said to be the* asymptotic ξ-semivalue *of v if, for every $T \in \mathcal{C}$ and every T-admissible sequence $(\pi_i)_{i=1}^{\infty}$, the following limit and equality exists:*

$$\lim_{k\to\infty} \psi_\xi v_{\pi_k}(T) = \varphi v(T).$$

Remarks: (1) A game v has an asymptotic ξ-semivalue if and only if for every S in C and every S-admissible sequence $\mathcal{P} = (\pi_i)_{i=1}^{\infty}$ the limit, $\lim_{k\to\infty} \psi_\xi v_{\pi_i}(S)$, exists and is independent of the choice of $\mathcal{P}$.

(2) For any given v, the asymptotic ξ-semivalue, if it exists, is clearly unique.

(3) The asymptotic ξ-semivalue φv of a game v is finitely additive and $\|\varphi v\|_{BV} \le \|v\|_{BV} \|d\xi/d\lambda\|_\infty$ (when $\xi \in W$).

(4) If v has an asymptotic ξ-semivalue φv, then v has a weak asymptotic ξ-semivalue ($= \varphi v$).

The space of all games of bounded variation that have an asymptotic ξ-semivalue is denoted $ASYMP(\xi)$.

Theorem 15 **(a)** *[Dubey 1980]. $pNA_\infty \subset ASYMP(\xi)$ and if φv denotes the asymptotic ξ-semivalue of $v \in pNA_\infty$, then $\|\varphi v\|_\infty \le 2\|v\|_\infty$.*

(b) *$pNA \subset pM \subset ASYMP(\xi)$ whenever $\xi \in W$.*

(c) *$bv'NA \subset bv'M \subset ASYMP(\xi)$ whenever $\xi \in W_c$.*

11 Partially Symmetric Values

Non-symmetric values (quasivalues) and partially symmetric values are generalizations and/or analogies of the Shapley value that do not necessarily obey the symmetry axiom. A (symmetric) value is covariant with respect to all automorphisms of the space of players. A partially symmetric value is covariant with respect to a specified subgroup of automorphisms. Of particular importance are the trivial group, the subgroups that preserve a finite (or countable) partition of the set of players and the subgroups that preserve a fixed population measure.

The group of all automorphisms that preserve a measurable partition Π, i.e., all automorphisms θ such that $\theta S = S$ for any $S \in \Pi$, is denoted $\mathcal{G}(\Pi)$. Similarly, if Π is a σ-algebra of coalitions ($\Pi \subset \mathcal{C}$) we denote by $\mathcal{G}(\Pi)$ the set of all automorphisms θ such that $\theta S = S$ for any $S \in \Pi$. The group of automorphisms that preserve a fixed measure μ on the space of players is denoted $\mathcal{G}(\mu)$.

11.1 Non-Symmetric and Partially Symmetric Values

Definition 11 *Let Q be a linear space of games. A* non-symmetric value *(*quasivalue*) on Q is a linear, efficient and positive map $\psi : Q \to FA$ that satisfies the projection axiom. Given a subgroup of automorphisms $\mathcal{H}$, an $\mathcal{H}$-*symmetric value *on Q is a non-symmetric value ψ on Q such that for every $\theta \in \mathcal{H}$ and $v \in Q$, $\psi(\theta_* v) = \theta_*(\psi v)$. Given a σ-algebra Π, a* Π-symmetric value *on Q is a $\mathcal{G}(\Pi)$-symmetric value. Given a measure μ on the space of players, a* μ-value *is a $\mathcal{G}(\mu)$-symmetric value.*

A *coalition structure* is a measurable, finite or countable, partition Π of I such that every atom of Π is an infinite set. The results below characterize all the $\mathcal{G}(\Pi)$-symmetric values, where Π is a fixed coalition structure, on finite games and on smooth games with a continuum of players. The characterizations employ path values which are defined by means of monotonic increasing paths in $B^1_+(I, \mathcal{C})$. A *path* is a monotonic function $\gamma : [0,1] \to B^1_+(I, \mathcal{C})$ such that for each $s \in I$, $\gamma(0)(s) = 0$, $\gamma(1)(s) = 1$, and the function $t \mapsto \gamma(t)(s)$ is continuous at 0 and 1. A path γ is called *continuous* if for every fixed $s \in I$ the function $t \mapsto \gamma(t)(s)$ is continuous; *NA-continuous* if for every $\mu \in NA$ the function $t \to \mu(\gamma(t))$ is continuous; and Π-*symmetric* (where Π is a partition of I) if for every $0 \leq t \leq 1$, the function $s \mapsto \gamma(t)(s)$ is

