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# Cost sharing: the nondifferentiable case☆

Ori Haimanko

*CORE, Universite Catholique de Louvain, Voie du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium*

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**Abstract**

We show existence and uniqueness of a cost allocation mechanism, satisfying standard axioms, on two classes of cost functions with major nondifferentiabilities. The first class consists of convex functions which exhibit nondecreasing marginal costs to scale, and the second of piecewise linear functions. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The allocation of a joint cost of producing a bundle of infinitely divisible consumption goods is a common practical problem with no obvious solution. Biller et al. (1978) applied the theory of nonatomic games, introduced by Aumann and Shapley (1974), to set equitable telephone billing rates which allocate the cost of service among users. Following this application Billera and Heath (1982) and Mirman and Tauman (1982) offered an axiomatic justification of the Aumann–Shapley (A–S) prices using economic terms only. It has been shown that the A–S prices are a natural extension of the average cost prices from a single product to an arbitrary finite number of products with nonseparable production cost functions (see Tauman (1988)).

The A–S prices satisfy four simple properties of the average cost prices — *cost sharing, additivity, rescaling invariance*, and *monotonicity* — and, in addition, another property, *consistency*, which ties pricing of multiple commodities with that of a single one. Furthermore,

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* *E-mail address:* haimanko@core.ucl.ac.be (O. Haimanko).


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the prices are *uniquely* determined by these five properties (axioms) on the class of differentiable cost functions with no fixed cost component, as was shown by Billera and Heath (1982) and Mirman and Tauman (1982). Other sets of axioms uniquely characterizing the A–S prices were also proposed in Young (1985), Hart and Mas-Colell (1989) and Monderer and Neyman (1988).

The A–S price mechanism, however, can only be applied when a cost function is differentiable, while typical cost functions possess major nondifferentiabilities. For instance, minimal factor cost functions, which compute the cost of producing a given bundle of output commodities by finding the least expensive configuration of input commodities, are generically non-differentiable. The natural question that arises is whether the A–S mechanism can be extended to classes non-differentiable cost functions, and still be uniquely characterized by the above five axioms.

In this paper we exhibit a price mechanism on the class of cost functions which are convex or piecewise linear, that satisfies the five axioms. These cost functions are important: all minimal factor cost functions are convex, and in many applications they are piecewise linear (see Samet et al. (1984)). Our main result is Theorem 1, which shows that this mechanism is the *only* one satisfying the axioms on the following two subclasses: (1) convex cost functions that exhibit non-decreasing marginal costs to scale; (2) piecewise linear cost functions.

Unlike most results in axiomatic cost sharing, our results follow only partially previous developments in the theory of values of nonatomic games. For instance, the uniqueness of the Aumann–Shapley value, shown by Aumann and Shapley (1974), was the inspiration behind the aforementioned uniqueness theorem of Billera and Heath (1982) and Mirman and Tauman (1982). Tauman (1988) gives a comprehensive account of other such instances. In the case of general convex cost problems, the extended A–S price mechanism is related to the Mertens (1988) value, defined on a space of nonatomic cooperative games containing all convex and concave functions of finitely many nonatomic measures on the space of players. For this reason we call our extension the Mertens mechanism. Uniqueness of the Mertens value is, however, unknown beyond the space of finite-type market games (as established in Haimanko (2000a)). Therefore, the uniqueness of Mertens mechanism has to be shown directly, albeit with some help drawn from Haimanko (2000a).

The paper is organized as follows. Section 2 provides the necessary definitions, notations and statement of the result. Its proof is brought in Section 3, technical parts of which are deferred to the Appendix A.

## 2. Definitions and statement of the result

In what follows $R^k$ is the $k$-dimensional Euclidean space, $R^k_+$ — a subset of all vectors with nonnegative coordinates, and $R^k_{++}$ — a subset of all vectors with strictly positive coordinates. A *cost problem* is a pair $(f, a)$, where, for some positive integer $k$, $a \in R^k_{++}$ and $f$ is a real valued function on $D_a = \{x \in R^k_+ | 0 \leq x \leq a\}$ with $f(0) = 0$. The vector $a$ is interpreted as quantities of commodities $1, \ldots, k$ that are produced, and $f$ as a cost function: for each $x \in D_a\, f(x)$ is viewed as the cost of producing the bundle $x = (x_1, \ldots, x_k)$ of commodities. Given a set (class) $F$ of cost problems we denote by $F_k$ the subset of problems $(f, a)$ for which $a \in R^k_{++}$.

A price mechanism on a class $F$ of cost problems is a function $\psi : F \to \cup_{k=1}^{\infty} R^k$ such that $\psi(f,a) \in R^k$ whenever $(f,a) \in F^k$. The $j$th coordinate of $\psi(f,a)$ is denoted by $\psi_j(f,a)$.

Before we introduce the axioms on $\psi$, several notations are needed. Given $x, y \in R^k$ we denote by $x \cdot y$ the internal product of $x$ and $y$, and $x * y$ is defined as $(x_1 y_1, \dots, x_k y_k)$; we also write $\alpha^{-1} = (1/\alpha_1, \dots, 1/\alpha_k)$ and $(\alpha * f)(x) = f(\alpha * x)$ for $\alpha \in R^k_{++}$ and a function $f$ on $R^k_+$. Given $m \geq k \geq 1$ and an ordered partition $\pi = (S_1, \dots, S_k)$ of $\{1, 2, \dots, m\}$, $\pi^* : R^m \to R^k$ is defined by $\pi^*(x)_i = \sum_{j \in S_i} x_j$ for every $x \in R^m$.

**Definition 1.** A price mechanism $\psi$ on $F$ is:

1. *cost sharing* if for every $k \geq 1$ and $(f,a) \in F^k$
$$a \cdot \psi(f,a) = f(a); \tag{2.1}$$
2. *additive* if for $(f_1, a), (f_2, a) \in F^k$ such that $(f_1 + f_2, a)$ is also in $F^k$
$$\psi(f_1 + f_2, a) = \psi(f_1, a) + \psi(f_2, a); \tag{2.2}$$
3. *consistent* if for every $m \geq k \geq 1$, $b \in R^m_{++}$, ordered partition $\pi = (S_1, \dots, S_k)$ of $\{1, 2, \dots, m\}$, cost function $f$ such that both $(f, \pi^*(b))$ and $(f \circ \pi^*, b)$ are in $F$, and $i, j$ with $j \in S_i$,
$$\psi_j(f \circ \pi^*, b) = \psi_i(f, \pi^*(b)); \tag{2.3}$$
4. *rescaling invariant* if for every $\alpha, a \in R^k_{++}$ and $f$ such that $(f,a), (\alpha * f, \alpha^{-1} * a) \in F^k$
$$\psi(\alpha * f, \alpha^{-1} * a) = \alpha * \psi(f,a); \tag{2.4}$$
5. *monotone* if for every $(f,a), (g,a) \in F^k$ are such that $(f-g)(x)$ is a nondecreasing function on $D_a$,
$$\psi_i(f,a) \geq \psi_i(g,a). \tag{2.5}$$

**Remark 1.** Given a permutation $\theta$ of $\{1, \dots, k\}$, denote $\theta^* x = (x_{\theta(1)}, \dots, x_{\theta(k)})$ for any $x \in R^k$. By taking $m = k$ in the definition of consistency one sees that any consistent price mechanism $\psi$ is *symmetric*, i.e. for any $(f,a) \in F^k$ and permutation $\theta$ of $\{1, \dots, k\}$
$$\psi_{\theta(j)}(f \circ \theta^*, b) = \psi_j(f, \theta^* b), \tag{2.6}$$
provided $(f \circ \theta^*, b)$ and $(f, \theta^* b)$ are in $F^k$.

