ELSEVIER

Journal of Mathematical Economics 36 (2001) 117–140

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Stable voting procedures for committees in economic environments

Hans Keiding [a,*], Bezalel Peleg [b]

[a] *Institute of Mathematics and Center for Rationality and Interactive Decision Theory, Hebrew University of Jerusalem, Hebrew, Denmark*

[b] *Institute of Economics, Studiestraede 6, DK-1455 Copenhagen K., Denmark*

Received 12 September 2000; accepted 4 May 2001

Abstract

A strong representation of a committee, formalized as a simple game, on a convex and closed set of alternatives is a game form with the members of the committee as players such that (i) the winning coalitions of the simple game are exactly those coalitions, which can get any given alternative independent of the strategies of the complement, and (ii) for any profile of continuous and convex preferences, the resulting game has a strong Nash equilibrium. In the paper, it is investigated whether committees have representations on convex and compact subsets of $\boldsymbol{R}^m$. This is shown to be the case if there are vetoers; for committees with no vetoers the existence of strong representations depends on the structure of the alternative set as well as on that of the committee (its Nakamura-number). Thus, if A is strictly convex, compact, and has smooth boundary, then no committee can have a strong representation on A. On the other hand, if A has non-smooth boundary, representations may exist depending on the Nakamura-number (if it is at least 7).

JEL classification: D71; C71.

Keywords: Committees; Simple games; Representation; Effectivity functions

1. Introduction

The study of committee decision making in political and economic environments has been the subject of many investigations since Black (1948). At this point we shall only mention Arrow (1951), Moulin (1980), Barbera and Peleg (1990), Zhou (1991) and Barbera et al. (1994). For a recent survey of closely related work the reader is referred to Sprumont (1995).

* Tel.: +45-3532-3089; fax: +45-3532-3085.
E-mail address: hans.keiding@pop.oko.ku.dk (H. Keiding).

PII: S0304-4068(01)00068-4

Black (1948) and Arrow (1951) introduced the class of single-peaked preferences on the real line. For this class of preferences, the analysis of the decision making of a simple majority committee leads to the median voter rule. This rule is (coalitionally) strategy-proof, anonymous, and Pareto efficient. In Moulin (1980) all voting rules (on the class of single peaked preference profiles) which are strategy-proof, anonymous, and efficient, have been characterized. Moulin's work has been refined by several authors (see Sprumont, 1995), so that now committee decision making on (a closed convex subset of) the real line with single-peaked preferences is fully understood (see also Section 8 in the present paper).

Clearly, assuming that choice set is one-dimensionsional is very restrictive. In many real-life problems we have to deal with several issues simultaneously or allocate the budget to several projects. However, the problem of extending Moulin's results to higher dimensions remained open till Zhou (1991). In his paper, Zhou generalizes the Gibbard-Satterthwaite Theorem (see Gibbard (1973) and Satterthwaite (1975)) to economies with pure public goods. In terms of the theory of committee decision making, Zhou's result can be described as follows. Let A be a closed and convex subset of $\boldsymbol{R}^m$, $m \geq 2$, and let the dimension of A be m. Further, let G be a non-dictatorial commitee (i.e. G is a non-dictatorial (monotonic and proper) simple game), and let f be a social choice function which has the following properties: (i) f induces the same power structure (among the players) as G, (ii) f is defined for (profiles of) continuous and convex preferences (on A). Then (under all the foregoing assumptions), f is manipulable.

The work of Zhou (1991) is the starting point of our investigation. Our method for finding satisfactory voting rules for pure public goods economies can be described in the following way. Let $G = (N, W)$ be a committee, and let A be a convex and compact subset of $\boldsymbol{R}^m$ of full dimension m, for $m \geq 2$. The pair (G, A) will be called a choice problem. The core of a choice problem (G, A) with respect to a profile $\succsim^N$ of preferences is defined in the usual way (see our Section 2 below). We mainly consider stable choice problems, i.e. choice problems (G, A) such that the core $\mathrm{C}(G, A, \succsim^N) \neq \emptyset$ for every profile $\succsim^N$ of continuous and convex preferences. We remark that if G is weak (i.e. G contains at least one vetoer), then (G, A) is stable (for every compact set A). For a non-weak committee G, (G, A) is stable iff $m \leq \nu(G) - 2$, where $\nu(G)$ is the number of G (see Greenberg (1979) and Le Breton (1987)).

Now, let (G, A) be a stable choice problem. We ask whether there exists a game form Γ with the following properties:

1. Γ (partially) implements the core $\mathrm{C}(G, A, \cdot)$ on the (restricted) domain of continuous and convex preferences in strong Nash equilibria;
2. the power structure induced by Γ on N (i.e. the set of members of G), is equal to G.

If a game form Γ satisfies the foregoing conditions (1) and (2) with respect to a (stable) choice problem (G, A), then we say that Γ is a strong representation of G on A. In this paper, we study the existence of strong representations of choice problems. Our study is motivated by the following two claims:

1. A strong representation of a choice problem (G, A) is a satisfactory (generalized) voting procedure that enables the committee G to choose a member of A.
2. There exist important families of choice problems which have strong representations.

We shall now elaborate on these two claims. Let (G, A) be a choice problem, and let Γ be a strong representation of G on A. As Γ is a game form, it may be considered as a generalized voting procedure for G (the ordinary voting procedures are given by social choice functions). Indeed, quite a few voting rules are given by game forms; approval voting is a well-known example. In addition, each voting game that is induced by Γ (in conjunction with a profile of continuous and convex preferences) has a strong Nash equilibrium. This strong stability property is not implied, for example, by the existence of equilibrium in dominant strategies. Finally, Γ truly reflects the power structure represented by G.

In order to justify claim (b) we shall mention two of our results. (1) If G is weak, then (G, A) has a strong representation for every A (that satisfies our assumptions); (2) assume that $A = \boldsymbol{R}^m$ and that $m \leq \nu(G) - 2$. If we restrict ourselves to continuous and convex preferences which are also bounded (i.e. the upper level sets are bounded; see Section 3), then (G, A) has a strong representation. We remark that preferences which are derived from a weighted Euclidean distance (see Enelow and Hinich, 1984), are bounded.

We can now summarize our approach. Using the nonemptiness of the core of a choice problem (G, A), we find strongly stable (generalized) voting procedures for G (i.e. procedures which are stable when combined with continuous and convex preferences on A). The manipulability problem is avoided because we use game forms (and not social choice functions). However, as expected, the voting games induced by our game forms are not solvable by dominant strategies; nevertheless, they have strong Nash equilbria.

Earlier works on existence of strong representations for committees considered choice problems with a finite set of alternatives. The following is a (partial) list of contributions to the theory of representation: Peleg (1978a, 1978b, 1984), Dutta and Pattanaik (1978), Ishikawa and Nakamura (1980) and Holzman (1986a, 1986b). These works proved existence of strong representations by social choice functions, whereas we only prove strong representation by game forms. We have to enlarge the set of possible representations because of the complexity of the representation problem in the continuous case. Indeed, we obtained some impossibility results for (the larger set of) game forms. Finally, we should mention the close relationship between representation theory and implementation in strong Nash equilibria (see Moulin and Peleg (1982) and Maskin (1985) for results on implementation by strong Nash equilibria).

We now briefly review the contents of this paper. Section 2 is devoted to definitions and notations. Existence of strong representations for spatial voting games is proved in Section 3. Choice problems for committees with vetoers are considered in Section 4, where it is proved that all such problems have strong representations. In Section 5, we state and prove the first impossibility result (Theorem 3) which may be formulated as follows. Let $A \subset \boldsymbol{R}^2$ be strictly convex, compact, and smooth, and let G be a committee without vetoers. Then G has no strong representation on A. This result can be generalized to higher dimensions (see Theorem 4). Small Nakamura numbers are considered in Section 6. If (G, A) is a choice problem and $\nu(G) \leq 6$, then G has no strong representation on A. The first case which is not excluded by our impossibility theorems is $G = (7, 6)$ (i.e. a special majority of 6 out of 7) and A is the (two-dimensional) standard unit simplex in $\boldsymbol{R}^3$ (notice that $\nu(G) = 7$ and A is not strictly convex). It is solved in full detail in Section 7. Finally, the classical case of dimension 1 is briefly discussed in Section 8. A general result, extending the basic result of Moulin and Peleg (1982) on representation of effectivity functions with finite sets of alternatives to effectivity functions with infinitely many alternatives, is used at several occassions to prove existence of representations. This result is stated and proved in an appendix.

2. Definitions and notations

Let A be a set of alternatives. Throughout this paper, excluding the appendix, A is a closed and convex subset of a Euclidean space $\boldsymbol{R}^m$, $m \geq 1$. We always assume that A is of dimension m. A preference ordering on A is a complete and transitive binary relation. We denote by $\mathcal{P}$ the set of all preference orderings on A. $\succsim \in \mathcal{P}$ is *continuous* if for each $x \in A$, the sets $\{y \in A | y \succsim x\}$ and $\{y \in A | x \succsim y\}$ are closed. We denote by $\mathcal{P}_c$ the set of all continuous preference orderings on A. $\succsim \in \mathcal{P}$ is convex if for each $x \in A$ the set $\{y \in A | y \succsim x\}$ is convex. We denote by $\mathcal{P}_{cc}$ the set of all continuous and convex preference orderings. Finally, if $\succsim \in \mathcal{P}$, then its asymmetric part $\succ$ is defined by

$$[x \succ y] \Leftrightarrow [x \succsim y \text{ and not } y \succsim x] \text{ for all } x, y \in A.$$

Let D be a set. We denote by $P(D)$ the set of all subsets of D, i.e. $P(D) = \{D' | D' \subset D\}$. Also, $2^D = P(D) \setminus \{\emptyset\}$ is the set of all non-empty subsets of D.

Let A be a set of alternatives and let $N = \{1, \dots, n\}$ be a finite set of players. An *effectivity function* (EF) is a function $E : P(N) \to P(P(A))$ that satisfies the following conditions;

$$E(N) = 2^A,$$
$$E(\emptyset) = \emptyset,$$
$$A \in E(S) \text{ for all } S \in 2^N,$$
$$\emptyset \notin E(S) \text{ for all } S \in P(N).$$

Let E be an EF. E is *superadditive* if it satisfies the following condition. If $S_i \in 2^N$, $B_i \in E(S_i)$, $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$. E is *maximal* if for all $S \in 2^N$ and $B \in 2^A$

$$B \notin E(S) \Rightarrow A \setminus B \in E(N \setminus S).$$

The core of E with respect to $\succsim^N \in \mathcal{P}^N$ is defined in the following way. Let $B \in 2^A$, $S \in 2^N$, and $x \in A \setminus B$. B *dominates* x *via* S *at* $\succsim^N$, if $B \in E(S)$ and $y \succ^i x$ for all $y \in B$ and $i \in S$. $x \in A$ is *dominated at* $\succsim^N$ if there exist $B \in 2^A$ and $S \in 2^N$ such that B dominates x via S at $\succsim^N$. The *core* of E with respect to $\succsim^N$, $\mathrm{C}(E, \succsim^N)$, is the set of all undominated alternatives at $\succsim^N$. If $\hat{\mathcal{P}}^N \subset \mathcal{P}^N$, then E is *stable* over $\hat{\mathcal{P}}^N$ if $\mathrm{C}(E, \succsim^N) \neq \emptyset$ for all $\succsim^N \in \hat{\mathcal{P}}^N$.