Π-measurable. Given an extension operator ψ on a linear space of games Q (for $v \in Q$ we use the notation $\bar{v} = \psi v$) and a path γ, the *γ-path extension* of v, $\bar{v}_\gamma$, is defined (Haimanko 2000c) by

$$\bar{v}_\gamma(f) = \bar{v}(\gamma^*(f))$$

where $\gamma^* : B^1_+(I, \mathcal{C}) \to B^1_+(I, \mathcal{C})$ is defined by $\gamma^*(f)(s) = \gamma(f(s))(s)$. For a coalition $S \in \mathcal{C}$ and $v \in Q$, define

$$\varphi^\varepsilon_\gamma(v)(S) = \frac{1}{\varepsilon} \int_\varepsilon^{1-\varepsilon} \left[\bar{v}_\gamma\left(tI + \varepsilon S\right) - \bar{v}_\gamma\left(tI\right)\right] dt$$

and $DIFF(\gamma)$ is defined as the set of all $v \in Q$ such that for every $S \in \mathcal{C}$ the limit $\varphi_\gamma(v)(S) = \lim_{\varepsilon \to 0+} \varphi^\varepsilon_\gamma(v)(S)$ exists and the game $\varphi_\gamma(v)$ is in FA.

Proposition 7 *[Haimanko 2000c]. $DIFF(\gamma)$ is a closed linear subspace of Q and the map φ_γ is a non-symmetric value on $DIFF(\gamma)$. If Π is a measurable partition (or a σ-algebra) and γ is Π-symmetric then φ_γ is a Π-symmetric value.*

In what follows we assume that $Q = bv'NA$ with the natural extension operator and Π is a fixed coalition structure.

Theorem 16 *[Haimanko 2000c]. (a) γ is NA-continuous $\iff$ $pNA \subset DIFF(\gamma)$.*
(b) Every non-symmetric value on $pNA(\lambda)$, $\lambda \in NA^1$, is a mixture of path values.
(c) Every Π-symmetric value (where Π is a coalition structure) on pNA (on FG) is a mixture of Π-symmetric path values.

Remarks: *[Haimanko 2000c].* 1) There are non-symmetric values on pNA which are not mixtures of path values.
2) Every linear and efficient map $\psi : pNA \to FA$ which is Π-symmetric with respect to a coalition structure Π obeys the projection axiom. Therefore, the projection axiom of a non-symmetric value is not used in (c) of Theorem 16.

The above characterization of all Π-symmetric values on $pNA(\lambda)$ as mixtures of path values relies on the non-atomicity of the games in pNA. A more delicate construction, to be described below, leads to the characterization of all Π-symmetric values on pM.

For every game v in $bv'M$ there is a measure $\lambda \in M^1$ such that $v \in bv'M(\lambda)$. The set of atoms of λ is denoted by $A(\lambda)$. Given two paths γ, γ', define for $v \in bv'M(\lambda)$, $f \in B^1_+(I, \mathcal{C})$

$$\bar{v}_{\gamma,\gamma'}(f) = \bar{v}_{\gamma''}$$

where $\gamma'' = \gamma\,[I - A(\lambda)] + \gamma'\,[A(\lambda)]$ and $v \to \bar{v}$ is the natural extension on $bv'M$. It can be verified that $\bar{v}_{\gamma,\gamma'}$ is well defined on $bv'M$, i.e., the definition is independent of the measure λ. For a coalition $S \in \mathcal{C}$ and $v \in bv'M$, define

$$\varphi^{\varepsilon}_{\gamma,\gamma'}(v)(S) = \frac{1}{\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \left[\bar{v}_{\gamma,\gamma'}\,(tI + \varepsilon S) - \bar{v}_{\gamma,\gamma'}\,(tI)\right] dt.$$

We associate with $bv'M$ and the pair of paths γ and γ' the set $DIFF(\gamma, \gamma')$ of all games $v \in bv'M$ such that for every $S \in \mathcal{C}$ the limit $\varphi_{\gamma,\gamma'}(v)(S) = \lim_{\varepsilon \to 0+} \varphi^{\varepsilon}_{\gamma''}(v)(S)$ exists and the game $\varphi_{\gamma,\gamma'}(v)$ is in FA.