**Remark 2.** Assume that whenever $(f,a) \in F^k$, the cost problem $(f_j^{\alpha}, (a, a_j))$, where
$$f_j^{\alpha}(x_1, \dots, x_k, x_{k+1}) = f(x_1, \dots, x_{j-1}, \alpha x_j + (1-\alpha) x_{k+1}, x_{j+1}, \dots, x_k), \tag{2.7}$$
$1 \leq j \leq k$ and $\alpha \in (0,1)$, is in $F^{k+1}$. Then any consistent and rescaling invariant price mechanism $\psi$ on $F$ also satisfies the *dummy* axiom — if $(f,a) \in F^k$ is such that $f(x)$ is constant in the $j$th coordinate, then
$$\psi_j(f,a) = 0. \tag{2.8}$$

Indeed, given such $(f, a)$, consider $(f_j^\alpha, (a, a_j))$ for $\alpha \neq (1/2)$. By symmetry,

$$\psi_j(f_j^\alpha, (a, a_j)) = \psi_{k+1}(f_j^\alpha, (a, a_j)). \tag{2.9}$$

On the other hand, by rescaling invariance and consistency,

$$\psi_j(f_j^\alpha, (a, a_j)) = \alpha\psi_j(f, a) \text{ and } \psi_{k+1}(f_j^\alpha, (a, a_j)) = (1-\alpha)\psi_j(f, a). \tag{2.10}$$

To reconcile (2.9) and (2.10), the equality (2.8) must hold.

**Remark 3.** If $F^k$ contains only nondecreasing cost functions, then additivity and monotonicity of $\psi$ imply that it is also 0-*homogeneous*: $\psi(f, a) = \psi(cf, a)$ for all $(f, a) \in F^k$ such that $(cf, a)$ is also in $F^k$ for any $c > 0$.

To define classes of cost problems that we will consider, some further notations are in order. Given $(f, a)$, a point $x$ in the interior of $D_a$, and $y \in R^k$, the directional derivative of $f$ at $x$ in the direction $y$ is

$$\mathrm{d}f(x, y) = \lim_{\varepsilon\downarrow 0}\frac{f(x+\varepsilon y) - f(x)}{\varepsilon}. \tag{2.11}$$

The limit exists for all convex functions $f$. For $k \geq 1$ denote by $F_+^k$ the set of all cost problems $(f, a)$ for which $a \in R_{++}^k$, $f$ is a convex and vanishing at 0 Lipschitz function with *nondecreasing* marginal costs to scale, i.e. for $0 < x < a$, $y \in R_+^k$ and $t \geq 1$ such that $tx < a$

$$\mathrm{d}f(x, y) \leq \mathrm{d}f(tx, y). \tag{2.12}$$

By $F_l^k$ we denote the set of cost problems $(f, a)$, $a \in R_{++}^k$, such that $f$ is nondecreasing, continuous, vanishing at 0, and piecewise linear (i.e. there is a partition of $R^k$ into finitely many regions of the form $\{x \in R^k | \forall l = 1, \ldots, n, x \cdot a_l \leq b_l\}$ for finite sequences $\{a_l\}_{l=1}^n \subset R^k$ and $\{b_l\}_{l=1}^n \subset R$, such that $f$ coincides with some affine function on the intersection of each region with $D_a$). Let $F_i = \cup_{k=1}^\infty F_i^k$ for $i \in \{+, l\}$.

We now proceed to show the existence of a mechanism on both $F_+$ and $F_l$ that is simultaneously cost sharing, additive, consistent, rescaling invariant and monotone. Given $(f, a) \in F_+^k$ and a point $x$ in the interior of $D_a$, the function $\mathrm{d}f(x, \cdot)$ is convex and finite on all $R^k$ (Theorem 23.1 of Rockafellar (1970)). By convexity, all its directional derivatives also exist and are finite. They will be denoted by

$$\mathrm{d}f(x, y, z) = \lim_{\varepsilon\downarrow 0}\frac{\mathrm{d}f(x, y+\varepsilon z) - \mathrm{d}f(x, y)}{\varepsilon}. \tag{2.13}$$

Directional derivatives $\mathrm{d}f(x, y)$ and $\mathrm{d}f(x, y, z)$ can be defined also for $x$ on the boundary of $D_a$, provided $x + \varepsilon y + \delta z \in D_a$ for all $\varepsilon, \delta > 0$ small enough.

The directional derivatives $\mathrm{d}f(x, y)$ and $\mathrm{d}f(x, y, z)$ also exist for all piecewise linear functions $f$, and in particular those in cost problems of the class $F_l$.

Consider now a price mechanism defined on $F_+^k$ and $F_l^k$ as follows. Let $X = (X_1, \ldots, X_k)$ be a vector of independent random variables, each one having the standard Cauchy distribution. Then define

$$(\varphi_{\mathrm{M}})_j(f,a) = E_X\left(\int_0^1 \mathrm{d}f(ta, a*X, e_j)\,\mathrm{d}t\right), \tag{2.14}$$

where $e_j$ is the $j$th unit vector, and $E_X$ stands for the expectation operator.

We call $\varphi_{\mathrm{M}}$ the Mertens mechanism, due to its close affinity to the Mertens value, defined in Mertens (1988). Indeed, consider a standard measurable space of players $I$, partitioned into a finite number of measurable uncountable (disjoint) sets, $I_1, \dots, I_k$ — the types of players. If $\mu_1, \dots, \mu_k$ are nonatomic probability measures, each supported on the corresponding type, and $f$ is a convex function on $[0,1]^k$, vanishing at 0, then $(\varphi_{\mathrm{M}})_j(f,(1,\dots,1))$ is precisely the Mertens value of the players of $j$th type, in the game $f(\mu_1, \dots, \mu_k)$.

The Mertens mechanism is clearly additive, rescaling invariant and monotone. Its cost sharing property is due to the efficiency axiom of the Mertens value. The consistency of $\varphi_{\mathrm{M}}$ follows from the fact that if $X = (X_1, \dots, X_k)$ are independent and have the standard Cauchy distribution, then, for each ordered partition $\pi = (S_1, \dots, S_k)$ of $\{1, 2, \dots, m\}$ and $a \in R^k_{++}$, the random variables $\pi^*(a*X)_1, \dots, \pi^*(a*X)_k$ are independent and distributed as $\pi^*(a)_1X_1, \dots, \pi^*(a)_kX_k$.

The following theorem states the reverse implication.

**Theorem 1.** *Let $\psi$ be a cost sharing, additive, consistent, rescaling invariant and monotone price mechanism on either $F_+$ or $F_l$. Then $\psi = \varphi_{\mathrm{M}}$.*

The proofs of the theorem for the two cases are rather similar. We give a complete proof for a price mechanism on $F_+$ in the next section, and the proof for $F_l$ is built upon the one for $F_+$ in Section A.4 of Appendix A.

## 3. Proof of Theorem 1 for $F_+$

For a positive integer $k \geq 2$, denote by $D^k$ the "diagonal" $\{x \in R^k | \forall i, j,\ x_i = x_j\}$, $\mathbf{1}_k = (1, \dots, 1) \in R^k$. Let $S^k_\perp$ be the "equator" of the unit sphere in $R^k$ perpendicular to $\mathbf{1}_k$, i.e. $\{x \in R^k | \|x\| = 1, x \cdot \mathbf{1}_k = 0\}$, where $\|\ \|$ stands for the Euclidean norm. Given $x \in R^k - D^k$, denote by $\Upsilon^k(x)$ the point of intersection of $S^k_\perp$ with the half-plane that contains the point $x$ and has $D^k$ as its boundary.

Endow the set $\Lambda^k = ([0,1]^k - D^k) \cup [0,1] \times S^k_\perp$ with topology defined by the following sequential requirements: let $(x_n)_{n=1}^\infty$ be a sequence in $\Lambda^k$. It converges to $x \in [0,1]^k - D^k$ only if it is in $[0,1]^k - D^k$ starting at some $n$, and it converges to $x$ in the Euclidean topology. It converges to $(t,y) \in [0,1] \times S^k_\perp$ if it can be partitioned into two subsequences such that:

1. the first one is of the form $(t_n, y_n) \subset [0,1] \times S^k_\perp$, either finite or converging to $(t,y)$ in the Euclidean topology;
2. the second one is of the form $(w_n) \subset ([0,1]^k - D^k)$, either finite or converging to $t\mathbf{1}_k$ in the Euclidean topology and $\Upsilon^k(w_n - t\mathbf{1}_k) \to_{n\to\infty} y$.