Let A be a set of alternatives, and let $N = \{1, \dots, n\}$ be a finite set of players. A *game form*(GF) is an $(n+2)$-tuple $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$, where

Σ^i is the set of *strategies* of player $i \in N$;

$\pi : \Sigma^1 \times \cdots \times \Sigma^n \to A$ is the *outcome function*.

Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a GF and let $S \in 2^N$. We denote $\Sigma^S = \prod_{i \in S} \Sigma^i$. Let, again, $S \in 2^N$ and let $B \in 2^A$. S is α-effective for B if there exists $\sigma^S \in \Sigma^S$ such that

$\pi(\sigma^S, \mu^{N\setminus S}) \in B$ for all $\mu^{N\setminus S} \in \Sigma^{N\setminus S}$. S is β-effective for B if for every $\mu^{N\setminus S} \in \Sigma^{N\setminus S}$ there exists $\sigma^S \in \Sigma^S$ such that $\pi(\sigma^S, \mu^{N\setminus S}) \in B$. The α-EF of Γ, E_α^Γ, is defined by $E_\alpha^\Gamma(\emptyset) = \emptyset$, and

$$E_\alpha^\Gamma(S) = \{B \in 2^A | S \text{ is } \alpha\text{-effective for } B\}, \quad \text{for } S \in 2^N.$$

The β-EF of Γ, E_β^Γ, is defined by $E_\beta^\Gamma(\emptyset) = \emptyset$, and

$$E_\beta^\Gamma(S) = \{B \in 2^A | S \text{ is } \beta\text{-effective for } B\}, \quad \text{for } S \in 2^N.$$

We remark that E_α^Γ is superadditive, and E_β^Γ is maximal (see, e.g. Abdou and Keiding, 1991).

Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a GF and let $\succsim^N \in \mathcal{P}^N$. The pair $(\Gamma, \succsim^N)$ defines, in an obvious way, a game in strategic form. We denote $\Sigma = \Sigma^N = \prod_{i=1}^n \Sigma^i$. $\sigma \in \Sigma$ is a *strong Nash equilibrium* (SNE) of $(\Gamma, \succsim^N)$ if for all $S \in 2^N$ and $\mu^S \in \Sigma^S$, there exists $i \in S$ such that

$$\pi(\sigma) \succsim^i \pi(\mu^S, \sigma^{N\setminus S}).$$

We remark that if σ is an SNE of $(\Gamma, \succsim^N)$, then $\pi(\sigma) \in \mathrm{C}(E_\beta^\Gamma, \succsim^N)$ (see Peleg, 1984). Γ is SNE-*consistent* over $\hat{\mathcal{P}}^N \subset \mathcal{P}^N$ if for each $\succsim^N \in \hat{\mathcal{P}}^N$ the game $(\Gamma, \succsim^N)$ has an SNE.

Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a GF and let $E : P(N) \to P(P(A))$ be an EF. Γ (partially) implements the core $\mathrm{C}(E, \succsim^N)$ over $\hat{\mathcal{P}}^N \subset \mathcal{P}^N$ if for every $\succsim^N \in \hat{\mathcal{P}}^N$ $(\pi(\mathrm{SNE}(\Gamma, \succsim^N)) \subset \mathrm{C}(E, \succsim^N))$ $\pi(\mathrm{SNE}(\Gamma, \succsim^N)) = \mathrm{C}(E, \succsim^N)$ (here $\mathrm{SNE}(\Gamma, \succsim^N)$ is the set of SNE's of the game $(\Gamma, \succsim^N)$).

Finally, we recall some properties of simple games. A simple game is a pair (N, W), where $N = \{1, \dots, n\}$ is a set of players, and $W \subset 2^N$ is a set of winning coalitions. Let $G = (N, W)$ be a simple game. G is *monotonic* if

$$[S \in W \text{ and } S \subset T \subset N] \Rightarrow T \in W,$$

G is *proper* if

$$S \in W \Rightarrow N \setminus S \notin W \text{ for all } S \in 2^N.$$

We only deal with monotonic and proper simple games. Let, again, $G = (N, W)$ be a simple game. G is *strong* if

$$S \notin W \Rightarrow N \setminus S \in W \text{ for all } S \in 2^N.$$

G is *symmetric* if G is an (n, k) game, i.e. there exists $(n/2) < k \le n$ such that $W = \{S \subset N || S| \ge k\}$ (if D is a finite set, then $|D|$ is the number of members of D). G is *weak* if

$$V = \cap\{S | S \in W\} \ne \emptyset.$$

V is the set of *vetoers* of G. If G is not weak, then the *Nakamura number* of G, $\nu(G)$, is given by

$$\nu(G) = \min\{|U| | U \subset W \text{ and } \cap \{S | S \in U\} = \emptyset\}$$

(see Nakamura, 1979).

Let $G = (N, W)$ be a simple game, let A be a set of alternatives, let $\succsim^N \in \mathcal{P}^N$, and let $x, y \in A$. x *dominates* y *at* $\succsim^N$ if there exists $S \in W$ such that $x \succ^i y$ for all $i \in S$. The *core* of G with respect to $\succsim^N$, $\mathrm{C}(G, \succsim^N)$, is the set of all undominated alternatives at $\succsim^N$. Assume now that A is a compact and convex subset of $\boldsymbol{R}^m$, the dimension of A is m, and G is not weak. Then $\mathrm{C}(G, \succsim^N) \neq \emptyset$ for all $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$ iff $\nu(G) \geq m+2$ (see Le Breton, 1987).

Let $E : P(N) \to P(P(A))$ be an EF. The simple game (N, W_E) which is associated with E is given by

$$W_E = \{S \in 2^N | E(S) = 2^A\}.$$

E is an *extension* of the simple game (N, W) if $W_E = W$. Let now $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a GF. The simple game which is associated with Γ is $G^\Gamma_\alpha = (N, W_{E^\Gamma_\alpha})$. Γ is a *representation* of $G = (N, W)$ if $G = G^\Gamma_\alpha$. Γ is a *strong* representation of G if (i) Γ is a representation of G, and (ii) Γ is SNE-consistent over $\mathcal{P}^N_{\mathrm{cc}}$.

3. Implementation of the core of spatial voting games

In this section, we consider a particular class of choice problems (G, A), namely such where the set of alternatives is Euclidean space of some dimension m and where the preferences are bounded in the sense that at each $a \in A$, the set of alternatives which are at least as good as a is a bounded set. The literature on spatial voting problems, see e.g. Enelow and Hinich (1984), treats particular cases of such decision problems; we use the term spatial voting games for the entire class.

Let $A = \boldsymbol{R}^m$, $m \geq 1$. We denote by $\mathcal{A}^*$ the set of all $D' \subset A$ such that D' is open, convex, and bounded. A preference relation $\succsim \in \mathcal{P}_{\mathrm{cc}} = \mathcal{P}_{\mathrm{cc}}(A)$ is *bounded* if for each $x \in A$ the set $\{y \in A | y \succ x\}$ is bounded. We denote by $\mathcal{P}_{\mathrm{ccb}}$ the set of bounded preferences in $\mathcal{P}_{\mathrm{cc}}$. A *spatial voting game* is an $(n+1)$-tuple $(G; \succsim^1, \ldots, \succsim^n)$, where $G = (N, W)$ is a proper and monotonic simple game, $N = \{1, \ldots, n\}$, and $\succsim^i \in \mathcal{P}_{\mathrm{ccb}}$ for $i = 1, \ldots, n$.

Theorem 1. *Let $G = (N, W)$ be a proper and monotonic game. Then there exists a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ with the following properties*:

1. *For every $\succsim^N \in \mathcal{P}^N_{\mathrm{ccb}}$, $\pi(SNE(\Gamma, \succsim^N)) = \mathrm{C}(G, \succsim^N)$ (thus, in particular, if $\mathrm{C}(G, \succsim^N) \neq \emptyset$ for all $\succsim^N \in \mathcal{P}^N_{\mathrm{ccb}}$, then Γ is SNE-consistent (on $\mathcal{P}^N_{\mathrm{ccb}}$).)*
2. *$G^\Gamma_\alpha = G$, that is Γ is a representation of G.*

Proof. Define an EF $E : P(N) \to P^2(A)$ by the following rules. If $S \in W$, then $E(S) = 2^A$, the set of non-empty subsets of A; if $S \subset N$, $S \neq \emptyset$, and $N \setminus S \in W$, then $E(S) = \{A\}$; further, we put $E(\emptyset) = \emptyset$, and finally, if $S \subset N$ is blocking, that is $S, N \setminus S \notin W$, then

$$E(S) = \{D \in 2^A | \exists D' \in \mathcal{A}^* : D \supset A \setminus D'\}.$$

Thus, if S is blocking and $B \in E(S)$, then B is unbounded. Now, if $S \subset N$, $S \neq \emptyset$, $\succsim^N \in \mathcal{P}^N_{\mathrm{ccb}}$, $x \in A$, and $B \subset \Pr(S, \succsim^N, x)$, then B is bounded. Hence $\mathrm{C}(E, \succsim^N) = \mathrm{C}(G, \succsim^N)$ for all $\succsim^N \in \mathcal{P}^N_{\mathrm{ccb}}$. We claim that E is superadditive and satisfies condition (CC) of Theorem 8.

To check superadditivity, let $S, T \subset N, S, T \neq \emptyset, S \cap T = \emptyset, B_1 \in E(S)$ and $B_2 \in E(T)$. If $T \in W$ (or $S \in W$), $B_1 = A$ ($B_2 = A$), and consequently $B_1 \cap B_2 \in E(S \cup T)$. Thus assume that both S and T are not winning. If S or T are losing, then again $B_1 \cap B_2 \in E(S \cup T)$ by the monotonicity of E. Hence, it remains to consider the possibility that both S and T are blocking. In this case there are $B'_1, B'_2 \in \mathcal{A}^*$ such that $B_i \supset A \setminus B'_i, i = 1, 2$. Let B' be the convex hull of $B'_1 \cup B'_2$. Then $B' \in \mathcal{A}^*$ is open, convex, and bounded, and $B_1 \cap B_2 \supset A \setminus B'$. As $A \setminus B' \in E(S \cup T)$, E is indeed superadditive.