Theorem 17 *[Haimanko 2000c]. (a)$DIFF(\gamma, \gamma')$ is a closed linear subspace of $bv'M$ and the map $\varphi_{\gamma,\gamma'} : DIFF(\gamma, \gamma') \to FA$ is a non-symmetric value.*
(b) Given a measurable partition (or a σ-algebra) Π, $\varphi_{\gamma,\gamma'}$ is Π-symmetric whenever $s \mapsto \gamma(t)(s)$ and $s \mapsto \gamma'(t)(s)$ are Π measurable for every t.
(c) If Π is a coalition structure, any Π-symmetric value on pM is a mixture of maps $\varphi_{\gamma,\gamma'}$ where the paths γ and γ' are continuous and Π-symmetric.

Remarks: *[Haimanko 2000c].* 1) If γ is NA-continuous and the discontinuities of the functions $t \mapsto \gamma'(t)(s)$, $s \in I$, are mutually disjoint, $pM \subset DIFF(\gamma, \gamma')$. In particular if γ and γ' are continuous, $pM \subset DIFF(\gamma, \gamma')$ and therefore if in addition $\gamma(t)$ and $\gamma'(t)$ are constant functions for each fixed t, γ and γ' are $\mathcal{G}$-symmetric and thus $\varphi_{\gamma,\gamma'}$ defines a value of norm 1 on pM. These infinitely many values on pM were discovered by Hart (1973) and are called the Hart values.
2) A surprising outcome of this systematic study is the characterization of all values on pM. Any value on pM is a mixture of the Hart values.
3) If Π is a coalition structure, any Π-symmetric linear and efficient map $\psi : bv'M \to FA$ obeys the projection axiom. Therefore, the projection axiom of a non-symmetric value is not used in the characterization (c) of Theorem 17.

Weighted values are non-symmetric values of the form φ_γ where γ is a path described by means of a measurable weight function $w : I \to \mathbb{R}_{++}$

(which is bounded away from 0): for every $\gamma(s)(t) = t^{w(s)}$. Hart and Monderer (1997) have successfully applied the potential approach of Hart and Mas-Colell (1989) to weighted values of smooth (continuously differentiable) games: to every game $v \in pNA_\infty$ the game $(P_w v)(S) = \int_0^1 \frac{\bar{v}(t^w S)}{t}\, dt$ is in pNA_∞ and $\varphi_\gamma v(S) = (\bar{P_w v})_{wS}(I)$. In addition they define an asymptotic w-weighted value and prove that every game in pNA_∞ has an asymptotic w-weighted value.

Another important subgroup of automorphisms is the one that preserves a fixed (population) non-atomic measure μ.

11.2 Measure-Based Values

Measure-based values are generalizations and/or analogues of the value that take into account the coalition worth function and a fixed (population) measure μ. This has stemmed partly from applications of value theory to economic models in which a given population measure μ is part of the models.

Let μ be a fixed non-atomic measure on $(I, \mathcal{C})$. A subset of games, Q, is *μ-symmetric* if for every $\Theta \in \mathcal{G}(\mu)$ (the group of μ-measure-preserving automorphisms), $\Theta_* Q = Q$. A map φ from a μ-symmetric set of games Q into a space of games is μ-symmetric if $\varphi \Theta_* v = \Theta_* \varphi v$ for every Θ_* in $\mathcal{G}(\mu)$ and every game v in Q.

Definition 12 *Let Q be a μ-symmetric space of games. A μ-*value* on Q is a map from Q into finitely additive games that is linear, μ-symmetric, positive, efficient, and obeys the dummy axiom (i.e., $\varphi v(S) = 0$ whenever the complement of S, S^c, is a carrier of v).*

Remarks: The definition of a μ-value differs from the definition of a value in two aspects. First, there is a weakening of the symmetry axiom. A value is required to be covariant with the actions of all automorphisms, while a μ-value is required to be covariant only with the actions of the μ-measure-preserving automorphism. Second, there is an additional axiom in the definition of a μ-value, the dummy axiom. In view of the other axioms (that φv is finitely additive and efficiency), this additional axiom could be viewed as a strengthening of the efficiency axiom; the μ-value is required to obey $\varphi v(S) = v(S)$ for any carrier S of the game v, while the efficiency axiom requires it only for the carrier $S = I$. In many cases of interest, the dummy axiom follows from the other axioms of the value (Aumann and Shapley 1974, Note 4, p.18). Obviously, the value on pNA obeys the dummy axiom and the restriction of any value that obeys the dummy axiom to a μ-symmetric subspace is a μ-value. In particular, the restriction of the unique value on pNA to $pNA(\mu)$, the closed (in the bounded variation norm) algebra generated by non-atomic probability measures that are absolutely continuous with respect to μ, is a μ-value. It turns out to be the unique μ-value on $pNA(\mu)$.