The set $\Lambda^k$ is metrizable and compact in this topology, being homeomorphic to the "tube" $[0,1] \times \{x \in R^k | 1 \leq \|x\| \leq 2, x \cdot \mathbf{1}_k = 0\}$.

Denote by $\overline{M}_+^k$ the space of all differences $f_1 - f_2$, where $f_1$, $f_2$ are convex nondecreasing Lipschitz functions on $[0,1]^k$ with $f_i(0) = 0$, both showing nondecreasing marginal costs to scale. Given $f \in \overline{M}_+^k$, we define a function $g(f)_i$ on $\Lambda^k$ as follows. First, let $g(f)_i|_{[0,1)^k - D^k}(x) = \mathrm{d}f(x, e_i)$, and $g(f)_i|_{(0,1)\times S_\perp^k}(t, y) = \mathrm{d}f(t\mathbf{1}_k, y, e_i)$. For $y \in S_\perp^k$ define $g(f)_i(0, y) = \mathrm{d}f(0, y + \mathbf{1}_k, e_i)$ and $g(f)_i(1, y) = \mathrm{d}f(\mathbf{1}_k, y - \mathbf{1}_k, e_i)$. If $x \in [0,1]^k - [0,1)^k$, define $g(f)_i(x) = -\mathrm{d}f(x, -e_i)$. Note that $g(f)_j$ can be defined in the same way for any function $f$ on $[0,1]^k$ for which all directional derivatives, and directional derivatives of directional derivatives, exist.

Let $\psi$ be a cost sharing, additive, consistent, rescaling invariant and monotone price mechanism on $F_+$. Observe that $\psi$ can be extended to the class of cost problems $(f, a)$, with $a * f \in \overline{M}_+^k$, by the formula

$$\psi(f, a) = \psi(f_1, a) - \psi(f_2, a), \tag{3.1}$$

where $(f_1, a), (f_2, a) \in F_+^k$ and $f = f_1 - f_2$. This extension is well-defined, as follows from the additivity of $\psi$. The extended $\psi$ is also cost sharing, additive, homogenous, consistent, rescaling invariant and monotone.

Denote by $B(\Lambda^k)$ the Banach space of bounded measurable functions on $\Lambda^k$, with the supremum norm, and by $C(\Lambda^k)$ the subspace of continuous functions. As any $f \in \overline{M}_+^k$ is completely determined by the knowledge of $g(f) = (g(f)_i)_{i=1}^k$, the function $\psi_j^k$ on a subspace $V = \{g(f) | f \in \overline{M}_+^k\}$ of $(B(\Lambda^k))^k$ is well-defined by the formula

$$\psi_j^k(g(f)) = \psi_j(f, \mathbf{1}_k). \tag{3.2}$$

It is, moreover, a linear functional, by additivity and homogeneity of $\psi$.

If the functions $g(f)$ are all nonnegative then $f$ is nondecreasing. By monotonicity of $\psi$, $\psi_j(f, \mathbf{1}_k)$ is nonnegative, and therefore $\psi_j^k$ is a positive functional. For any $c = (c_1, \ldots, c_k) \in R^k$

$$\psi_j^k(c) = c_j, \tag{3.3}$$

if $c$ is a $k$-tuple of constant functions, in which the $i$th function has the value $c_i$. Indeed, by linearity of $\psi_j^k$, it suffices to consider the case of $c = e_i$. The claim follows by taking $f(x) = x_i$, and applying the dummy and cost sharing axioms.

If $(B(\Lambda^k))^k$ is viewed as the Banach space of bounded measurable functions on the union of $k$ disjoint copies of $\Lambda^k$, $\psi_j^k$ is a positive projection on it. The functional $\psi_j^k$ can thus be extended to a positive functional on the entire $(B(\Lambda^k))^k$, by Kantorovitch theorem (Zaanen (1988), Theorem 83.15). It is a sum of $(\psi_j^k)_i$ — positive functionals on each coordinate of $(B(\Lambda^k))^k$, i.e. on the space $B(\Lambda^k)$. By the property expressed in (3.3), $(\psi_j^k)_i(c) = 0$ for any constant function $c$ and $i \neq j$, and this, together with positivity, implies $(\psi_j^k)_i = 0$ for $i \neq j$. Therefore, $\psi_j^k = (\psi_j^k)_j$ is a function of the $j$th coordinate alone.

From Riez representation theorem, applied to the restriction of $\psi_j^k$ to $C(\Lambda^k)$, there is a nonnegative measure $\lambda_j^k \in M(\Lambda^k)$ such that for every $g \in C(\Lambda^k)$

$$\psi_j^k(g) = \int g(x)\,\mathrm{d}\lambda_j^k(x). \tag{3.4}$$

As $\psi_j^k(e_j) = 1$, by (3.3), $\lambda_j^k$ is a probability measure.

**Lemma 1.** *For every $f$ such that $g(f)_j$ exists, and is continuous $\lambda_j^k$ — almost everywhere, and bounded,*

$$\psi_j^k(g(f)_j) = \int g(f)_j(x)\,\mathrm{d}\lambda_j^k(x). \tag{3.5}$$

**Proof.** The proof is a verbatim repetition of the proof of Lemma 3.5 in Haimanko (2000a). □

The following Lemma gives some further properties of the measure $\lambda_j^k$.

**Lemma 2.**

1. *The measure $\lambda_j^k$ is supported on the set $[0, 1] \times S_\perp^k$.*
2. *For any $t \in [0, 1]$,*

$$\lambda_j^k([0, t] \times S_\perp^k) = t.$$

3. *For any $t \in (0, 1]$ and Borel set $S \subset S_\perp^k$,*

$$\lambda_j^k([0, t] \times S) = \lambda_j^k([0, t] \times (-S)), \tag{3.6}$$

*where $-S = \{-x | x \in S\}$.*

**Proof.** See Section A.1 of the Appendix A. □

By part 1 of Lemma 2, $\lambda_j^k$ can be viewed as a measure on $[0, 1] \times S_\perp^k$. We denote by ${}^t\lambda_j^k$ the measure on $[0, t] \times S_\perp^k$ given by

$${}^t\lambda_j^k(S) = \lambda_j^k([0, t] \times S). \tag{3.7}$$

For $\alpha \in [0, 1]$, and $0 \leq j \leq k$, let $\Gamma_j^\alpha : S_\perp^{k+1} \to R^k$ be the map given for $x = (x_1, \ldots, x_{k+1})$ by $\Gamma_j^\alpha(x)_l = x_l$ for $l \neq j$, and $\Gamma_j^\alpha(x)_j = \alpha x_{k+1} + (1-\alpha)x_j$. Also denote $\Upsilon_j^\alpha = \Upsilon^k \circ \Gamma_j^\alpha$ whenever it is defined. The map $[0, 1] \times S_\perp^{k+1} \to [0, 1] \times S_\perp^k$, given by $(t, y) \mapsto (t, \Upsilon_j^\alpha(y))$, whenever defined, is also denoted by $\Upsilon_j^\alpha$.

**Lemma 3.** *If $0 \leq j \leq k$ then for almost every $\alpha \in (0, 1]$*

$$\lambda_j^k = \Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1}, \tag{3.8}$$

*i.e. for any Borel set* $K \subset [0,1] \times S_{\perp}^{k}$, $\lambda_j^k(K) = \lambda_{k+1}^{k+1}((\Upsilon_j^{\alpha})^{-1}(K))$. *Also*, (3.8) *holds for* $\alpha = 0$, *provided* $k \geq 3$.