It remains to prove (CC). Let $S \subset N$, $S \neq \emptyset, N$. Further, let $\succsim^N \in \mathcal{P}^N_{\text{ccb}}$, and $x \in A$. We should prove that

$$\Pr(S, \succsim^N, x) \notin E(S) \Rightarrow A \setminus \Pr(S, \succsim^N, x) \in E(N \setminus S). \quad (1)$$

If $S \in W$ or $N \setminus S \in W$, then (1) is obviously true. Thus, let S be blocking. By our assumptions, $\Pr(S, \succsim^N, x)$ is open, convex, and bounded. Hence, $A \setminus \Pr(S, \succsim^N, x) \in E(N \setminus S)$ because $N \setminus S$ is blocking.

We may now apply Theorem 8 to obtain a GF $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ that implements $\mathrm{C}(E, \cdot)$ in SNE's (over $\mathcal{P}^N_{\text{ccb}}$). Clearly, Γ implements $\mathrm{C}(G, \cdot)$ in SNE's. Furthermore, for $S \in W$, $E^\Gamma_\alpha(S) = E(S)$, that is $E^\Gamma_\alpha(S) = 2^A$. Also, if $S \notin W$, then clearly $E^\Gamma_\alpha(S) \neq 2^A$. Thus, $G^\Gamma_\alpha = G$. □

Remark 1. Theorem 1 can be generalized in two directions. (i) It is possible to replace $\boldsymbol{R}^m$ by a closed, convex, and *unbounded* subset of $\boldsymbol{R}^m$. (ii) $\mathcal{P}_{\text{ccb}}$ may be replaced by $\mathcal{P}_{\text{cb}}$, the set of continuous and bounded preferences over $\boldsymbol{R}^m$.

4. Representations of weak games

In this and the following sections, we consider choice problems (G, A) for which A is a convex and *compact* subset of some Euclidean space. In the present section, we consider the case where the game is weak, that is there is a vetoer. It will be shown that in this case the representation problem has a solution for all convex and compact sets of alternatives A.

Thus, let A be a convex and compact subset of $\boldsymbol{R}^m$, $m \geq 1$. Assume that $\text{aff}(A) = \boldsymbol{R}^m$ (here, $\text{aff}(A)$ is the affine hull of A).

Theorem 2. *Let $G = (N, W)$ be a weak game, that is $V = \cap\{S | S \in W\} \neq \emptyset$. Assume that $N = \{1, \dots, n\}$ and $1 \in V$. Then there exists a strong representation of G (on A), i.e. there exists a GF $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ with the following properties*:

1. *$G^\Gamma_\alpha = G$, and*
2. *Γ is SNE-consistent.*

Remark 2. If Γ is a strong representation of G, then for each $\succsim^N \in \mathcal{P}^N_{\text{cc}}$

$$\pi(\mathrm{SNE}(\Gamma, \succsim^N)) \subset \mathrm{C}(E^\Gamma_\beta, \succsim^N) \subset \mathrm{C}(E^\Gamma_\alpha, \succsim^N) \subset \mathrm{C}(G^\Gamma_\alpha, \succsim^N) = \mathrm{C}(G, \succsim^N)$$

Hence, Γ partially implements the core $\mathrm{C}(G, \cdot)$.

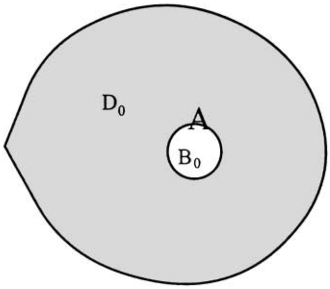

Fig. 1.

Proof of Theorem 2. We define an EF E that extends G by the following rules: $E(S) = 2^A$ if $S \in W$ (here $2^A = \{B \subset A | B \neq \emptyset\}$), $E(S) = \{A\}$ if $N \setminus S \in W$ and $S \neq \emptyset$, and $E(\emptyset) = \emptyset$. In order to complete the definition of E we choose an open ball B_0 such that $\mathrm{cl} B_0 \subset \mathrm{int} A$, and let $D_0 = A \setminus B_0$. If $S \subset N$ is blocking and $1 \in S$, then we define

$$E(S) = \{D | D \subset A \text{ and } D \cap D_0 \neq \emptyset\} \tag{2}$$

and

$$E(N \setminus S) = \{D | D \subset A \text{ and } D \supset D_0\}. \tag{3}$$

This completes the definition of E (Fig. 1).

We claim that $\mathrm{C}(E, \succsim^N) \neq \emptyset$ for all $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$. Indeed, if $S \subset N$ is blocking and $1 \notin S$, then domination of $a \in A$ via S is impossible due to the convexity of preferences, since $\Pr(S, \succsim^N, a)$ is convex and does not contain a, and S is not effective for any set B for which $\mathrm{conv}(B) \neq A$. Thus, every alternative in $\mathrm{argmax}_A \succsim^1$ belongs to $\mathrm{C}(E, \succsim^N)$.

Next, we prove that E is superadditive. Let $S_i \subset N$, $i = 1, 2$, $S_1 \cap S_2 = \emptyset$, and let $B_i \in E(S_i)$, $i = 1, 2$. If S_1 or S_2 is winning, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$. Similarly, if S_1 or S_2 are losing, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$ by the monotonicity of E. Finally, if both S_1 and S_2 are blocking, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$ by Eqs. (2) and (3).

Finally, we check that E satisfies (CC) (see Theorem 8). Let $S \subset N$, $S \neq \emptyset, N$, let $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$, and let $x \in A$. We must show

$$\Pr(S, \succsim^N, x) \notin E(S) \Rightarrow A \setminus \Pr(S, \succsim^N, x) \in E(N \setminus S). \tag{4}$$

If $S \in W$ or $N \setminus S \in W$, then (4) is true, Thus, let S be blocking; (4) now follows from Eqs. (2) and (3).

By Theorem 8 there exists a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ such that

$$\pi(\mathrm{SNE}(\Gamma, \succsim^N)) = \mathrm{C}(E, \succsim^N) \tag{5}$$

for every $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$, and

$$E^\Gamma_\alpha(S) \supset E(S) \text{ for all } S \in P(N). \tag{6}$$

By Eqs. (5), Γ is strongly consistent, because E is stable. Finally, by Eqs. (2), (3) and (6), and the proof of Theorem 8, $G^\Gamma_\alpha = G$. □

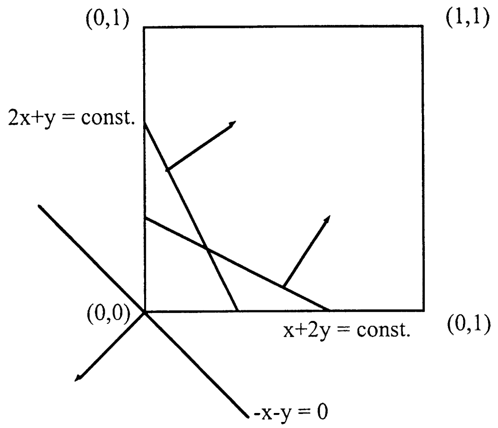

Fig. 2.

The core of a weak game may not be (fully) implementable in SNE's by strong representations. This is shown by the following example (Fig. 2).

Example 1. Let $G = [3; 2, 1, 1]$ and let $A = [0, 1]^2$. Further, let $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}(A)$ be given by the following utility functions:

$$u^1(x, y) = -x - y, \qquad u^2(x, y) = 2x + y, \qquad u^3(x, y) = x + 2y.$$

Then $\mathrm{C}(G, u^N) = \{(0, 0), (1, 1)\}$.

Let $\Gamma = (\Sigma^1, \Sigma^2, \Sigma^3; \pi; A)$ be a strong representation of G. We claim that $(1, 1) \notin \pi(\mathrm{SNE}(\Gamma, u^N))$. Indeed, if there is an SNE σ^N of (Γ, u^N) such that $\pi(\sigma^N) = (1, 1)$, then $\{2, 3\}$ is α-effective for $(1, 1)$. Consider now the following profile:

$$\hat{u}^1(x, y) = -y, \qquad \hat{u}^2(x, y) = x + y, \qquad \hat{u}^3(x, y) = 2y - x.$$

As the reader may check, $\mathrm{C}(E^\Gamma_\alpha, \hat{u}^N) = \emptyset$, because $\{(1, 1)\} \in E^\Gamma_\alpha(\{2, 3\})$ and $G^\Gamma_\alpha = G$. However, this contradicts the SNE-consistency of Γ.

5. Strictly convex sets of alternatives

For games without vetoers, the structure of the set A will matter for the existence of a strong representation on A of a committee G. Indeed, in the present section we show that if the set A is strictly convex and has a smooth boundary, then (G, A) has no strong representation. We start by treating the special case of $m = 2$; the impossibility result derived in this context may then be extended to an impossibility result for arbitrary dimension m (Fig. 3).

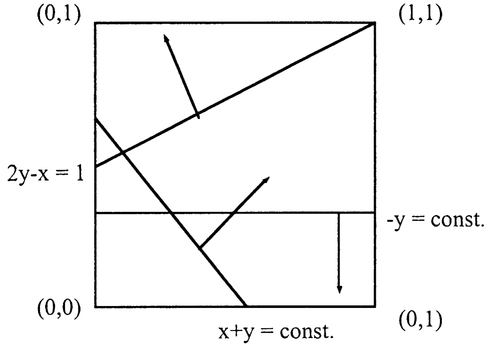

Fig. 3.

Let $A \subset \boldsymbol{R}^2$ be a convex and compact set of alternatives. We assume that $\text{aff}(A) = \boldsymbol{R}^2$. Let $G = (N, W)$ be a proper and monotonic simple game. Furthermore, let $\Gamma = (\Sigma^1, \dots, \Sigma^n; A; \pi)$ be a strong representation of G, i.e. Γ is SNE-consistent and $G_\alpha^\Gamma = G$. A coalition $S \in 2^N$ is *almost winning* with respect to Γ if for every $x \in \text{bd}A$ and every $\varepsilon > 0$, $\{y \in A | \|y - x\| < \varepsilon\} \in E_\alpha^\Gamma(S)$ (here, bd A is the boundary of A). We now recall that a (proper) face of A is a set $F \subset \text{bd}A$, $F \neq \emptyset$, with the following property. There exist $p \in \boldsymbol{R}^2 \setminus \{0\}$ and $\alpha \in \boldsymbol{R}$ such that

$$F = \{x \in A | p \cdot x = \alpha\}$$

(if a face is a singleton, then it consists of an exposed point). Using this term we can now introduce our last (new) concept in this section. A coalition $S \in 2^N$ is *weakly* winning with respect to Γ if every open neighborhood of every face of A is in $E_\alpha^\Gamma(S)$.