Theorem 18 *[Monderer 1986, Theorem A]. There exists a unique μ-value on $pNA(\mu)$; it is the restriction to $pNA(\mu)$ of the unique value on pNA.*

Remarks: In the above characterization, the dummy axiom could be replaced by the projection axiom (Monderer 1986, Theorem D): there exists a unique linear, μ-symmetric, positive and efficient map $\varphi : pNA(\mu) \to FA$ that obeys the projection axiom; it is the unique value on $pNA(\mu)$. The characterizations of the μ-value on $pNA(\mu)$ is derived from Monderer (1986, Main Theorem), which characterizes all μ-symmetric, continuous linear maps from $pNA(\mu)$ into FA.

Similar to the constructive approaches to the value, one can develop corresponding constructive approaches to μ-values. We start with the asymptotic approach.

Given T in $\mathcal{C}$ with $\mu(T) > 0$, a *$\mu-T$-admissible sequence* is a T-admissible sequence $(\pi_1, \pi_2, \ldots)$ for which $\lim_{k\to\infty}(\min\{\mu(A) : A \text{ is an atom of } \pi_k\}/\max\{\mu(A) : A \text{ is an atom of } \pi_k\}) = 1$.

Definition 13 *A game φv is said to be the* μ-asymptotic value *of v if, for every T in $\mathcal{C}$ with $\mu(T) > 0$ and every $\mu - T$-admissible sequence $(\pi_i)_{i=1}^{\infty}$, the following limit and equality exists*

$$\lim_{k\to\infty} \psi v_{\pi_k}(T) = \varphi v(T).$$

Remarks: (1) A game v has a μ-asymptotic value if and only if for every T in $\mathcal{C}$ with $\mu(T) > 0$ and every $\mu - T$-admissible sequence $\mathcal{P} = (\pi_i)_{i=1}^{\infty}$, the limit, $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists and is independent of the choice of $\mathcal{P}$.

(2) For a given game v, the μ-asymptotic value, if it exists, is clearly unique.

(3) The μ-asymptotic value φv of a game v is finitely additive and $\|\varphi v\| \leq \|v\|$ whenever v has bounded variation.

The space of all games of bounded variation that have a μ-asymptotic value is denoted by $ASYMP[\mu]$.

Theorem 19 *[Hart 1980, Theorem 9.2]. The set of all games having a μ-asymptotic value is a linear μ-symmetric space of games and the operator mapping each game to its μ-asymptotic value is a μ-value on that space. $ASYMP[\mu]$ is a closed μ-symmetric linear subspace of BV which contains all functions of the form $f \circ (\mu_1, \ldots, \mu_n)$ where $\mu_1, \ldots, \mu_n$ are absolutely continuous w.r.t. μ and with Radon-Nikodym derivatives, $d\mu_i/d\mu$, in $L_2(\mu)$ and f is concave and homogeneous of degree 1.*

12 The Value and the Core

The Banach space of all bounded set functions with the supremum norm is denoted BS, and its closed subspace spanned by pNA is denoted pNA'. The set of all games in pNA' that are homogeneous of degree 1 (i.e., $\bar{v}(\alpha f) = \alpha\bar{v}(f)$ for all f in $B_1^+(I, \mathcal{C})$ and all α in $[0,1]$), and superadditive (i.e., $v(S \cup T) \geq v(S) + v(T)$ for all S, T in $\mathcal{C}$), is denoted H'. The subset of all games in H' which also satisfy $v \geq 0$ is denoted H'_+. The core of a game v in H' is non-empty and for every S in $\mathcal{C}$, the minimum of $\mu(S)$ when μ runs over all elements μ in core (v) is attained. The exact cover of a game v in H' is the game $\min\{\mu(S) : \mu \in \text{ Core } (v)\}$ and its core coincides with the core of v. A game $v \in H'$ that equals its exact cover is called exact. The closed (in the bounded variation norm) subspace of BV that is generated by pNA and $DIAG$ is denoted by $pNAD$.