**Proof.** See Section A.2 of the Appendix A. □

**Lemma 4.** *For every* $k \geq 3$, *the measure* $\lambda_j^k$ *is independent of j; it will be denoted by* $\lambda^k$ *from now on. Additionally, for every permutation* $\theta$ *of* $\{1, \ldots, k\}$,

$$\mathrm{d}\lambda^k(t, y) = \mathrm{d}\lambda^k(t, \theta^* y). \tag{3.9}$$

**Proof.** See Section A.2 of the Appendix A. □

Properties of the measures $\lambda^k$, stated in the above lemmas, determine these measures uniquely, as we show next. Let $(U_i)_{i=1}^{\infty}$ be a sequence of identically independently distributed (i.i.d.) random variables, each with a standard Cauchy distribution, and denote by $\kappa^k$ the probability measure on $S_{\perp}^k$ that describes the distribution of $\Upsilon^k(U_1, \ldots, U_k)$. Lemma 3.14 of Haimanko (2000a) provides sufficient and necessary conditions on a sequence $(\eta^k)_{k=3}^{\infty}$ of probability measures to be equal to $(\kappa^k)_{k=3}^{\infty}$ :

1. $\eta^k = \Upsilon_1^{\alpha} \circ \eta^{k+1}$ for every $k$ and almost every $\alpha$ in $[0, 1]$;
2. specifically, $\eta^k = \Upsilon_1^{0} \circ \eta^{k+1}$;
3. $\eta^k(S) = \eta^k(-S)$ for every $k$ and Borel subset $S$ of $S_{\perp}^k$;
4. $\mathrm{d}\eta^k(y) = \mathrm{d}\eta^k(\theta^* y)$ for every $k$ and permutation $\theta$ of $\{1, \ldots, k\}$.

By part 2 of Lemma 2, $({}^t\lambda^k/t)_{k=3}^{\infty}$ is a sequence of probability measures. By Lemma 3, part 3 of Lemma 2 and 4, for each $t \in (0, 1]$ the sequence of measures $({}^t\lambda^k/t)_{k=3}^{\infty}$ obeys (1)–(4) above, and so it is equal to $(\kappa^k)_{k=3}^{\infty}$. Therefore, $\lambda^k$ is a product measure — $\lambda^k = L \times \kappa^k$, where $L$ is the Lebesgue measure on $[0, 1]$. Note that now $\Upsilon_j^{\alpha} \circ \lambda^{k+1}$ is continuous in $\alpha$, and so (3.8) actually holds for *all* $\alpha$ and $k \geq 2$. Thus, in particular, $\lambda^2 = \Upsilon_1^{0} \circ (L \times \kappa^3) = L \times \kappa^2$.

**Proposition 1.** *Let* $f \in \overline{M}_{+}^{k}$.

1. *The functions* $\mathrm{d}f(\cdot \mathbf{1}_k, \cdot, z)$, $z \in R^k$, *are continuous at* $\lambda^k$ — *almost every* $(t, y) \in (0, 1) \times S_{\perp}^k$.
2. *For* $\lambda^k$ — *almost every* $(t, y) \in (0, 1) \times S_{\perp}^k$ *and every* $z \in R^k$, *if* $x_n \to t\mathbf{1}_k$ *and* $\Upsilon^k(x_n - t\mathbf{1}_k) \to y$, *then* $\mathrm{d}f(x_n, z) \to \mathrm{d}f(t\mathbf{1}_k, y, z)$.

**Proof.** See Section A.3 of the Appendix A. □

From our characterization of the measures $\lambda^k$ and Proposition 1, the function $g(f)_i$ is continuous at $\lambda^k$ — almost every point of $\Lambda^k$ for every $f \in \overline{M}_{+}^{k}$ and $1 \leq i \leq k$. Lemmas 1 and 2 now yield

$$\psi(f, \mathbf{1}_k) = \varphi_{\mathrm{M}}(f, \mathbf{1}_k) \tag{3.10}$$

for all $f \in \overline{M}^k_+$, $k \geq 2$, since we have shown that any two price mechanisms that obey the axioms induce the same measures $\lambda^k$. By the cost sharing axiom, (3.10) also holds for all $(f, 1) \in F^1_+$. Rescaling invariance of both $\psi$ and $\varphi_{\mathrm{M}}$ concludes the proof of Theorem 1 for the class $F_+$.

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## Appendix A

Before presenting the proofs, we introduce several notations and facts from the theory of convex functions. A vector $p \in R^k$ is called a *subgradient* of $f$ at an interior point $x \in [0, 1]^k$ if

$$f(y) - f(x) \geq p \cdot (y - x) \tag{A.1}$$

for all $y \in R^k$. The set of all such $p$ will be denoted by $\partial f(x)$. Directional derivatives $\mathrm{d}f(x, y)$ are given by

$$\mathrm{d}f(x, y) = \max_{p \in \partial f(x)} p \cdot y \tag{A.2}$$

(Theorem 23.4 of Rockafellar (1970)). The set of subgradients of the convex function $\mathrm{d}f(x, \cdot)$ at $y$ is

$$[\partial f(x)]_y = \{p' \in \partial f(x) | p' \cdot y = \max_{p \in \partial f(x)} p \cdot y = \mathrm{d}f(x, y)\}, \tag{A.3}$$

and

$$\mathrm{d}f(x, y, z) = \max_{p \in [\partial f(x)]_y} p \cdot z \tag{A.4}$$

(Theorem 23.4 and Corollary 23.5.3 of Rockafellar (1970)).

### A.1. *Proof of Lemma 2*

1. Observe first that any polynomial $p$ on $[0, 1]^k$ is in $\overline{M}^k_+$. Indeed, if $f$ is given by $f(x) = \sum_{j=1}^k (x_j + 1)^2 - k$, then $p + Kf$ is a nondecreasing convex function with increasing marginal costs to scale for large enough $K > 0$. Therefore, $p = (p + Kf) - Kf \in \overline{M}^k_+$.
   Theorem 1.1 of Mirman and Tauman (1982), or proof of the main theorem in Billera and Heath (1982), show that the axioms of cost sharing, additivity, rescaling invariance, consistency and monotonicity suffice to determine price mechanism on the class of cost problems with polynomial cost functions as the A–S mechanism. The restriction of $\psi$

to this class is therefore uniquely determined. By the definition of A–S mechanism, for any polynomial $p \in \overline{M}^k_+$

$$\psi^k_j(p, \mathbf{1}_k) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x_j} p(t\mathbf{1}_k)\,\mathrm{d}t. \tag{A.5}$$

Let $x \notin D^k$. For any $\varepsilon > 0$ there is an open neighborhood $O$ of $x$, disjoint from $D^k$, and a nondecreasing polynomial $p \in \overline{M}^k_+$ such that $(\mathrm{d}/\mathrm{d}x_j)p|_O > (1/2)$ and $(\mathrm{d}/\mathrm{d}x_j)p|_{D^k} < \varepsilon$. This fact, together with (A.5), (3.5) and the positivity of $\lambda^k_j$ imply that $\lambda^k_j(O) = 0$, and so $\lambda^k_j$ is supported on $[0, 1] \times S^k_\perp$.