Throughout the rest of this section we assume that A is smooth, that is, at each $x \in \text{bd}A$, A has a unique tangent.

The following lemma uses all the concepts introduced previously:

Lemma 1. *Let $A \subset \boldsymbol{R}^2$ convex and compact with* $\text{aff}(A) = \boldsymbol{R}^2$, *such that A is smooth, let $G = (N, W)$ be a proper and monotonic simple game, and let Γ be a strong representation of G.*

If S_1 and S_2 *are almost winning coalitions with respect to Γ, then $T = S_1 \cap S_2$ is weakly winning with respect to Γ.*

Proof. Let $F \subset \text{bd}A$ be a (proper) face of A, and let U be an open set (in A) containing F. We shall prove that $U \in E_\alpha^\Gamma(T)$. Two cases must be distinguished:

1. F is a point, and
2. F is an interval (with positive length) (Fig. 4).

We shall deal only with case (2) (the proof in case (1) is similar to that of case (2)).

Thus, let $F = [a, b]$ with $a \neq b$. There exists $p \in \boldsymbol{R}^2 \setminus \{0\}$ such that (i) $F = \{y \in A | p \cdot y = p \cdot a\}$, and (ii) $p \cdot a \geq p \cdot x$ for all $x \in A$. We now choose $\delta > 0$ such that $U \supset \{x \in A | p \cdot x > p \cdot a - \delta\}$. We also can choose $q_1, q_2 \in \boldsymbol{R}^2 \setminus \{0\}$ with the following

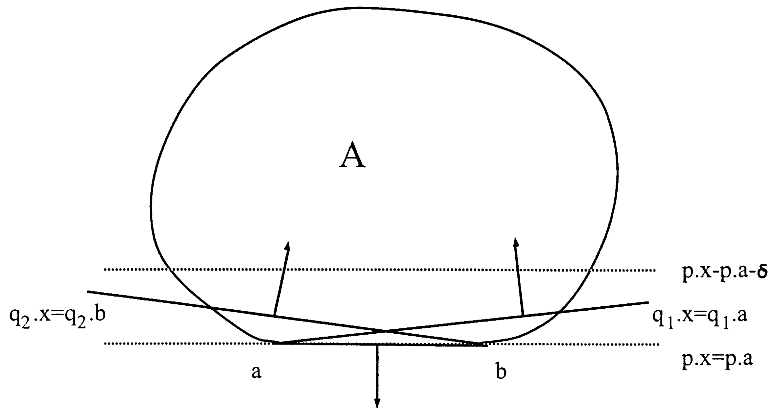

Fig. 4.

properties: $q_i \cdot p < 0, i = 1, 2, q_1 \cdot q_2 < 0$, and $\{x \in A | q_1 \cdot x = q_1 \cdot a\} \cup \{x \in A | q_2 \cdot x = q_2 \cdot b\}$ is contained in $\{x \in A | p \cdot x > p \cdot a - \delta\}$. Now we define the utility profile u^N by

1. $u^i(y) = p \cdot y, i \in T$,
2. $u^i(y) = q_1 \cdot y, i \in S_1 \setminus T$, and
3. $u^i(y) = q_2 \cdot y, i \in N \setminus S_1$.

As the reader may check, $\mathrm{C}(E_\alpha^\Gamma, u^N) \subset [a, b]$ (notice that $\mathrm{C}(G, u^N) = [a, b]$). The game (Γ, u^N) has an SNE σ because Γ is SNE-consistent (Fig. 5). Clearly,

$$x = \pi(\sigma) \in \mathrm{C}(E_\beta^\Gamma, u^N) \subset \mathrm{C}(E_\alpha^\Gamma, u^N) \subset [a, b].$$

Now $\{y \in A | y \succ^i x \text{ for all } i \in N \setminus T\} \supset \{y \in A | p \cdot y \leq p \cdot a - \delta\}$. Hence,

$$\pi(\sigma^T, \mu^{N \setminus T}) \in \{y \in A | p \cdot y > p \cdot a - \delta\} \subset U \text{ for all } \mu^{N \setminus T} \in \Sigma^{N \setminus T}.$$

Therefore, $U \in E_\alpha^\Gamma(T)$. □

Lemma 1 has an important corollary. First we recall that A is strictly convex if for all $x, y \in A, x \neq y$, and all $0 < \alpha < 1, \alpha x + (1 - \alpha) y \in \mathrm{int}\, A$.

Theorem 3. *Let $A \subset \boldsymbol{R}^2$ be strictly convex, compact, and smooth, with $\mathrm{aff}(A) = \boldsymbol{R}^2$, and let $G = (N, W)$ be a proper and monotonic simple game without veto players. Then G has no strong representation on A.*

Proof. Assume, on the contrary, that G has a strong representation Γ on A. If $S \in 2^N$ is weakly winning with respect to Γ, then S is almost winning, because every $x \in \mathrm{bd}\, A$ is an exposed point of A. Thus, by Lemma 1, the intersection of two almost winning coalitions is almost winning. Now every $S \in W$ is winning with respect to Γ, since $G_\alpha^\Gamma = G$; hence, in particular, it is almost winning. By assumption, $\cap\{S | S \in W\} = \emptyset$. Hence, there exist two disjoint almost winning coalitions (with respect to Γ). Clearly, this contradicts the superadditivity of E_α^Γ. □

We now generalize Theorem 3 to $m \geq 3$.

Theorem 4. *Let $A \subset \boldsymbol{R}^m$, $m \geq 3$, be strictly convex, compact, and smooth, and let* aff $(A) = \boldsymbol{R}^m$. *Let $G = (N, W)$ be a proper and monotonic simple game without veto players. Then G has no strong representation on A.*

Proof. Assume, on the contrary, that G has a strong representation $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ on A. For $a = (x_1, x_2, x_3, \ldots, x_m) \in \boldsymbol{R}^m$, let $x_1 = x, x_2 = y$, and $z = (x_3, \ldots, x_m)$. Denote by p the projection of $\boldsymbol{R}^m$ on the subspace given by $z = 0$, i.e.

$$p(x, y, z) = (x, y, 0).$$

Let $\hat{A} = p(A) = \{p(a) | a \in A\}$. Then $\hat{A}$ is strictly convex, compact, and smooth. Furthermore aff $(\hat{A})$ is two-dimensional. Now, define a GF $\hat{\Gamma}$ on $\hat{A}$ by $\hat{\Gamma} = (\Sigma^1, \ldots, \Sigma^n; p \circ \pi; \hat{A})$, and let $\hat{G} = G_\alpha^{\hat{\Gamma}}$. By our assumption, $G_\alpha^\Gamma = G$. Hence, if $\hat{G} = (N, \hat{W})$, then $\hat{W} \supset W$. Thus, $\hat{G}$ has no vetoers. Also, by its definition, $\hat{G}$ is proper and monotonic. We shall conclude the proof by showing that $\hat{\Gamma}$ is SNE-consistent.

Let $(u^i(x, y, 0))_{i \in N}$, be a utility profile for $\hat{\Gamma}$, that is, each u^i is continuous and quasi-concave on $\hat{A}$. Let $v^i(x, y, z) = u^i(x, y, 0)$ for all $(x, y, z) \in A$ and $i \in N$. Then v^N is a utility profile for Γ. Thus, the game (Γ, v^N) has an SNE. Moreover, as $v^i(x, y, z) = u^i(p(x, y, z))$ for all $i \in N$ and $(x, y, z) \in A$, $(\Gamma, v^N) = (\hat{\Gamma}, u^N)$. Thus, $\hat{\Gamma}$ is SNE-consistent. Therefore, $\hat{\Gamma}$ is a strong representation of $\hat{G}$, contradicting Theorem 3. □

6. An impossibility result for small Nakamura numbers

The choice problems (G, A) considered in the previous section do not exhaust the possibilities for choice problems with no strong representation. We show in this section that if the Nakamura number of G is less than 6, then no strong representation exists; this impossibility result holds for general convex and compact sets of alternatives.

Let $A \subset \boldsymbol{R}^2$ be a convex and compact set, let aff $(A) = \boldsymbol{R}^2$, and let $G = (N, W)$ be a proper and monotonic simple game. We will prove in this section the following impossibility result. If $\nu(G) \in \{4, 5, 6\}$, then G has no strong representation on A. We start with the following lemma.

Lemma 2. *Let $A \subset \boldsymbol{R}^2$ be convex and compact, let* aff $(A) = \boldsymbol{R}^2$, *let $G = (N, W)$ be a proper and monotonic simple game, and let the GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a strong representation of G. If $S_1, S_2 \in W$, then $T = S_1 \cap S_2$ is weakly winning with respect to Γ.*

Proof. Let $F \subset \text{bd}A$ be a (proper) face of A and let U be an open set (in A) containing F. We shall prove that $U \in E_\alpha^\Gamma(T)$.

Let $a \in F$. There exists $p \in \boldsymbol{R}^2 \setminus \{0\}$ such that

1. $F = \{x \in A | p \cdot x = p \cdot a\}$; and

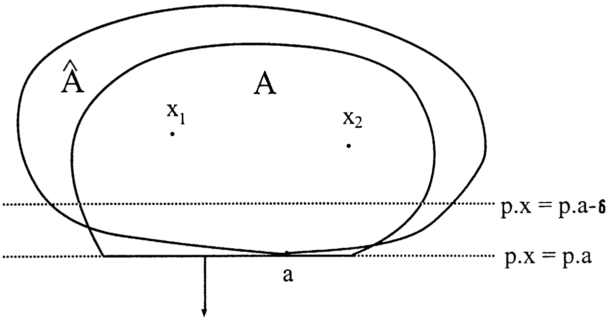

Fig. 5.

2. $p \cdot a \geq p \cdot x$ for all $x \in A$.

Clearly, there exists $\delta > 0$ such that

$$U \supset \{x \in A | p \cdot x > p \cdot a - \delta\}.$$

We now choose a convex, compact, and smooth set $\hat{A}$ such that

1. $\{x \in A | p \cdot x \leq p \cdot a - \delta\} \subset \mathrm{int}\hat{A}$, and
2. $a \in \mathrm{bd}\hat{A}$ and the line $p \cdot x = p \cdot a$ is tangent to $\hat{A}$ at a.

Using $\hat{A}$ we define two utility functions in the following way. Let $x_i \in \mathrm{int}A \cap \mathrm{int}\hat{A}$, $i = 1, 2$, such that a, x_1, and x_2 are not on the same line. Denote by $\|\cdot; \hat{A}, x_i\|$, $i = 1, 2$, the gauge determined by $\hat{A}$ with center x_i, that is

$$\|y; \hat{A}, x_i\| = \inf\{\lambda > 0 | x_i + \lambda^{-1}(y - x_i) \in \hat{A}\}, i = 1, 2.$$

Then we define $u_i(y) = -\|y; \hat{A}, x_i\|$, $i = 1, 2$.

