Journal of Economic Theory **106**, 17–39 (2002)
doi:10.1006/jeth.2001.2826

Payoffs in Nondifferentiable Perfectly Competitive TU Economies[1]

Ori Haimanko

CORE, Voie du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium; and Department of Economics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
haimanko@core.ucl.ac.be

Received January 10, 2000; final version received February 15, 2001; published online November 7, 2001

We show that a single-valued solution of nonatomic finite-type market games (or perfectly competitive TU economies underling them) is uniquely determined as the Mertens value by four plausible value-related axioms. Since the Mertens value is always in the core of an economy, this result provides an axiomatization of a core-selection (or, alternatively, a competitive payoff selection). *Journal of Economic Literature* Classification Numbers: C71, D51, D61.

Key Words: market games; TU economies; core selection; Mertens value.

1. INTRODUCTION

Among the solution concepts of cooperative game theory, the core is one of the most straightforward and widely used. However, stability conditions defining it are too strong for some games, and too weak for others: large classes of games possess no core, and when the core is nonempty, it is usually multivalued.

The Shapley [14] value avoids these two shortcomings. Defined as a mapping that assigns a single payoff vector to any game, subject to certain plausible axioms, it exists and is unique, on finite games [14], and when agents form a nonatomic continuum (a natural assumption in models of perfect competition), appropriately modified axioms also lead to a unique value on important spaces of nonatomic games (Aumann and Shapley [2]). In computational terms, these values are based on the principle of assigning an agent his average marginal contribution to coalitions he may join, in certain models of random ordering of the agents.

[1] This work is a part of the author's Ph.D. thesis done at the Hebrew University of Jerusalem under the supervision of Professor A. Neyman, whose advice and guidance are gratefully appreciated.

0022-0531/01 $35.00

On general classes of nonatomic games there may be no value, and the core may be empty. This changes, however, if we restrict our attention to the class of market games, derived from transferable utility (TU) perfectly competitive economies. The core of a market game is always nonempty. Moreover, if all agents have differentiable utility functions, then the (Aumann–Shapley) value is also well defined, and this value turns out to be the unique element of the core.

Without the differentiability assumption, the core of a market game is multivalued in general. But even in this scenario, all known values have the property of being elements of the core. Indeed, all (asymptotic) measure-based values (of Hart [8]) of a nonatomic market game are contained in its core, and so is the value of Mertens [10] (which is, in fact, defined on a large space of games[2] that includes *all* nonatomic market games).[3]

As a mechanism that selects an element of the core for every market game, the value has obvious merits. Arbitrary core selections can lead to distortions, e.g., significant differences in payoffs in "close" economies, discrimination between agents, and inconsistencies between choices. This is not the case with values, where the choices of payoff measures in different games are linked to each other via the value axioms, in a consistent and economically meaningful way. The important and natural question that arises is whether the properties of the value can, in fact, determine a unique core selection on the domain of market games.

In this paper we show that the following four properties (axioms) can accomplish this job on the set of nonatomic *finite-type* market games (i.e., those induced by economies with finitely many types of utility functions and endowments):

(1) **efficiency**—the total payoff to the agents equals their joint payoff in the market game, i.e., the maximal aggregate utility they can achieve by redistributing initial endowments;

(2) **anonymity**—relabeling the agents does not affect their payoffs, which are simply relabeled accordingly;

(3) **separability**—if a market game corresponds to an economy with two separate parts, then any agent's payoff in it is the sum of his payoffs in the games derived from each part;

(4) **contraction**—the distance between the payoffs does not exceed the distance between the market games.

[2] In contrast to the above mentioned uniqueness of Shapley and Aumann–Shapley values, it is not known if the Mertens value is uniquely determined by value axioms on spaces where the Aumann–Shapley value is not defined, including the space generated by market games.

[3] In [11], the Mertens value is given by an explicit formula as a baricenter of the core.

Axioms (1) and (2) precisely correspond to the value axioms of *efficiency* and *symmetry*. Separability (3) is a weaker version of *additivity*, and is tailored to fit the context of market games (which do not form a linear space, as is typical of the domains considered in value theory). Finally, contraction (4) is an adapted form of continuity, satisfied by most known values. The solution that is categorically determined by (1)–(4) is precisely the Mertens value. Since the Mertens value is an element of the core, we effectively obtained a core selection.

Note that, with the exception of (1), the axioms do not involve any assumption on the range of the solution concept; in particular, inclusion in the core is not implied by any of them individually. It is thus striking that core selection is determined by axioms posited on entirely unrelated grounds, and even more striking is the fact that our axioms collectively imply not only inclusion in the core, but complete determination by the core: if two market games have the same core, they also have the same Mertens value.