2. It follows from (A.5), (3.5), and the density of functions $(\mathrm{d}/\mathrm{d}x_j)p(t\mathbf{1}_k)$ for polynomials $p \in \overline{M}^k_+$ in $C([0, 1])$.
3. Let $c^1, c^2 \in R^k_+$ with $c^i \cdot \mathbf{1}_k = 1$, $t \in (0, 1]$, and $\pi = (S_+, S_-)$ — a partition of $\{1, \ldots, k\}$ such that for $i \in S_+ a_i = c^1_i - c^2_i$ is nonnegative, and for $i \in S_- a_i = c^2_i - c^1_i$ is positive. Let $f \in \overline{M}^k_+$ be the sum $f_1 + f_2$, where $f_1(x) = \min((1/2)(c^1 + c^2 - a) \cdot x, (1 - (1/2)\sum_{i=1}^k a_i)t)$, and $f_2(x) = g(\pi^*(a*x))$, for $g(x_1, x_2) = \min(x_1, x_2, (\sum_{i=1}^k a_i)(1/2))$. By formula (3.5) and part 1 of this lemma, $\psi_j(f_1, \mathbf{1}_k) = (1/2)(c^1_j + c^2_j - a_j)$. By efficiency, consistency and rescaling invariance of $\psi$, $\psi_j(f_2, \mathbf{1}_k) = (t/2)a_j$. Therefore

$$\psi(f, \mathbf{1}_k) = \tfrac{1}{2}t(c^1 + c^2). \tag{A.6}$$

Denote $w = c^1 - c^2$. Observe that

$$\mathrm{d}f(t_0\mathbf{1}_k, y, e_j) = \begin{cases} c^1_j, & \text{if } w \cdot y > 0; \\ c^2_j, & \text{if } w \cdot y < 0. \end{cases} \tag{A.7}$$

if $t_0 < t$, and

$$\mathrm{d}f(t_0\mathbf{1}_k, y, e_j) = 0 \tag{A.8}$$

if $t_0 > t$. Lemma 1 and (A.6) thus implies that

$${}^t\lambda^k_j(\{y | w \cdot y \geq 0\}) = {}^t\lambda^k_j(\{y | w \cdot y \leq 0\}) = \tfrac{1}{2} \tag{A.9}$$

provided

$${}^t\lambda^k_j(\{y | w \cdot y = 0\}) = 0. \tag{A.10}$$

Since (A.10) holds for almost every $w$, (A.9) also holds almost everywhere. It is well-known that (3.6) follows from this fact (e.g. Lemma 2.3 in Rubin (1999)).

### A.2. Proofs of Lemmas 3 and 4

The proofs necessitate some preparations. Denote by $\Delta^k$ the $(k-1)$-dimensional simplex in $R^k$, i.e. the set $\{x \in R^k_+ | x \cdot \mathbf{1}_k = 1\}$. Given a convex and compact subset $C$ of $\Delta^k$ and $t \in [0, 1]$, denote by $f_C$ the function on $R^k_+$ defined by $f_C(x) = \min_{c \in C} x \cdot c$, and by $h_{C,t}$ the function $h_{C,t}(x) = \min(f_C(x), f_C(t\mathbf{1}_k))$. Let $M^k$ be the linear subspace of $\overline{M}^k_+$ spanned by all functions $h_{C,t}$ with a compact $C \subset \Delta^k$ which is *strictly* convex (i.e. the extreme points of $C$ coincide with its relative boundary in $\Delta^k$) and $t \in [0, 1]$, and denote by $HM^k$ the subspace of $M^k$ spanned by functions $f_C$ for $C$ as above.

**Lemma 5.** *$\lambda^k_j$ is the unique measure such that* (3.5) *holds for all $f \in M^k$.*

**Proof.** For a compact and strictly convex $C \subset \Delta^k$ the function $f_C$ is continuously differentiable on $R^k - D^k$. The function $\mathrm{d}f_C(\cdot\mathbf{1}_k, \cdot, e_j)$ is 0-homogeneous in $t$ and continuous at all $y \in S^k_\perp$ (it follows from the fact that $\mathrm{d}f_C(t\mathbf{1}_k, y, e_j) = \mathrm{d}f_C(y, e_j)$). As

$$\mathrm{d}h_{C,t}(t\mathbf{1}_k, y, e_j) = \begin{cases} \partial f_C(t'\mathbf{1}_k, y, e_j), & \text{if } t' \in [0, t]; \\ 0, & \text{otherwise,} \end{cases} \tag{A.11}$$

discontinuities of $\mathrm{d}h_{C,t}(\cdot\mathbf{1}_k, \cdot, e_j)$ are all confined to the set $\{t\} \times S^k_\perp$. Therefore, (3.5) holds for all $h_{C,t}$, for compact and strictly convex $C \subset \Delta^k$ and $t \in [0, 1]$ (and thus the entire space $M^k$), by part 2 of Lemma 2 and Lemma 1.

To show uniqueness of $\lambda^k_j$, note that it suffices to do so for every ${}^t\lambda^k_j$, $t \in (0, 1]$, defined in (3.7). By (A.11) and (3.4),

$$\int \mathrm{d}f_C(t\mathbf{1}_k, y, e_j)\, \mathrm{d}({}^t\lambda^k_j)(y) = \psi^k_j(g(h_{C,t})). \tag{A.12}$$

It was shown in Proposition 3.8 of Haimanko (2000a) that the space $\{\mathrm{d}f(t\mathbf{1}_k, \cdot, e_j) | f \in HM^k\}$ is dense in $C(S^k_\perp)$, and therefore the measure ${}^t\lambda^k_j$ is determined uniquely by the knowledge of $\psi^k_j$ on $\mathrm{d}f(\cdot\mathbf{1}_k, \cdot, e_j)$, or $\psi$ on all $(f, \mathbf{1}_k)$, $f \in M^k$. □

We are now in a position to prove Lemmas 3 and 4.

**Proof of Lemma 3.** Observe that for almost every $\alpha \in (0, 1]$

$${}^1\lambda^{k+1}_{k+1}(\{y \in S^{k+1}_\perp | \Gamma^\alpha_j(y) \in D^k\}) = 0, \tag{A.13}$$

since the sets $(\Gamma^\alpha_j)^{-1}(D^k)$ are disjoint for different $\alpha$. For $f \in M^k$, let $f^\alpha_j \in \overline{M}^{k+1}_+$ be given by $f^\alpha_j(x) = f(\Gamma^\alpha_j(x))$. The set $[0, 1] \times \{y | \Gamma^\alpha_j(y) \in D^k\} \cup N \times S^{k+1}_\perp \subset [0, 1] \times S^{k+1}_\perp$ contains all possible discontinuity points of $\mathrm{d}f^\alpha_j(\cdot\mathbf{1}_{k+1}, \cdot, e_{k+1})$, for some finite subset $N$ of $[0, 1]$. For $\alpha$ that satisfies (A.3) this set has $\lambda^{k+1}_{k+1}$-measure 0, as follows from the definition of ${}^1\lambda^{k+1}_{k+1}$ and parts 1 and 2 of Lemma 2.

Given $f \in M^k$ and $\alpha > 0$ that satisfies (A.13), it follows from the consistency and rescaling invariance of $\psi$ that

$$\psi^k_j(g(f)_j) = \frac{1}{\alpha}\psi^{k+1}_{k+1}(g(f^\alpha_j)_{k+1}). \tag{A.14}$$

Thus, by applying Lemma 1 to $f_j^\alpha$,

$$\psi_j^k(g(f)_j) = \frac{1}{\alpha}\int \mathrm{d}f_j^\alpha(t\mathbf{1}_{k+1}, (y_1, \ldots, y_{k+1}), e_{k+1})\,\mathrm{d}\lambda_{k+1}^{k+1}(t, (y_1, \ldots, y_{k+1})) \tag{A.15}$$

$$= \int \mathrm{d}f(t\mathbf{1}_k, \Gamma_j^\alpha(y_1, \ldots, y_{k+1}), e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, (y_1, \ldots, y_{k+1})). \tag{A.16}$$

Using Remark 4 (to be shown in the proof of Proposition 1), the last integral is

$$\int \mathrm{d}f(t\mathbf{1}_k, \Upsilon_j^\alpha(y_1, \ldots, y_{k+1}), e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, (y_1, \ldots, y_{k+1})) \tag{A.17}$$

$$= \int \mathrm{d}f(t\mathbf{1}_k, (y_1, \ldots, y_k), e_j)\,\mathrm{d}(\Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1})(t, (y_1, \ldots, y_k)). \tag{A.18}$$

On the other hand, by Lemma 5, if

$$\psi_j^k(f, \mathbf{1}_k) = \int \mathrm{d}f(t\mathbf{1}_k, y, e_j)\,\mathrm{d}\lambda(t, y) \tag{A.19}$$

holds for all $f \in M^k$ for a probability measure $\lambda$, then $\lambda = \lambda_j^k$. Therefore, $\Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1} = \lambda_j^k$, and hence (3.8) holds for almost every $\alpha \in (0, 1]$.