Consider now the utility profile u^N, where

1. $u^i(y) = p \cdot y, i \in T$,
2. $u^i(y) = u_1(y), i \in S_1 \setminus T$, and
3. $u^i(y) = u_2(y), i \in N \setminus S_1$.

As the reader may check, $\mathrm{C}(G, u^N) = F$. Hence, because $G_\alpha^\Gamma = G$, $\mathrm{C}(E_\alpha^\Gamma, u^N) \subset F$. The game (Γ, u^N) has an SNE σ because Γ is SNE-consistent. Clearly,

$$x = \pi(\sigma) \in \mathrm{C}(E_\beta^\Gamma, u^N) \subset \mathrm{C}(E_\alpha^\Gamma, u^N) \subset \mathrm{C}(G_\alpha^\Gamma, u^N) = \mathrm{C}(G, u^N) = F.$$

Also,

$$\{y \in A | y \succ^i x \text{ for all } i \in N \setminus T\} \supset \{y \in A | p \cdot y \leq p \cdot a - \delta\}.$$

Hence, for every $\mu^{N \setminus T} \in \Sigma^{N \setminus T}$,

$$\pi(\sigma^T, \mu^{N \setminus T}) \in \{y \in A | p \cdot y > p \cdot a - \delta\} \subset U.$$

Therefore, $U \in E_\alpha^\Gamma(T)$. □

We proceed with the following result.

Lemma 3. *Let $A \subset \boldsymbol{R}^2$ be convex and compact, let* $\text{aff}(A) = \boldsymbol{R}^2$, *let $G = (N, W)$ be a proper and monotonic simple game, and let $\Gamma = (\Sigma^1, \dots, \Sigma^n : \pi; A)$ be a strong representation of G on A. If T_i, $i = 1, 2, 3$ are weakly winning with respect to Γ, then $T_1 \cap T_2 \cap T_2 \neq \emptyset$.*

Proof. Assume, on the contrary, that $T_1 \cap T_2 \cap T_3 = \emptyset$. We shall prove that there exists $\succsim^N \in \mathcal{P}^N_{\text{cc}}$ such that $\text{C}(E^\Gamma_\alpha, \succsim^N) = \emptyset$, and thereby we arrive at the desired contradiction. Two cases must be distinguished:

(a) A is strictly convex. Let x_i, $i = 1, 2, 3$, be three distinct points of bd A. Choose $q_i \in \boldsymbol{R}^2$, $i \in \{1, 2, 3\}$ such that

$$q_i \cdot x_i < q_i \cdot x_j = q_i \cdot x_k, \text{ where } \{i, j, k\} = \{1, 2, 3\}.$$

Now define a utility profile u^N by

$$u^i(a) = \begin{cases} q_2 \cdot a & \text{for } i \in T_1 \setminus T_2; \\ q_3 \cdot a & \text{for } i \in T_2 \setminus T_3; \\ q_1 \cdot a & \text{for } i \in (T_3 \setminus T_1) \cup N \setminus (T_1 \cup T_2 \cup T_3). \end{cases} \tag{7}$$

Denote by H_k the convex cone spanned by $\{x_i - x_k, x_j - x_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$. Then $x_i \in E^\Gamma_\alpha(T_i)$ (because x_i is an exposed point of A and T_i is weakly winning), and $u^k(x_i) > u^k(a)$ for all $a \in \text{int}[\{x_i\} + H_i]$ and $k \in T_i$, $i = 1, 2, 3$. Thus, if $y \in A \setminus \{x_1, x_2, x_3\}$, then $y \notin \text{C}(E^\Gamma_\alpha, u^N)$. Finally, let $i, j \in \{1, 2, 3\}$, $i \neq j$. By strict convexity of A there exists an exposed point x'_j near x_j such that $x_i \in \text{int}[\{x'_j\} + H_j]$. Hence, $x_i \notin \text{C}(E^\Gamma_\alpha, u^N)$. Therefore, $\text{C}(E^\Gamma_\alpha, u^N) = \emptyset$.

(b) A is not strictly convex. Then bd A contains a one-dimensional face $[x_1, x_2]$. Here two subcases must be distinguished:

(b.1) x_1 and x_2 are exposed points of A. Let x_3 be a third exposed point of A (here we use the assumption that $\text{aff}(A) = \boldsymbol{R}^2$). Now we choose three linear utility functions in the following way. Let $q_2 \in \boldsymbol{R}^2$ satisfy

$$q_2 \cdot x_1 > q_2 \cdot x_3 > q_2 \cdot x_2.$$

It is possible now to choose $y \in$ bd A near x_3 on the arc of bd A that connects x_1 and x_3 (and does not contain x_2) such that

$$q_2 \cdot x_3 < q_2 \cdot y < q_2 \cdot x_1.$$

Let $q_1 \in \boldsymbol{R}^2$ satisfy

$$q_1 \cdot x_1 < q_1 \cdot x_2 = q_1 \cdot y < q_1 \cdot x_3$$

(see Fig. 6). Again, we may choose $z \in$ bd A near x_2 such that

$$q_1 \cdot x_2 < q_1 \cdot z < q_1 \cdot x_3.$$

Finally, choose $q_3 \in \boldsymbol{R}^2$ such that

$$q_3 \cdot x_3 < q_3 \cdot x_1 = q_3 \cdot z < q_3 \cdot x_2.$$

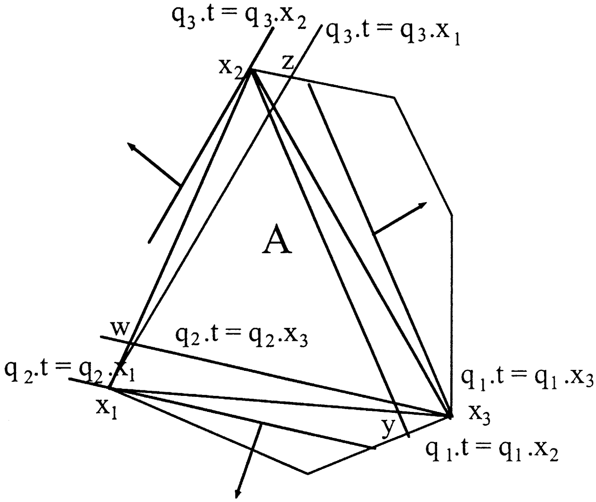

Fig. 6.

As in case (a) we define a utility profile u^N by (7). As the reader may check, the following three open sets,

$$H_1 = \{a \in A | q_2 \cdot a < q_2 \cdot x_1 \text{ and } q_3 \cdot a < q_3 \cdot x_1\},$$
$$H_2 = \{a \in A | q_1 \cdot a < q_1 \cdot x_2 \text{ and } q_3 \cdot a < q_3 \cdot x_2\}, \text{ and}$$
$$H_3 = \{a \in A | q_2 \cdot a < q_2 \cdot x_3 \text{ and } q_1 \cdot a < q_1 \cdot x_3\}$$

cover A. Moreover, if $x \in H_i$, then x_i dominates x via T_i, $i = 1, 2, 3$. Thus $\mathrm{C}(E_\alpha^\Gamma, u^N) = \emptyset$.

(b.2) x_1 or x_2 are not exposed points of A. Let $w \in [x_1, x_2]$ satisfy $q_2 \cdot x_3 = q_2 \cdot w$. Then $q_3 \cdot x_2 > q_3 \cdot w$. We can choose exposed points x_1', x_2' of A near x_1 and x_2, respectively, such that

$$q_2 \cdot x_1' > q_2 \cdot x_3 > q_2 \cdot x_2' \text{ and } q_2 \cdot x_3 < q_2 \cdot y < q_2 \cdot x_1'.$$

Now choose $q_1' \in \boldsymbol{R}^2$ such that

$$q_1' \cdot x_1' < q_1' \cdot x_2' = q_1' \cdot y < q_1' \cdot x_3.$$

Clearly, if x_2' is sufficiently close to x_2, then $q_1' \cdot x_2' < q_1' \cdot z < q_1' \cdot x_3$. Finally, choose $q_3' \in \boldsymbol{R}^2$ such that

$$q_3' \cdot x_3 < q_3' \cdot x_1' = q_3' \cdot z < q_3' \cdot x_2'.$$

Again, we may choose x_1', x_2', q_1' and q_3' such that $q_3' \cdot x_2' > q_3' \cdot w$. We now define u^N and H_i, $i = 1, 2, 3$, as in the case (b.1) and obtain, again, that $\mathrm{C}(E_\alpha^\Gamma, u^N) = \emptyset$. □

The main result of this section can now be proved.

Theorem 5. *Let $m \geq 2$, let $A \subset \boldsymbol{R}^m$ be convex and compact, let* $\mathrm{aff}(A) = \boldsymbol{R}^m$, *and let $G = (N, W)$ be a proper and monotonic simple game. If $\nu(G) \leq 6$, then G has no strong representation on A.*

Proof. As in Section 5, it is sufficient to consider the case $m = 2$. Assume now, on the contrary, that G has a strong representation Γ on A. As $\nu(G) \leq 6$, there exist three weakly winning coalitions with respect to Γ, T_1, T_2 and T_3, such that $T_1 \cap T_2 \cap T_3 = \emptyset$ (see Lemma 2). By Lemma 3 we obtain the desired contradiction. □

Notice that the only new cases which are excluded by Theorem 5 are: $m = 2$, $\nu(G) \in \{4, 5, 6\}$, $m = 3$, $\nu(G) \in \{5, 6\}$, and $m = 4$ and $\nu(G) = 6$. In all other cases impossibility is implied by Le Breton (1987).

7. A game with no vetoers that has a strong representation

In the light of the impossibility results obtained in the previous sections, one might be tempted to believe that impossibility hold generally, that is for all choice problems (G, A) where G has no vetoers. It is shown in this section that this is not the case; indeed we find a strong representation of the game $(7, 6)$ on a particular set of alternatives, namely the standard simplex in $\boldsymbol{R}^3$, and the method can be applied to give a strong representation of any game $(n, n-1)$ on this set alternatives.

Let $X = \{x_1, x_2, x_3\}$ be a set of three affinely independent points in $\boldsymbol{R}^2$, let $A = \mathrm{conv}(X)$ (the convex hull of X), and let $G = (N, W) = (7, 6)$, i.e. $N = \{1, \ldots, 7\}$ and

$$W = \{S \subset N \,||S| \geq 6\}.$$

We shall prove that G has a strong representation on A.