Since a market game is only a derivative concept, it is important to note that the value can be viewed as a solution defined per se not on the market games, but on the set of underlying TU economies. Axioms (1)–(4) can be restated in the new setting in a straightforward way. Our result then implies that any solution on finite type TU economies that satisfies the axioms is the Mertens value. (Indeed, it is immediate from the contraction axiom that the solution of an economy is determined via the corresponding market game, and so the result for economies becomes a simple corollary.)

In the differentiable setting, the question of axiomatic determination of a (possibly multivalued) solution for perfectly competitive and differentiable TU economies was previously investigated by Dubey and Neyman [4]. While two of their axioms are close to our (3) and (4), the other two are somewhat different: (1) is replaced by the *inessential economy* axiom,[4] while in (2) all relabellings are required to preserve the distribution of agents. These axioms lead to a unique outcome (the single-valued core) in the differentiable setting, but not outside of it, as is pointed out in Section 6 of [4]: both the (single-valued) Mertens value and the (possibly multivalued) core satisfy the axioms. However, what we show is that the situation changes when a solution is assumed to be single-valued, and the axioms are slightly modified.

Our result can also be considered from a complementary, noncooperative point of view. A noncooperative solution of an economy is provided by the notion of competitive equilibrium. The competitive

[4] In an inessential economy, any coalition of agents achieves the maximal aggregate utility by sticking to its initial endowment of resources. According to the axiom, in such an economy each agent is getting precisely the utility derived from his initial endowment.

equilibrium is akin to Nash equilibrium in that both, despite being highly tractable, transparent and appealing solutions, suffer from the same non-uniqueness problem. And as Harsanyi and Selten [7] put it in the introduction to their theory of Nash equilibrium selection, "a theory that can only predict that the outcome ... is an equilibrium point—without specifying what the equilibrium point is—is an extremely weak and uninformative theory."

It follows from the core equivalence principle for perfectly competitive economies [1, 2] that the Mertens value of a market game is a competitive payoff. Thus, we have in effect provided an axiomatically based selection from competitive payoffs. Although this selection does not determine a competitive equilibrium itself, it does determine a set of equilibria that lead to the same payoff for agents.

Finally, let us mention that in the differentiable case axioms of a similar nature can characterize the competitive payoff correspondence for perfectly competitive economies with nontransferable utilities [5]. We hope that our approach can be translated into the setting of NTU economies as well.

2. DEFINITIONS, AXIOMS, AND THE MAIN RESULT

Let (T, C) be a standard measurable space. T is the set of agents, and C is the σ-algebra of coalitions. The set of nonatomic probability measures on (T, C) is denoted by NA^1. An *economy* E is a triple (u, a, η), where u: $T \times R^k_+ \to R$ and a: $T \to R^k_+$ for some k, and $\eta \in NA^1$. For each $t \in T$, $a_t = a(t)$ is agent t's initial endowment of commodities 1, 2, ..., k, $u_t = u(t, \cdot)$ is his utility function on the space of commodity bundles R^k_+, and η is the population measure.

Given $\eta \in NA^1$ we denote by $\mathfrak{E}_F(\eta)$ the set of *finite type* economies, i.e., the set of $E = (u, a, \eta)$ for which there exists a finite subfield H of C with the following properties:

(i) a is H-measurable;

(ii) u is $H \times B$-measurable, where B denotes the Borel σ-field of R^k_+.

We further assume that for every $t \in T$:

(iii) u_t is monotonic (nondecreasing);

(iv) $u_t(x) = o(\|x\|)$ as $\|x\| \to \infty$;

(v) u_t is continuous;

(vi) $u_t(0) = 0$;

(vii) $\int_T a_t \, d\eta(t) > 0$ (i.e., strictly positive in all coordinates).

Conditions (i) and (ii) formalize the finite type assumption: there is a finite partition of T such that agents in the same element of the partition (type) have the same initial endowments and the same utility functions. Condition (vii) requires a positive amount of each commodity to be present in the market. We denote $\mathfrak{E}_F = \bigcup_{\eta \in NA^1} \mathfrak{E}_F(\eta)$.

A *game* is determined by its characteristic function $v\colon C \to R$ that vanishes on the empty set. Given $E = (u, a, \eta) \in \mathfrak{E}_F$, the *market game* corresponding to E, v_E, is given by

$$v_E(S) = \max \left\{ \int_X u_t(x_t)\, d\eta \,\middle|\, x\colon T \to R^n_+,\ x_\eta(S) = a_\eta(S) \right\}, \tag{1}$$

where $y_\eta(S)$ abbreviates $\int_S y(t)\, d\eta(t)$. The maximum above is attained by Proposition 36.1 of Aumann and Shapley [2]. The set of all finite-type market games corresponding to economies in $\mathfrak{E}_F(\eta)$ will be denoted by $MG(\eta)$, and $MG \equiv \bigcup_{\eta \in NA^1} MG(\eta)$.