Theorem 20 *[Aumann and Shapley 1974, Theorem I]. The core of a game v in $pNAD \cap H'$ (which obviously contains $pNA \cap H'$) has a unique element which coincides with the (asymptotic) value of v.*

Theorem 21 *[Hart 1977a]. Let $v \in H'_+$. If v has an asymptotic value φv, then φv is the center of symmetry of the core of v. The game v has an asymptotic value φv when either the core of v contains a single element ν (and then $\varphi v = \nu$), or v is exact and the core of v has a center of symmetry ν (and then $\varphi v = \nu$).*

Remarks: (1) The second theorem provides a necessary and sufficient condition for the asymptotic value of an exact game v in H'_+ to exist: the symmetry of its core. There are (non-exact) games in H'_+ whose cores have a center of symmetry which do not have an asymptotic value (Hart 1977a, Section 8).

(2) A game v in H'_+ has a unique element in the core if and only if $v \in pNAD$.

(3) The conditions of superadditivity and homogeneity of degree 1 are satisfied by all games arising from non-atomic (i.e., perfectly competitive) economic markets (see chapter 35 in this Handbook). However, the games arising from these markets have a finite-dimensional core while games in H'_+ may have an infinite-dimensional core.

The above result shows that the asymptotic value fails to exist for games in H'_+ whenever the core of the game does not have a center of symmetry. Nevertheless, value theory can be applied to such games, and when it is, the value turns out to be a "center" of the core. It will be described as an average of the extreme points of the core of the game $v \in H'_+$.

Let $v \in H'_+$ have a finite-dimensional core. Then the core of v is a convex closed subset of a vector space generated by finitely many non-atomic probability measures, $\nu_1, \ldots, \nu_n$. Therefore the core of v is identified with a convex subset K of $I\!R^n$, by $a = (a_1, \ldots, a_n) \in K$ if and only if $\sum a_i \mu_i \in$ Core (v). For almost every z in $I\!R^n$ there exists a unique $a(z) = (a_1(z), \ldots, a_n(z))$ in K with $\sum_{i=1}^n a_i(z) z_i = \min\{\sum_{i=1}^n a_i z_i : (a_1, \ldots, a_n) \in K\}$.

Theorem 22 *[Hart 1980]. If $\nu_1, \ldots, \nu_n$ are absolutely continuous with respect to μ and $d\nu_i/d\mu \in L_2(\mu)$ then $v \in ASYMP[\mu]$ and its μ-asymptotic value, $\varphi_\mu v$ is given by*

$$\varphi_\mu v = \int_{I\!R^n} \sum a_i(z) \nu_i dN(z)$$

where N is a normal distribution on $I\!R^n$ with 0 *expectation and a covariance matrix that is given by*

$$E_N(z_i z_j) = \int_I (\frac{d\nu_i}{d\mu})(\frac{d\nu_j}{d\mu})d\mu.$$

Equivalently, N is the distribution on $I\!R^n$ whose Fourier transform φ_N is given by $\varphi_N(y) = \exp(-\int_I(\sum_{i=1}^n y_i(\frac{d\nu_i}{d\mu}))^2 d\mu)$.

Theorem 23 *[Mertens 1988]. The Mertens value of the game $v \in H'_+$ with a finite-dimensional core, φv, is given by*

$$\varphi v = \int_{I\!R^n} (\sum a_i(z)\nu_i)dG(z)$$

where G is the distribution on $I\!R^n$ whose Fourier transform φ_G is given by $\varphi_G(y) = \exp(-N_\nu(y))$, $\nu = (\nu_1, \ldots, \nu_n)$ and N_ν is defined as in the section regarding formulas for the value.

It is not known whether there is a unique continuous value on the space spanned by all games in H'_+ which have a finite-dimensional core. We will state however a result that proves uniqueness for a natural subspace. Let M_F be the closed space of games spanned by vector measure games $f \circ \mu \in H'_+$ where $\mu = (\mu_1, \ldots, \mu_k)$ is a vector of mutually singular measures in NA^1.

Theorem 24 *[Haimanko 2001a]. The Mertens value is the only continuous value on M_F.*

13 Comments on Some Classes of Games

13.1 Absolutely Continuous Non-Atomic Games

A game v is *absolutely continuous with respect to a non-atomic measure* μ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every increasing sequence of coalitions $S_1 \subset S_2 \subset \ldots \subset S_{2k}$ with $\sum_{i=1}^k \mu(S_{2i}) - \mu(S_{2i-1}) < \delta$, $\sum_{i=1}^k |v(S_{2i}) - v(S_{2i-1})| < \varepsilon$. A game v is *absolutely continuous* if there is a measure $\mu \in NA^+$ such that v is absolutely continuous with respect to μ. A game of the form $f \circ \mu$ where $\mu \in NA^1$ is absolutely continuous (with respect to μ) if and only if $f : [0, \mu(I)] \to I\!R$ is absolutely continuous. An absolutely continuous game has bounded variation and the set of all absolutely continuous games, AC, is a closed subspace of BV (Aumann and Shapley 1974). If v and u are absolutely continuous with respect to $\mu \in NA^+$ and $\nu \in NA^+$ respectively, then vu is absolutely continuous with respect to $\mu + \nu$. Thus AC is a closed subalgebra of BV.