Define an EF $E : P(N) \to P(P(A))$ as follows. For $S \subset N$, let

$$E(S) = \begin{cases} 2^A & \text{if } |S| \geq 6, \\ \{B \,||B \cap X| \geq 1\} & \text{if } |S| = 5, \\ \{B \,||B \cap X| \geq 2\} & \text{if } 3 \leq |S| \leq 4, \\ \{B | B \supset X\} & \text{if } |S| = 2, \\ \{A\} & \text{if } |S| = 1, \\ \emptyset & \text{if } S = \emptyset. \end{cases}$$

As the reader may check, E is superadditive and maximal. We shall prove that E is stable, i.e. $\mathrm{C}(E, \succsim^N) \neq \emptyset$ for every $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$.

Let $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$ and let $u^i : A \to \boldsymbol{R}$ be a representation of $\succsim^i$ for each $i \in N$. Without loss of generality, $\min_{x \in A} u^i(x) = 0$ for all $i \in N$. We define now an NTU game (N, V) by

$$V(S) = \{(y^1, \ldots, y^n) \in \boldsymbol{R}^N | \exists B \in E(S) \text{ such that } \inf_{x \in B} u^i(x) \geq y^i,\ i \in S\}.$$

We shall prove that the core of V, $\mathrm{C}(N, V)$, is nonempty. Clearly, this will imply — that $\mathrm{C}(E, \succsim^N) \neq \emptyset$ and complete the proof of the stability of E.

We now observe that if $3 \leq |S| \leq 5$, then $V(S)$ is a union of three corners, that is

$$V(S) = \bigcup_{j=1}^{3} \{y \in \boldsymbol{R}^N | y^S \leq c_j^S\},$$

where $c_j^S \in \boldsymbol{R}_+^S$, $j = 1, 2, 3$. We call the corner c_j^S *idle* if $\min_{i \in S} c_j^i = 0$. We replace each idle corner c_j^S, $3 \leq |S| \leq 5$, $1 \leq j \leq 3$, by 0^S and obtain a new game $(N, \hat{V})$. Observe that

$$\min_{1 \leq j \leq 3} u^i(x_j) = 0 \text{ for all } i \in N, \tag{8}$$

(u^i is quasiconcave for $i \in N$ and $\min_{x \in A} u^i(x) = 0$).

Hence, for $|S| = 3, 4$, $\hat{V}(S)$ is a union of at most two corners (where one of them is determined by 0^S). Similarly, if $|S| = 5$, then $\hat{V}(S)$ is the union of at most three corners, and one of them is determined by 0^S (in the foregoing discussion we did not distinguish between a corner $\{y \in \boldsymbol{R}^N | y^S \leq c^S\}$ and the vector c^S). We remark that $\mathrm{C}(N, V) = \mathrm{C}(N, \hat{V})$ ($\hat{V}(S) = V(S)$ for $|S| \notin \{3, 4, 5\}$).

We shall prove that $\mathrm{C}(N, \hat{V}) \neq \emptyset$ by showing that $(N, \hat{V})$ is balanced. Thus, let $\mathcal{B} \subset 2^N$ be a balanced collection with balancing weights $(\lambda_S)_{S \in \mathcal{B}}$, and let

$$\bar{y} = (\bar{y}^1, \ldots, \bar{y}^n) \in \bigcap_{S \in \mathcal{B}} \hat{V}(S).$$

We must show that $\bar{y} \in \hat{V}(N)$ $(=V(N))$.

Denote

$$A^i = \{x \in A | u^i(x) \geq \bar{y}^i\}, i \in N.$$

Then A^i is a nonempty convex set for each $i \in N$. Also, if $\bar{y}^i \leq 0$, then $A^i = A$. We shall prove the following claim.

Lemma 4. *If $S \subset N$ and $|S| = 3$, then $\cap_{i \in S} A^i \neq \emptyset$.*

Let $S = \{i_1, i_2, i_3\} \subset N$ and let $S^+ = \{i \in S | \bar{y}^i > 0\}$. If there exists $S' \in \mathcal{B}$ such that $S' \supset S^+$, then Lemma 4 is true. Therefore, we shall assume in the sequel:

$$\text{There exists no } S' \in \mathcal{B} \text{ such that } S' \supset S^+. \tag{9}$$

The following result will be used in the proof of Lemma 4.

Lemma 5. *If the following condition is satisfied,*

$$[S' \cap \{i_1, i_2\} \neq \emptyset \text{ and } S' \in \mathcal{B}] \Rightarrow |S'| \geq 5, \tag{10}$$

then there exists $S' \in \mathcal{B}$ such that $i_1, i_2 \in S'$ and $|S'| = 5$.

Proof of Lemma 5. Assume, on the contrary, that there exists no $S' \in \mathcal{B}$ such that $i_1, i_2 \in S'$ and $|S'| = 5$. Let $T = N \setminus S$ and for each j, $j = 1, 2, 3$, let

$$a_j = \sum\{\lambda_{S'} | S' \in \mathcal{B},\ i_j \in S',\ T \subset S',\ \text{and } |S'| = 5\},$$
$$b_j = \sum\{\lambda_{S'} | S' \in \mathcal{B},\ i_j \in S',\ |T \cap S'| = 3,\ \text{and } |S'| = 5\}, \qquad c_j = \sum_{k \neq j} \lambda_{N\setminus\{i_k\}},$$

where, by convention, $\lambda_{N\setminus\{i_k\}} = 0$ if $N \setminus \{i_k\} \notin \mathcal{B}$. By Eqs. (9) and (10),

$$a_j + b_j + c_j = 1, \quad j = 1, 2. \tag{11}$$

Also, by (all) the foregoing assumptions, there exists $j \in T$ such that

$$1 = \sum\{\lambda_{S'} | j \in S' \text{ and } S' \in \mathcal{B}\} \geq a_1 + a_2 + a_3 + \tfrac{1}{2}(c_1 + c_2 + c_3) + \tfrac{3}{4}(b_1 + b_2) \tag{12}$$

We now consider the following possibilities:

(a) $a_1 = a_2 = b_1 = b_2 = c_3 = 0$. In this case $N \setminus \{i_3\}$ is the unique set of $\mathcal{B}$ which contains i_1 and i_2. Also, as $c_3 = 0$ and $\bar{y}^{i_3} > 0$, there exists $S' \in \mathcal{B}$ such that $i_3 \in S'$ and $3 \leq |S'| \leq 5$. As $S' \cap (N \setminus \{i_3\}) \neq \emptyset$, this contradicts the balancedness of $\mathcal{B}$. Thus, (a) is impossible.

(b) $a_1 + a_2 + b_1 + b_2 + c_3 > 0$. By Eqs. (11) and (12),

$$1 > \tfrac{1}{2}(a_1 + a_2) + \tfrac{1}{2}(c_1 + c_2) + \tfrac{1}{2}(b_1 + b_2) = 1.$$

Hence, (b) is also impossible, and the desired contradiction has been obtained. We conclude that there exists $S' \in \mathcal{B}$ such that $|S'| = 5$ and $i_1, i_2 \in S'$. □

We shall now prove Lemma 4.

Proof of Lemma 4. Let $\mathcal{T} = \{S' \in \mathcal{B} | |S'| \in \{3, 4\}\}$. We say that $i \in S$ is *covered by* $\mathcal{T}$ if there exists $S' \in \mathcal{T}$ such that $i \in S'$. We distinguish the following possibilities:

(a) Every $i \in S^+$ is covered by $\mathcal{T}$. Here, we further have to consider the following subcases:

(a.1) There exist two sets $S'_1, S'_2 \in \mathcal{T}$ such that $S'_1 \cup S'_2 \supset S^+$. We have for each j two extreme points $x_{j,1}, x_{j,2}$ such that

$$u^i(x_{j,h}) \geq \bar{y}^i \text{ for all } i \in S^+ \cap S'_j \text{ and } h = 1, 2.$$

Let $x \in \{x_{1,1}, x_{1,2}\} \cap \{x_{2,1}, x_{2,2}\}$. Then $u^i(x) \geq \bar{y}^i$ for all $i \in S$.

(a.2) There exist three sets S'_j in $\mathcal{T}$ such that $\{i_j\} = S^+ \cap S'_j$, $j = 1, 2, 3$. Again, for each j there exist two extreme points $x_{j,1}, x_{j,2}$, such that

$$u^i(x_{j,h}) \geq \bar{y}^i \text{ for all } i \in S'_j \text{ and } h = 1, 2.$$

Clearly, there exist $1 \leq j_1, j_2 \leq 3$, $j_1 \neq j_2$, such that $S'_{j_1} \cap S'_{j_2} \neq \emptyset$. Without loss of generality $j_1 = 1$ and $j_2 = 2$. Let $k \in S'_1 \cap S'_2$. Then

$$\min\{u^{i_1}(x_{1,1}), u^{i_1}(x_{1,2})\} \geq \bar{y}^{i_1} > 0$$

implies, by the construction of $\hat{V}$, that $\min\{u^k(x_{1,1}), u^k(x_{1,2})\} > 0$. Similarly, $\min\{u^k(x_{2,1}), u^k(x_{2,2})\} > 0$. Therefore, by (4), $\{x_{1,1}, x_{1,2}\} = \{x_{2,1}, x_{2,2}\}$. As $\{x_{1,1}, x_{1,2}\} \cap \{x_{3,1}, x_{3,2}\} \neq \emptyset$, we have that $\cap_{i \in S} A^i \neq \emptyset$.

(b) Only two members of S^+, say i_1 and i_2, are covered by $\mathcal{T}$: there exist $S'_1, S'_2 \in \mathcal{T}$ such that $i_j \in S^+ \cap S'_j$, $j = 1, 2$ ($S'_1 = S'_2$ is not excluded). Thus, $i_3 \in S^+ \cap S'$ for some $S' \in \mathcal{B}$ with $|S'| \geq 5$. We distinguish the following subcases:

(b.1) There exists $S' \in \mathcal{B}$ such that $i_3 \in S'$ and $|S'| = 5$. There is an extreme point x such that $u^{i_3}(x) \geq \bar{y}^{i_3} > 0$. Clearly, $S'_1 \cap S' \neq \emptyset$. Let $k \in S'_1 \cap S'$. We have $u^k(x) > 0$ by the construction of $\hat{V}$. By Eq. (8) and the definition of $\hat{V}$, $u^{i_1}(x) \geq \bar{y}^{i_1}$. Similarly, $u^{i_2}(x) \geq \bar{y}^{i_2}$. Thus, $x \in \cap_{x \in S} A^i$.