We show now that (3.8) holds for $\alpha = 0$, provided $k \geq 3$. By the first part of the lemma, there is a sequence $\alpha_n > 0$ satisfying (A.13), such that $\alpha_n \to 0$. Let $1 \leq j \leq k$ and $f \in M^k$. By (A.15) and (A.16), for every $n$

$$\psi_j(f, \mathbf{1}_k) = \lim_{n\to\infty}\int \mathrm{d}f(t\mathbf{1}_{k+1}, \Gamma_j^{\alpha_n}(y_1, \ldots, y_{k+1}), e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, (y_1, \ldots, y_{k+1})), \tag{A.20}$$

and a limit as $n \to \infty$ of the right-hand side can be taken. Since for almost every $t$, $\mathrm{d}f(t\mathbf{1}_k, \cdot, e_j)$ is continuous on $R_+^k - D_k$, the bounded convergence theorem yields

$$\psi_j^k(f, \mathbf{1}_k) = \int_{(y_1, y_2, \ldots, y_k)\notin D^k} \mathrm{d}f(t\mathbf{1}_k, (y_1, \ldots, y_k), e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, (y_1, \ldots, y_{k+1})) \tag{A.21}$$

$$+\int_0^1 \mathrm{d}f(t\mathbf{1}_k, a_j^k, e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, \{a_{k+1}^{k+1}\}) + \int_0^1 \mathrm{d}f(t\mathbf{1}_k, -a_j^k, e_j)\,\mathrm{d}\lambda_{k+1}^{k+1}(t, \{-a_{k+1}^{k+1}\}) \tag{A.22}$$

where

$$a_j^k \in S_\perp^k,\ (a_j^k)_l = \begin{cases} \dfrac{k-1}{\sqrt{k^2-k}}, & l = j, \\ -\dfrac{1}{\sqrt{k^2-k}}, & \text{otherwise.} \end{cases}$$

Since $\psi$ is cost sharing, $\sum_{j=1}^{k}\psi_j^k(f,\mathbf{1}_k)=f(\mathbf{1}_k)$. If, in addition, $f\in HM^k$,

$$\sum_{j=1}^{k}\mathrm{d}f(t\mathbf{1}_k,y,e_j)=f(\mathbf{1}_k) \tag{A.23}$$

for all $t\in[0,1]$, $y\in R^k$. Therefore, summing the right-hand and the left-hand sides of the equality (A.22) over $j=1,\dots,k$, we obtain

$$f(\mathbf{1}_k)=f(\mathbf{1}_k)^1\lambda_{k+1}^{k+1}(\{y|(y_1,y_2,\dots,y_k)\notin D^k\}) \tag{A.24}$$

$$\sum_{j=1}^{k}[\mathrm{d}f(t\mathbf{1}_k,a_j^k,e_j)^1\lambda_{k+1}^{k+1}(\{a_{k+1}^{k+1}\})+\mathrm{d}f(t\mathbf{1}_k,-a_j^k,e_j)^1\lambda_{k+1}^{k+1}(\{-a_{k+1}^{k+1}\})]. \tag{A.25}$$

If $k\geq 3$ one can easily find closed and strictly convex $C_1,C_2\subset\Delta^k$ such that $\partial f_{C_1}(\epsilon a_j^k)=\partial f_{C_2}(\epsilon a_j^k)$ for $j\neq 1$ and $\epsilon\in\{1,-1\}$, $\partial f_{C_1}(-a_1^k)=\partial f_{C_2}(-a_1^k)$, and $\partial f_{C_1}(a_1^k)$ differs from $\partial f_{C_2}(a_1^k)$ in the first coordinate (e.g. take a strictly convex $C_1\subset\Delta^k$ and modify it slightly at a small neighborhood of the point $c_1$ at which $\max_{y\in C_1}y_1=c_1$ to create a set $C_2$; now use the fact that $\partial f_{C_i}(y)=\{c\in C_i|c\cdot y=f_{C_i}(y)\}$, $i=1,2$). By (A.4), the function $f=f_{C_1}-f_{C_2}$ violates the equality in (A.24) and (A.25) if $\lambda_{k+1}^{k+1}(\{a_{k+1}^{k+1}\})>0$, and so $\lambda_{k+1}^{k+1}(\{a_{k+1}^{k+1}\})=0$. Similarly it can be shown that $\lambda_{k+1}^{k+1}(\{-a_{k+1}^{k+1}\})=0$.

Therefore, the summands in (A.22) are zero, and thus the equality in (A.21) and (A.22) can be rewritten as

$$\psi_j^k(f,\mathbf{1}_k)=\int \mathrm{d}f(t\mathbf{1}_k,(y_1,\dots,y_k),e_j)\,\mathrm{d}\Upsilon_j^0\circ\lambda_{k+1}^{k+1}(t,(y_1,\dots,y_k)). \tag{A.26}$$

Since it holds for all $f\in M^k$,

$$\lambda_j^k=\Upsilon_j^0\circ\lambda_{k+1}^{k+1},$$

by Lemma 5. □

**Proof of Lemma 4.** By Lemma 3, for any $j=1,\dots,k$

$$\lambda_1^k=\Upsilon_1^0\circ\lambda_{k+1}^{k+1}=\Upsilon_j^0\circ\lambda_{k+1}^{k+1}=\lambda_j^k \tag{A.27}$$

(since clearly $\Upsilon_1^0=\Upsilon_j^0$). Therefore, $\lambda_j^k=\lambda^k$ is independent of $j$.

By Remark 1, for every $f\in M^{k+1}$ and every permutation $\theta$ of $\{1,\dots,k,k+1\}$ that fixes $k+1$

$$\psi_{k+1}^{k+1}(g(f)_{k+1})=\psi_{k+1}^{k+1}(g(f\circ\theta^*)_{k+1}). \tag{A.28}$$

From application of (3.5) to both sides of (A.10) and Lemma 5 it follows that

$$\mathrm{d}\lambda_{k+1}^{k+1}(t,y)=\mathrm{d}\lambda_{k+1}^{k+1}(t,\theta^*y). \tag{A.29}$$

Now (A.29) imply that for every permutation $\theta$ of $\{1,\dots,k\}$

$$\mathrm{d}\lambda^k(t,y)=\mathrm{d}\lambda^k(t,\theta^*y). \tag{A.30}$$

□

*A.3. Proof of Proposition 1*

It clearly suffices to give the proof only for convex $f \in \overline{M}_+^k$ with nondecreasing marginal costs to scale. [1]

**Remark 4.** Observe that

$$\mathrm{d}f(t\mathbf{1}_k, \mathbf{1}_k) = -\mathrm{d}f(t\mathbf{1}_k, -\mathbf{1}_k) \tag{A.31}$$

for almost every $t \in (0, 1)$. Fix any $t$ for which this equality holds. Then

$$\max_{p \in \partial f(t\mathbf{1}_k)} p \cdot \mathbf{1}_k = \mathrm{d}f(t\mathbf{1}_k, \mathbf{1}_k) = -\mathrm{d}f(t\mathbf{1}_k, -\mathbf{1}_k) = \min_{p \in \partial f(t\mathbf{1}_k)} p \cdot \mathbf{1}_k, \tag{A.32}$$

and so $p \cdot \mathbf{1}_k$ is constant for all $p \in \partial f(t\mathbf{1}_k)$. Therefore, by (A.4), the function $\mathrm{d}f(t\mathbf{1}_k, \cdot, z)$ is constant on every half-plane that has $D^k$ as its boundary, for almost every $t \in (0, 1)$; in particular, $\mathrm{d}f(t\mathbf{1}_k, y, z) = \mathrm{d}f(t\mathbf{1}_k, \Upsilon^k(y), z)$ for all $y \in R^k - D^k$.