It is not known whether there is a value on AC.

13.2 Games in pNA

The space pNA played a central role in the development of values of non-atomic games. Games arising in applications are often either vector measure games or approximated by vector measure games. Researchers have thus looked for conditions on vector (or scalar) measure games to be in pNA. Tauman (1982) provides a characterization of vector measure games in pNA. A vector measure game $v = f \circ \mu$ with $\mu \in (NA^1)^n$ is in pNA if and only if the map that assigns to a vector measures ν with the same range as μ the game $f \circ \nu$, is continuous (at μ), i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\nu \in (NA^1)^n$ has the same range as the vector measure μ and $\sum_{i=1}^n \|\mu_i - \nu_i\| < \delta$ then $\|f \circ \mu - f \circ \nu\| < \varepsilon$. Proposition 10.17 of Aumann and Shapley (1974) provides a sufficient condition, expressed as a condition on a continuous non-decreasing real-valued function $f : I\!R^n \to I\!R$, for a game of the form $f \circ \mu$ where μ is a vector of non-atomic measures to be in pNA. Theorem C of Aumann and Shapley (1974) asserts that the scalar measure game $f \circ \mu$ ($\mu \in NA^1$) is in pNA if and only if the real-valued function $f : [0,1] \to I\!R$ is absolutely continuous. Given a signed scalar measure μ with range $[-a, b]$ $(a, b > 0)$, Kohlberg (1973) provides a sufficient and necessary condition on the real-valued function $f : [-a, b] \to I\!R$ for the scalar measure game $f \circ \mu$ to be in pNA.

13.3 Games in $bv'NA$

Any function $f \in bv'$ has a unique representation as a sum of an absolutely continuous function f^{ac} that vanishes at 0 and a singular function f^s in bv', i.e., a function whose variation is on a set of measure zero. The subspace of all singular functions in bv' is denoted s'. The closed linear space generated by all games of the form $f \circ \mu$ where $f \in s'$ and $\mu \in NA^1$ is denoted $s'NA$. Aumann and Shapley (1974) show that $bv'NA$ is the algebraic direct sum of the two spaces pNA and $s'NA$, and that for any two games, $u \in pNA$

and $v \in s'NA$, $\|u+v\| = \|u\| + \|v\|$. The norm of a function $f \in bv'$, $\|f\|$, is defined as the variation of f on $[0,1]$. If $\mu \in NA^1$ and $f \in bv'$ then $\|f \circ \mu\| = \|f\|$. Therefore, if $(f_i)_{i=1}^{\infty}$ is a sequence of functions in bv' with $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and $(\mu_i)_{i=1}^{\infty}$ is a sequence of non-atomic probability measures, the series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ converges to a game $v \in bv'NA$. If the functions f_i are absolutely continuous the series converges to a game in pNA. However, not every game in pNA has a representation as a sum of scalar measure games. If the functions f_i are singular, the series $\sum_{i=1}^{\infty} f_i \circ \mu_i$ converges to a game in $s'NA$, and every game $v \in s'NA$ is a countable (possibly finite) sum, $v = \sum_{i=1}^{\infty} f_i \circ \mu_i$, where $f_i \in s'$ with $\|v\| = \sum_{i=1}^{\infty} \|f_i\|$ and $\mu_i \in NA^1$.

A simple game is a game v where $v(S) \in \{0,1\}$. A weighted majority game is a (simple) game of the form

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) \geq q \\ 0 & \text{if} \quad \mu(S) < q \end{cases}$$

or

$$v(S) = \begin{cases} 1 & \text{if} \quad \mu(S) > q \\ 0 & \text{if} \quad \mu(S) \leq q \end{cases}$$

where $\mu \in FA^+$ and $0 < q < \mu(I)$. The finitely additive measure μ is called the *weight measure* and q is called the *quota*. Every simple monotonic game in $bv'NA$ is a weighted majority game, and the weighted measure is non-atomic.

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