(b.2) $i_3 \in S' \in \mathcal{B} \Rightarrow |S'| = 6$. By Eq. (5), i_3 is contained in at most two members S' of $\mathcal{B}$ with $|S'| = 6$. Let these coalitions be T_1 and T_2 (where $T_1 = T_2$ is possible). As $|T_1 \cap T_2| \geq 5$, $T_1 \cap T_2 \cap S'_2 \neq \emptyset$ contradicting our assumption that $\mathcal{B}$ is balanced (we assume $\lambda_T > 0$ for $T \in \mathcal{B}$). Thus (b.2) is impossible.

(c) Only one member of S^+, say i_1, is covered by $\mathcal{T}$. By Lemma 5 there exists $S' \in \mathcal{B}$ such that $i_2, i_3 \in S'$ and $|S'| = 5$. Without loss of generality $i_2 \in S^+$. There is an extreme point x such that $u^{i_2}(x) \geq \bar{y}^{i_2} > 0$, and $u^{i_3}(x) \geq \bar{y}^{i_3}$. Let $i_1 \in S'_1 \in \mathcal{T}$. Then $S' \cap S'_1 \neq \emptyset$. Let $k \in S' \cap S'_1$. Then $u^k(x) > 0$ by the definition of $\hat{V}$. By Eq. (8) and the definition of $\hat{V}$, $u^{i_1}(x) \geq \bar{y}^{i_1} > 0$. Thus, $x \in \cap_{i \in S} A^i$.

(d) No member of S^+ is covered by $\mathcal{T}$. By Lemma 5 and Eq. (9) there exist three coalitions $S'_j \in \mathcal{B}$ such that $|S'_j| = 5$ and $S'_j \supset S^+ \setminus \{i_j\}$ for $j = 1, 2, 3$. For each j there exists an extreme point z_j such that

$$u^i(z_j) \geq \bar{y}^i > 0 \text{ for } i \in S^+ \setminus \{i_j\}.$$

Let $k \in S'_1 \cap S'_2 \cap S'_3$. By the definition of $\hat{V}$, $u^k(z_j) > 0$ for $j = 1, 2, 3$. Hence, by Eq. (8), there exist $j_1, j_2 \in \{1, 2, 3\}$, $j_1 \neq j_2$, such that $z_{j_1} = z_{j_2}$. Clearly, $z_{j_1} \in \cap_{i \in S} A^i$. □

Now we can prove the following lemma.

Lemma 6. (N, V) *has a nonempty core.*

Proof. Using the previous notation it is sufficient to prove that $(N, \hat{V})$ is balanced. By Lemma 4 and Helly's Theorem (see Rockafellar, 1970, Theorem 21.6), $\cap_{i \in N} A^i \neq \emptyset$. Hence, $\bar{y} \in \hat{V}(N)$ $(=V(N))$ and $(N, \hat{V})$ is balanced. By Scarf (1967), $\mathrm{C}(N, \hat{V}) \neq \emptyset$. Finally, by the definition of $\hat{V}$, $\mathrm{C}(N, \hat{V}) = \mathrm{C}(N, V)$. □

Theorem 6. *The game* (7,6) *has a strong representation on A.*

Proof. Using the previous notations we have that the EF E is superadditive, maximal, and stable. By Theorem 8 in the Appendix there exists a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^7; \pi; A)$ that implements the core of E. As the reader may check, if Γ is constructed according to the proof of Theorem 8, then Γ is a strong representation of (7, 6). □

The following generalization of Theorem 6 is true:

Theorem 7. *Let $X = \{x_1, x_2, x_3\}$ be a set of three affinely independent points in $\boldsymbol{R}^2$, let $A = \mathrm{conv}(X)$, and let $G = (n, n-1)$, $n \geq 7$. Then G has a strong representation on A.*

Proof. Let $N = \{1, \ldots, n\}$ be the set of players of G. Further, let

$$n_1 = \left[\frac{n}{3}\right] + 1, n_2 = \left[\frac{2n}{3}\right] + 1,$$

and let $N_1 = \{k \in \boldsymbol{N} | 2 \leq k < n_1\}$, $N_2 = \{k \in \boldsymbol{N} | n_1 \leq k < n_2\}$, and $N_3 = \{k \in \boldsymbol{N} | n_2 \leq k < n-1\}$. Define an EF $E : P(N) \to P(P(A))$ by the following rules. for $S \subset N$, let

$$E(S) = \begin{cases} 2^A & \text{if } |S| \geq n-1, \\ \{B || B \cap X| \geq 1\} & \text{if } |S| \in N_3, \\ \{B || B \cap X| \geq 2\} & \text{if } |S| \in N_2, \\ \{B | B \supset X\} & \text{if } |S| \in N_1, \\ \{A\} & \text{if } |S| = 1, \\ \emptyset & \text{if } S = \emptyset. \end{cases}$$

As the reader may check, E is superadditive and maximal. For the rest of the proof of this theorem, the reader should precisely follow the steps of the proof of Theorem 6 (with obvious changes in the notation). We have checked that the details remain valid. □

8. The case *m*=1

The last class of choice problems to be considered in this paper is that consisting of choice problems (G, A) with A a convex and compact subset of one-dimensional Euclidean space, that is an interval on the real line, where the strong representation problem always has a solution.

Let $A = [\underline{a}, \bar{a}]$, $\underline{a} < \bar{a}$, be an interval, and let $G = (N, W)$ be a proper and monotonic simple game. For the sake of completeness we shall indicate how to implement the core $\mathrm{C}(G; A; \succsim^N) = \mathrm{C}(G, \succsim^N)$ when $\succsim^N \in \mathcal{P}^N_{\mathrm{cc}}$. If G is not strong, then the core $\mathrm{C}(G, \cdot)$ is a set-valued function. Therefore, we need to construct a special GF, albeit simple, to implement the core.

Theorem 8. *Let $A = [\underline{a}, \bar{a}]$, where $\underline{a} < \bar{a}$, and let $G = (N, W)$ be a proper and monotonic simple game. Then there exists a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ that partially implements the core $\mathrm{C}(G, \cdot)$ and is also a representation of G.*

Proof. Let $\Sigma^i = A$ for every $i \in N$, and let

$$\pi(x^1, \ldots, x^n) = \min\{y \in A | \{i \in N | x^i \leq y\} \in W\}$$

for all $(x^1, \ldots, x^n) \in A^N$. π is the well-known committee rule (for the committee G). The reader is referred to Barbera et al. (1994) for a study of committee rules. Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$. We first check that $G^{\Gamma}_{\alpha} = G$. Clearly, if $S \in W$, then $E^{\Gamma}_{\alpha}(S) = 2^A$.

Now, let $S \in 2^N \setminus W$, and let $\underline{a} \le x < \bar{a}$. Then S is not α-effective for x. Thus, $G_\alpha^\Gamma = G$, and for every $\succsim^N \in \mathcal{P}_{\rm cc}^N$

$$\pi(\mathrm{SNE}(\Gamma, \succsim^N)) \subset \mathrm{C}(E_\beta^\Gamma, \succsim^N) \subset \mathrm{C}(E_\alpha^\Gamma, \succsim^N) \subset \mathrm{C}(G_\alpha^\Gamma, \succsim^N) = \mathrm{C}(G, \succsim^N).$$

It remains to prove that Γ is SNE-consistent. Let $\succsim^N \in \mathcal{P}_{\rm cc}^N$. Define the "right peak" of i by

$$p_R^i = \max\{y \in A | y \succsim^i x \text{ for all } x \in A\},$$

and let

$$p_R = \min\{p_R^i | \{j | p_R^j \le p_R^i\} \in W\}.$$

As the reader may verify, $p_R \in \mathrm{C}(G, \succsim^N)$. Let $\sigma = (p_R, \dots, p_R)$. We claim that σ is an SNE of $(\Gamma, \succsim^N)$. Indeed, no $S \in W$ has a profitable deviation from σ, because $p_R \in \mathrm{C}(G, \succsim^N)$. Now, let $x \in A$, $x \ne p_R$. If $x > p_R$, then $\{i | x \succ^i p_R\} = S$ is losing (i.e. $N \setminus S \in W$). Therefore, $\pi(\mu^S, \sigma^{N \setminus S}) = p_R$ for every $\mu^S \in \Sigma^S$. Finally, if $x < p_R$ then $\{i | x \succ^i p_R\} = S$ is not winning. Therefore, $\pi(\mu^S, \sigma^{N \setminus S}) \ge p_R$ for every $\mu^S \in \Sigma^S$. Thus, σ is an SNE of $(\Gamma, \succsim^N)$. □

9. Concluding remarks

In this section, we first summarize the main results of our work, and then comment on possible future continuations. We start with a formulation of a result which is implied by Zhou (1991). Let A be a convex and compact subset of $\boldsymbol{R}^m$, $m \ge 2$, let $\mathrm{aff}(A) = \boldsymbol{R}^m$, and let $G = (N, W)$ be a non-dictatorial committee (that is $\{i\} \notin W$ for all $i \in N$, and $N \in W$; thus, in particular, $n \ge 2$). Further, let $f : \mathcal{P}_{\rm cc}^N \to A$ be a social choice function, and let $G_\alpha^f = G$ (notice that G_α^f is well defined because f is a GF). Then f is manipulable (the reader may notice that every representation of G is surjective, because $N \in W$). Thus, the choice problem (G, A) has no strategy-proof representations. However, we have shown that when (G, A) is stable (that is the core $\mathrm{C}(G, A, \succsim^N) \ne \emptyset$ for all profiles $\succsim^N$ of continuous and convex preferences), it may be possible to (partially) implement the core of G, and thereby obtain a strongly stable representation of G on A.

We have obtained the following results on the existence of strongly stable representations of committees in economic environments. Let $G = (N, W)$ be a committee with $N \in W$, and let A be a closed convex subset of $\boldsymbol{R}^m$, $m \ge 2$, with $\mathrm{aff}(A) = \boldsymbol{R}^m$.

1. If G contains a vetoer and A is (in addition) compact, then G has a strong representation on A.
2. If $A = \boldsymbol{R}^m$, preferences are bounded, and $\nu(G) - 2 \ge m$, then strong representations exist.
3. If A is compact, smooth, and strictly convex, and G has no vetoers, then strong representations do not exist.
4. If A is (in addition) compact and $\nu(G) \le 6$, then there are no strong representations.
5. Every symmetric game $(n, n-1)$, $n \ge 7$, has a strong representation on the standard simplex in $\boldsymbol{R}^3$.

We remark that, in our view, (i) and (ii) are important and useful results. Clearly, (iii) and (iv) impose restrictions on our method. However, (v) shows that if the Nakamura number of a committee exceeds 6 and the set of alternatives is not strictly convex, then strong representations may exist (even when there are no vetoers). The general problem of existence of strong representations of committees (without vetoers) on polyhedral sets of alternatives is left as a subject for future research.