1. We first prove that functions $\{\mathrm{d}f(\cdot\mathbf{1}_k, w)\}_{w \in R^k}$ are continuous at almost every $t \in (0, 1)$. Indeed, given $w \in R_+^k$, the function $\mathrm{d}f(\cdot\mathbf{1}_k, w)$ is monotone by the assumption of nondecreasing marginal costs to scale, and thus continuous at almost every $t$ — say, at all $t \in T$, where $T \subset (0, 1)$ has the Lebesgue measure 1. In fact, it can be assumed that the functions $\mathrm{d}f(\cdot\mathbf{1}_k, w)$ are all continuous at every $t \in T$ for $w \in W$ — a dense countable subset of $R_+^k$, and that (A.31) holds for all $t \in T$. Let $(t_n)_{n=1}^\infty$ be a sequence converging to some $t_0 \in T$, and define a sequence $(g_n)_{n=0}^\infty$ of convex functions on $R^k$ by

$$g_n(w) = \mathrm{d}f(t_n\mathbf{1}_k, w). \tag{A.33}$$

Thus, $(g_n)_{n=1}^\infty$ converges to $g_0$ pointwise on $W$, and therefore on the entire $R_+^k$ (Theorem 10.8 of Rockafellar (1970)).

Given $w \in R^k$ let $w' \in R_+^k$ be such that $w' - w = t\mathbf{1}_k$ for some $t \geq 0$. Since $t_0 \in T$,

$$\lim_{n\to\infty} \mathrm{d}f(t_n\mathbf{1}_k, \mathbf{1}_k) = -\lim_{n\to\infty} \mathrm{d}f(t_n\mathbf{1}_k, -\mathbf{1}_k) = \mathrm{d}f(t_0\mathbf{1}_k, \mathbf{1}_k). \tag{A.34}$$

By (A.2)

$$\lim_{n\to\infty} \max_{p\in\partial f(t_n\mathbf{1}_k)} p \cdot \mathbf{1}_k = \lim_{n\to\infty} \min_{p\in\partial f(t_n\mathbf{1}_k)} p \cdot \mathbf{1}_k = \mathrm{d}f(t_0\mathbf{1}_k, \mathbf{1}_k). \tag{A.35}$$

Therefore, using (A.12),

$$\lim_{n\to\infty} \mathrm{d}f(t_n\mathbf{1}_k, w) = \lim_{n\to\infty} \max_{p\in\partial f(t_n\mathbf{1}_k)} p \cdot w \tag{A.36}$$

$$= \lim_{n\to\infty} \max_{p\in\partial f(t_n\mathbf{1}_k)} [p \cdot w' - p \cdot (t\mathbf{1}_k)] = (\lim_{n\to\infty} \max_{p\in\partial f(t_n\mathbf{1}_k)} p \cdot w') - \mathrm{d}f(t_0\mathbf{1}_k, t\mathbf{1}_k) \tag{A.37}$$

[1] This proposition is the only spot in the proof of Theorem 1 that uses the assumption on nondecreasing marginal costs to scale of cost functions in $F_+$.

$$= \lim_{n\to\infty} \mathrm{d}f(t_n\mathbf{1}_k, w') - \mathrm{d}f(t_0\mathbf{1}_k, t\mathbf{1}_k) = \mathrm{d}f(t_0\mathbf{1}_k, w') - \mathrm{d}f(t_0\mathbf{1}_k, t\mathbf{1}_k), \qquad \text{(A.38)}$$

where the last equality in (A.38) follows from the previous paragraph. By Remark 4, $\mathrm{d}f(t_0\mathbf{1}_k, \cdot)$ is linear, and so right-hand side of (A.38) is $\mathrm{d}f(t_0\mathbf{1}_k, w)$. This shows that the functions $(g_n)_{n=1}^{\infty}$ converge to $g_0$ pointwise on the entire $R^k$.

Note that the set $[\partial f(t_0\mathbf{1}_k)]_y$ contains only one point for ${}^1\lambda^k$-almost every $y \in S_{\perp}^k$ (since it is so for Lebesgue almost every $y \in R^k$, and $[\partial f(t_0\mathbf{1}_k)]_y = [\partial f(t_0\mathbf{1}_k)]_{\Upsilon^k(y)}$ by Remark 4), and fix one such $y = y_0$. Also let $t_0 \in T$. Given sequences $(t_n)_{n=1}^{\infty} \subset (0, 1)$ and $(y_n)_{n=1}^{\infty} \subset S_{\perp}^k$ converging to $t_0$ and $y_0$, respectively, let $(g_n)_{n=1}^{\infty}$ be defined as in (A.33) above. Then $\partial g_0(y_0) = [\partial f(t\mathbf{1}_k)]_{y_0}$ contains only one element, and $g_n$ converges to $g_0$ pointwise. This yields

$$\mathrm{d}g_n(y_n, z) \to \mathrm{d}g_0(y_0, z), \qquad \text{(A.39)}$$

after applying Theorem 24.5 of Rockafellar (1970) to the sequence $(\mathrm{d}g_n(y_n, z))_{n=1}^{\infty}$. We are done since $\mathrm{d}g_n(y_n, z) = \mathrm{d}f(t_n\mathbf{1}_k, y_n, z)$ for $n \geq 0$.

2. For any $(t, y) \in (0, 1) \times S_{\perp}^k$ and a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to t\mathbf{1}_k$ and $\Upsilon^k(x_n - t\mathbf{1}_k) \to y$, denote $t_n = \sum_{i=1}^{k}(x_n)_i$, and define functions

$$f_n(x) = f(x + (t_n - t)\mathbf{1}_k). \qquad \text{(A.40)}$$

Observe that $\mathrm{d}f_n(w_n, z) = \mathrm{d}f(x_n, z)$ where $w_n = x_n + (t - t_n)\mathbf{1}_k$. It suffices to show that for all $t \in T$ (defined in part 1 of the proof) and $y$ such that $[\partial f(t\mathbf{1}_k)]_y$ contains only one point (and this is the case ${}^1\lambda^k$ — almost everywhere)

$$\lim_{n\to\infty} \mathrm{d}f_n(w_n, z) = \mathrm{d}f(t\mathbf{1}_k, y, z). \qquad \text{(A.41)}$$

Since $(w_n - t\mathbf{1}_k)/\|w_n - t\mathbf{1}_k\| \to_{n\to\infty} y$, (A.41) follows by Theorem 24.6 of Rockafellar (1970). Although this theorem requires a fixed function $f$, rather than a sequence $(f_n)_{n=1}^{\infty}$, its proof goes through provided $\lim_{n\to\infty}\mathrm{d}f_n(t\mathbf{1}_k, y_n) = \mathrm{d}f(t\mathbf{1}_k, y)$, where $y_n = (w_n - t\mathbf{1}_k)/\|w_n - t\mathbf{1}_k\|$. This equality is indeed satisfied by part 1 of the proposition, or its proof.

### A.4. Proof of Theorem 1 for $F_l$

Denote by $\overline{M}_l^k$ the space of all piecewise linear functions on $[0, 1]^k$ that vanish at 0, and replace $\overline{M}_+^k$ throughout the proof in Section 3 by the space $\overline{M}_l^k$. Extend given price mechanism $\psi$ on $F_l$, satisfying axioms (1)–(5), to the class of cost problems $(f, a)$ with $a * f \in \overline{M}_l^k$, as in Section 3. Apply now all the steps of the proof in Section 3, with the following modifications:

1. The proof of part 1 of Lemma 2 requires a change, but remains similar to that for $\overline{M}_+^k$. Samet et al. (1984) prove that axioms of cost sharing, additivity, rescaling invariance, consistency and monotonicity suffice to determine a unique price mechanism, defined on the class of piecewise linear cost functions differentiable at all but finitely many points of

$D^k$ (denote this class by $(\overline{M}^k)'$), as the A–S mechanism. The restriction of $\psi$ to $(\overline{M}^k)'$ is therefore uniquely determined: for any $f \in (\overline{M}^k)'$

$$\psi(f, \mathbf{1}_k) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x_j} f(t\mathbf{1}_k)\,\mathrm{d}t. \tag{A.42}$$

As in the original proof of this lemma, one shows that the measure $\lambda_j^k$ is supported on $[0, 1] \times S_\perp^k$, using functions $f \in (\overline{M}^k)'$ for which $g(f)_j$ is continuous $\lambda_j^k$ — almost everywhere instead of polynomials $p$. The second claim of the lemma follows from (A.42), Lemma 1, and the fact that any function in $C([0, 1])$ can be uniformly approximated by functions $(\mathrm{d}/\mathrm{d}x_j) f(t\mathbf{1}_k)$, for $f \in (\overline{M}^k)'$ which are differentiable at all but finitely many points of $D^k$, and $g(f)_j$ is continuous $\lambda_j^k$-almost everywhere.