We conclude with some comments on the complexity of our GF's. Let (G, A) be a stable choice problem. We construct a strong representation of G on A by the following two-stage procedure. (I) We first check whether G can be extended to a superadditive, maximal, and stable EF E; and (II) if an extension E can be found, then we implement the core of E in SNE's, and thereby obtain the desired representation. We now remark that in the results (1), (2), and (5) mentioned above, the *definition* of the extending EF was simple and intuitive. Furthermore, in Theorem 8, which was applied at stage (II), the definition of the implementing GF is relatively simple. Thus, in all the above three cases, our GF's are explicitly defined in an uncomplicated and intuitively acceptable way.

Appendix A. Implementing the core of an EF on a restricted domain

In this appendix, we state and prove a general result on implementation of effectivity functions when there are restrictions on the set of admissible preferences.

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of voters and let A be a set of alternatives (with at least two members). Further, let $\mathcal{A}$ be a feasible sets structure on A, that is $\mathcal{A} \subset P(A)$, and $\emptyset$, $A \in \mathcal{A}$. We denote $\mathcal{A}^0 = \mathcal{A} \setminus \emptyset$. Also, we assume that $\{a\} \in \mathcal{A}$ for all $a \in A$. Let $\mathcal{P}$ be a set of complete and transitive binary relations on A. $\mathcal{P}$ specifies our restrictions on the preferences of the players. Finally, let $E : P(N) \to P(\mathcal{A}^0)$ be an EF.

In order to formulate our result we need the following notation. Let $S \subset N$, $S \neq \emptyset$, let $\succsim^N \in \mathcal{P}^N$, and let $a \in A$. We denote

$$\Pr(S, \succsim^N, a) = \{y \in A | y \succ^i a \text{ for all } i \in S\}.$$

Now we can formulate our result.

Theorem 9. *Assume that the following conditions are satisfied: (S) E is superadditive, (CC) the pair $(E, \mathcal{P})$ satisfies the complementarity condition. If $S \subset N$, $S \neq \emptyset, N$, and $\succsim^N \in \mathcal{P}^N$, then for each $x \in A$,*

$$\Pr(S, \succsim^N, x) \notin E(S) \Rightarrow A \setminus \Pr(S, \succsim^N, x) \in E(N \setminus S).$$

Then there exists a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ with the following properties:

1. *Γ implements* $\mathrm{C}(E, \cdot)$ *in SNE's on $\mathcal{P}^N$, that is*

$$\pi(\mathrm{SNE}(\Gamma, \succsim^N)) = \mathrm{C}(E, \succsim^N)$$

for every $\succsim^N \in \mathcal{P}^N$ (thus, in particular, if E is stable, then Γ is SNE-consistent),

2. *$E^{\Gamma}_{\alpha}(S) \supset E(S)$ for all $S \in P(N)$.*

Proof. For $i \in N$ let

$$\tau^i = \{S \subset N | i \in S\}$$

and define

$$\Sigma^i = \{\sigma^i : \tau^i \to \mathcal{A} | \sigma^i(S) \in E(S) \text{ for all } S \in \tau^i\}.$$

To define the outcome function π of Γ we need the following notation. Let $\sigma = (\sigma^1, \dots, \sigma^n) \in \Sigma^1 \times \cdots \times \Sigma^n$ and let $T \subset N$, $T \neq \emptyset$. Then $i, j \in T$ are σ-equivalent, written $i \sim j$, if $\sigma^i(T) = \sigma^j(T)$. We denote by $Q(T, \sigma)$ the set of equivalence classes of $\sim$.

Now let $\sigma = (\sigma^1, \dots, \sigma^n) \in \Sigma^1 \times \cdots \times \Sigma^n$. We define inductively a sequence of partitions of N. Let $P_0(\sigma) = \{N\}$. If $P_k(\sigma) = \{T_1, \dots, T_{r_k}\}$, $k \geq 0$, then $P_{k+1}(\sigma) = Q(T_1, \sigma) \cup \cdots \cup Q(T_{r_k}, \sigma)$. Because N is finite there exists a natural number h such that $P_h(\sigma) = P_{h+1}(\sigma)$. $P_h(\sigma) = \{T_1, \dots, T_{r_h}\}$ is called the partition associated with σ. Now we choose a selection ρ from $\mathcal{A}^0$, that is $\rho : \mathcal{A}^0 \to A$ and $\rho(B) \in B$ for all $B \in \mathcal{A}^0$. Because $P_h(\sigma) = P_{h+1}(\sigma)$, $\sigma^i(T_j) = \sigma^l(T_j)$ for all $i, l \in T_j$, $j = 1, \dots, r_h$. Hence, we may denote $\sigma(T_j) = \sigma^i(T_j)$, where $i \in T_j$, $j = 1, \dots, r_h$. With these notations define the outcome function π by

$$\pi(\sigma) = \rho\left(\cap_{j=1}^{r_h} \sigma(T_j)\right).$$

$\sigma(T_j) \in E(T_j)$ for all j. As E is superadditive, $\cap_{j=1}^{r_h} \sigma(T_j) \neq \emptyset$, and π is well-defined. This completes the definition of Γ.

Let now $\succsim^N \in \mathcal{P}^N$ and $a \in \mathrm{C}(E, \succsim^N)$. If $S \subset N$, $S \neq \emptyset, N$, then $\Pr(S, \succsim^N, a) \notin E(S)$. Hence, by (CC), $A \setminus \Pr(S, \succsim^N, a) \in E(N \setminus S)$. Define an n-tuple of strategies $\sigma = (\sigma^1, \dots, \sigma^n)$ by the following rules: $\sigma^i(N) = \{a\}$ for all $i \in N$; and $\sigma^i(S) = A \setminus \Pr(N \setminus S, \succsim^N, a)$ if $i \in S$ and $S \neq N$. Clearly, $\pi(\sigma) = a$. We claim that σ is an SNE of $(\Gamma, \succsim^N)$. As $\{b\} \in E(N)$ for all $b \in A$, and $a \in \mathrm{C}(E, \succsim^N)$, a is Pareto-optimal, and N cannot improve upon a. Suppose now that $\mu^T \in \Sigma^T$ is a deviation of a coalition $T \subset N$, $T \neq \emptyset, N$, from σ. Let $\{T_1, \dots, T_r\}$ be the partition associated with $(\sigma^{N \setminus T}, \mu^T)$. By the definition of σ, there exists $1 \leq j \leq r$ such that $N \setminus T \subset T_j$. Without loss of generality, $N \setminus T \subset T_1$. Again, by the definition of σ, $(\sigma^{N \setminus S}, \mu^T)(T_1) = A \setminus \Pr(N \setminus T_1, \succsim^N, a)$. Now $N \setminus T_1 \subset T$; hence $\Pr(T, \succsim^N, a) \subset \Pr(N \setminus T_1, \succsim^N, a)$. Thus,

$$A \setminus \Pr(N \setminus T_1, \succsim^N, a) \subset A \setminus \Pr(T, \succsim^N, a).$$

By the definitions of π and σ,

$$\pi(\sigma^{N \setminus T}, \mu^T) \in A \setminus \Pr(N \setminus T_1, \succsim^N, a).$$

Thus, $\pi(\sigma^{N \setminus T}, \mu^T) \in A \setminus \Pr(T, \succsim^N, a)$, and σ is an SNE.

Clearly, by the definition of Γ, $E_\alpha^\Gamma(S) \supset E(S)$ for all $S \in P(N)$ (notice that E is monotonic with respect to the players). Hence,

$$\pi(\mathrm{SNE}(\Gamma, \succsim^N)) \subset \mathrm{C}(E_\beta(\Gamma), \succsim^N) \subset \mathrm{C}(E_\alpha(\Gamma), \succsim^N) \subset \mathrm{C}(E, \succsim^N).$$

Thus, $\pi(\mathrm{SNE}(\Gamma, \succsim^N)) = \mathrm{C}(E, \succsim^N)$ for all $\succsim^N \in \mathcal{P}^N$. □

References

Abdou, J., Keiding, H., 1991. Effectivity Functions in Social Choice. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Arrow, K.J., 1951. Social Choice and Individual Values. Wiley, New York.

Barberà, S., Gul, F., Stachetti, E., 1994. Generalized median voter schemes and committees. Journal of Economic Theory 61, 262–289.

Barberà, S., Peleg, B., 1990. Strategy-proof voting schemes with continuous preferences. Social Choice and Welfare 7, 31–38.

Black, D., 1948. On the rationale of group decision making. Journal of Political Economy 56, 23–34.

Dutta, B., Pattanaik, P.K., 1978. On nicely consistent voting systems. Econometrica 46, 163–170.

Enelow, J.M., Hinich, M.J., 1984. The Spatial Theory of Voting. Cambridge University Press, Cambridge.

Gibbard, A., 1973. Manipulation of voting schemes: a general result. Econometrica 41, 587–601.

Greenberg, J., 1979. Consistent majority rules over compact sets of alternatives. Econometrica 47, 627–636.

Holzman, R., 1986a. On strong represetnations of games by social choice functions. Journal of Mathematical Economics 15, 39–57.

Holzman, R., 1986b. The capacity of a committee. Mathematical Social Sciences 12, 139–157.

Ishikawa, S., Nakamura, K., 1980. Representations of characteristic function games by social choice functions. International Journal of Game Theory 9, 191–199.

Le Breton, M., 1987. On the core of voting games. Social Choice and Welfare 4, 295–305.

Maskin, E.S., 1985. The theory of implementation in Nash equilibrium: a survey. In: Hurwicz, L., Schmeidler, D., Sonnenschein, H. (Eds.), Social Goals and Social Organization: Essays in Memory of Elisha Pazner. Cambridge University Press, Cambridge, pp. 173–204.

Moulin, H., 1980. On strategy-proofness and single-peakedness. Public Choice 35, 437–455.

Moulin, H., Peleg, B., 1982. Cores of effectivity functions and implementation theory. Journal of Mathematical Economics 10, 115–145.

Nakamura, K., 1979. The vetoers in a simple game with ordinal preferences. International Journal of Game Theory 8, 55–61.

Peleg, B., 1978a. Consistent voting systems. Econometrica 46, 153–161.

Peleg, B., 1978b. Representations of simple games by social choice functions. International Journal of Game Theory 7, 81–94.

Peleg, B., 1984. Game Theoretic Analysis of Voting in Committees. Cambridge University Press, Cambridge.

Rockafellar, R.T., 1970. Convex Analysis. Princeton University Press, Princeton, NJ.

Satterthwaite, M.A., 1975. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10, 187–217.

Scarf, H., 1967. The core of an n-person game. Econometrica 35, 50–69.

Sprumont, Y., 1995. Strategy-proof collective choice in economic and political environments. Canadian Journal of Economics 28, 68–107.

Zhou, L., 1991. Impossibility of strategy-proof mechanisms in economies with pure public goods. Review of Economic Studies 58, 107–119.