2. The second part of the proof of Lemma 5 remains correct once it is established that $\psi_j^k(h)$, for $h \in M^k$, is determined by the knowledge of all $\psi_j^k(g(f)_j)$, $f \in \overline{M}_l^k$ (in Section 3 this step was trivial since $M^k \subset \overline{M}_+^k$). This is what we show next.

**Remark 5.** If $C \subset \Delta^k$ is a polytope, then $f_C$, $h_{C,t}$ are all in $\overline{M}_l^k$.

**Lemma 6.** *For any strictly convex and compact $C \subset \Delta^k$ there is a sequence $(C_n)_{n=1}^\infty$ of polytopes in $\Delta^k$ such that*

$$\lim_{n\to\infty} \psi_j^k(g(h_{C_n,t_0})) = \psi_j^k(g(h_{C,t_0})) \tag{A.43}$$

*for every $t_0 \in [0, 1]$.*

**Proof.** By Gruber (1993) any such $C$ can be approximated by a sequence $(C_n)_{n=1}^\infty$ of convex closed polytopes in $\Delta^k$ in a way that $f_{C_n} \to_{n\to\infty} f_C$ pointwise. It is easy to deduce from this, using Theorem 24.5 in Rockafellar (1970), that $g(f_{C_n}) \to_{n\to\infty} g(f_C)$ in the supremum norm. It follows that $g(h_{C_n,t_0}) \to_{n\to\infty} g(h_{C,t_0})$ in the supremum norm outside an open set $O_\varepsilon = (t_0-\varepsilon, t_0+\varepsilon) \times S_\perp^k \cup \{x \in [0, 1]^k \,|\, |f_C(x) - t_0| < \varepsilon\}$ for any given $\varepsilon > 0$. By part 1 of Lemma 2 $\lambda_j^k(O_\varepsilon) \to_{\varepsilon\to 0} 0$. On the other hand, given $\varepsilon > 0$ there is a function $g \in C(\Lambda^k)$ with values in $[\varepsilon, 1+\varepsilon]$, such that $g|_{O_\varepsilon} \equiv 1+\varepsilon$ and $g|_{(O_{2\varepsilon})^c} \equiv \varepsilon$, and a natural number $N$ such that for $n \geq N$ both $g(h_{C,t_0})_j - g(h_{C_n,t_0})_j + g$ and $g(h_{C_n,t_0})_j - g(h_{C,t_0})_j + g$ are nonnegative. Clearly $\int g(x)\,\mathrm{d}\lambda_j^k(x) < 4\varepsilon(1+\varepsilon)$. By the positivity of $\psi_j^k$, the definition of $\lambda_j^k$, and arbitrariness of $\varepsilon$,

$$\lim_{n\to\infty} \psi_j^k(g(h_{C_n,t_0})) = \psi_j^k(g(h_{C,t_0})). \tag{A.44}$$

3. Equality (A.14) in the proof of Lemma 3 no longer follows from consistency and rescaling invariance of $\psi$, since $M^k$ is not a subset $\overline{M}_l^k$. However, as in the proof of Lemma 6 it can

be shown that

$$\lim_{n\to\infty}\psi_{k+1}^{k+1}(g((h_{C_n,t_0})_j^{\alpha})_{k+1}) = \psi_{k+1}^{k+1}(g((h_{C,t_0})_j^{\alpha})_{k+1}). \tag{A.45}$$

Since

$$\psi_{k+1}^{k+1}(g((h_{C_n,t_0})_j^{\alpha})_{k+1}) = \alpha\psi_j^k(g(h_{C_n,t_0})_j) \tag{A.46}$$

by consistency and rescaling invariance, (A.45) and Lemma 6 yield

$$\psi_{k+1}^{k+1}(g((h_{C,t_0})_j^{\alpha})_{k+1}) = \alpha\psi_j^k(g(h_{C,t_0})_j). \tag{A.47}$$

4. Proposition 1 remains valid, and its proof for $\overline{M}_l^k$, which is easier than that for $\overline{M}_+^k$, is given in the next subsection of the Appendix A. □

**Remark 6.** Theorem 1 can also be proved if $F_+$ is replaced by $F_h$, the subclass [2] in which cost functions show constant returns to scale (i.e. homogeneous of degree 1). The proof follows in a similar, and often much easier, fashion: one only has to uniquely characterize the measure ${}^1\lambda_j^k$, and not $\lambda_j^k$, because of the homogeneity of cost functions from $F_h$. Therefore, Lemma 2 must be restated accordingly (replace $\lambda_j^k$ with ${}^1\lambda_j^k$, and $M^k$ with $HM^k$ — the subspace of homogeneous functions). The other noticeable change occurs in proofs of parts 1 and 2 of Lemma 5. In the proof of part 1, Haimanko (2000b) characterization of the A–S price mechanism for continuously differentiable, homogeneous and convex cost problems is required, instead of the result of Mirman and Tauman (1982). Instead of polynomials $p$ as in the proof of Lemma 2, functions constructed in Lemma 3.2 of Haimanko (2000a) must be used, which are in the span of continuously differentiable, homogeneous and convex functions by Lemma 3.3 there. Part 2 of the lemma becomes redundant, since only ${}^1\lambda_j^k$ must be determined.

### *A.5. Proof of Proposition 1 for $\overline{M}_l^k$*

We show the proposition first for piecewise linear functions $f$ which are affine on every half-plane that has $D^k$ as its boundary ($f$ may or may not satisfy $f(0) = 0$). In this case $\mathrm{d}f(t\mathbf{1}_k, y) = f(y) - f(0)$ and $\mathrm{d}f(t\mathbf{1}_k, y, z) = \mathrm{d}f(y, z)$ for every $y$, $z$. Also note that the regions in the definition of piecewise linear function can all be assumed to have the form $\{x \in R^k | \forall l = 1, \ldots, n,\ x \cdot a_l \le 0\}$, where $\{a_l\}_{l=1}^n \subset S_\perp^k$, and that on the interior of each region the functions $\mathrm{d}f(\cdot, z)$ are constant. It is clear now that if $y \in S_\perp^k$ happens to be in the interior of one of the regions (and this is the case for almost every $y$), then assertions of the proposition hold for all $t$.

Now let $f$ be any function in $\overline{M}_l^k$. There are at most finitely many regions in the definition of piecewise linear function $f$ whose boundaries intersect, but do not contain, the diagonal $D^k$. It follows that for almost every $t_0 \in (0, 1)$ there is an open neighborhood $O$ of $t_0\mathbf{1}_k$ in $[0, 1]^k$, such that the restriction of $f$ to $O$ coincides with some piecewise linear function

[2] The characterization in Theorem 1 is of no use here, because uniqueness of a price mechanism on a class of cost problems does not, in general, imply uniqueness of the mechanism on a subclass.

$h$, which is affine on every half-plane that has $D^k$ as its boundary. As $\mathrm{d}f(t\mathbf{1}_k, y, z) = \mathrm{d}h(t\mathbf{1}_k, y, z)$ for all $y \in O$ and any $z$, assertions of the proposition are proved for such $t_0$ and almost every $y \in S_{\perp}^k$ by what we have shown in the previous paragraph.

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