A *value* is a mapping[5] $\Psi\colon MG \to FA_+$ (FA_+ is the set of bounded, finitely additive, and nonnegative measures on (T, C)), subject to the four conditions (axioms) that we state below.

Axiom 1 (Efficiency). For any $v \in MG$,

$$\Psi(v)(T) = v(T). \tag{2}$$

Denote by Θ the set of all automorphisms of (T, C), and, for any game (or a measure) v define the game θv by $(\theta v)(S) = v(\theta S)$. Note that if v is a market game corresponding to $E = (u, a, \eta) \in \mathfrak{E}_F$ and $\theta \in \Theta$, then θv is the market game corresponding to $\theta E = (\theta u, \theta a, \theta \eta) \in \mathfrak{E}_F$, where $(\theta u)_t(x) = u_{\theta(t)}(x)$ and $(\theta a)(t) = a_{\theta(t)}$. This lays the ground for the following axiom:

Axiom 2 (Anonymity). For any $v \in MG$ and $\theta \in \Theta$,

$$\Psi(\theta v) = \theta \Psi(v). \tag{3}$$

Note that a sum of two market games in $MG(\eta)$ is itself in $MG(\eta)$. Indeed, if $E = (u, a, \eta)$, $E' = (u', a', \eta) \in \mathfrak{E}_F(\eta)$, where $a\colon T \to R^k_+$, $a'\colon T \to R^{k'}_+$, we define the disjoint sum of the economies, $E \oplus E' \in \mathfrak{E}_F$, as $(u \oplus u', a \oplus a', \eta)$, where $u \oplus u'\colon T \times R^{k+k'}_+ \to R$, $a \oplus a'\colon T \to R^{k+k'}_+$ are given by $(a \oplus a')(t) = (a(t), a'(t))$, and $(u \oplus u')(t, (x, y)) = u(t, x) + u'(t, y)$. Observe that the sum $v_E + v_{E'}$ is precisely $v_{E \oplus E'} \in MG(\eta)$.

[5] In games with a continuum of players, the term "value" is traditionally reserved for *linear* operators defined on *linear* spaces of games. Nevertheless, because of the similarities between axioms (1)–(4) below and the value axioms, we opted to call Ψ a value.

Axiom 3 (Separability). For any $\eta \in NA^1$ and v, $v' \in MG(\eta)$,

$$\Psi(v+v') = \Psi(v) + \Psi(v'). \tag{4}$$

A game v is monotonic if for any $S \subset W \in C$, $v(S) \leqslant v(W)$, and a game representable as a difference of two monotonic games is said to be of bounded variation. For a game v of bounded variation we define its norm, $\|v\|$, by $\|v\| = \inf(u(T) + w(T))$, where the infimum is taken over all monotonic games u and w with $v = u - w$. Differences of two market game and measures in FA_+ are clearly of bounded variation.

Axiom 4 (Contraction). If v, $v' \in MG$ then

$$\|\Psi(v) - \Psi(v')\| \leqslant \|v - v'\|. \tag{5}$$

Remark 1. The contraction axiom is related to separability and efficiency via the following observation. If $\eta \in NA^1$ and v, v', u, $w \in MG(\eta)$ are such that $v - v' = u - w$, then $\Psi(v) - \Psi(v') = \Psi(u) - \Psi(w)$ by separability. Therefore

$$\|\Psi(v) - \Psi(v')\| = \|\Psi(u) - \Psi(w)\| \leqslant \Psi(u)(T) + \Psi(w)(T) \tag{6}$$

$$= u(T) + w(T), \tag{7}$$

where the last equality follows from efficiency. Thus

$$\|\Psi(v) - \Psi(v')\| \leqslant \inf(u(T) + w(T)), \tag{8}$$

where the infimum is taken over all u, $w \in MG(\eta)$ such that $v - v' = u - w$. The contraction axiom is therefore a strengthening of (8): it precisely requires that (8) remain valid when the infimum is taken over the larger set of all monotonic games u, w subject to $v - v' = u - w$.

Remark 2. An alternative continuity requirement that can be imposed on Ψ is the Lipschitz property: there is $K > 0$ such that

$$\|\Psi(v) - \Psi(v')\| \leqslant K \|v - v'\| \tag{9}$$

for any two v, $v' \in MG$. This notion of continuity was used in [4] in the characterization of the basic (equivalent) solution concepts on differentiable perfectly competitive economies; it is obviously weaker than contraction. Our results will remain intact if the contraction axiom is replaced by the Lipschitz property, provided the following positivity axiom is also imposed on Ψ: $\Psi(v) - \Psi(v')$ is a nonnegative measure whenever $v - v'$ is monotonic.

Positivity is crucial in our analysis, but there is no need to assume it explicitly in the presence of contraction axiom: positivity is a simple corollary of efficiency and contraction (Proposition 4.6 of Aumann and Shapley [2]).

Axioms 1–4 are among the basic properties of Mertens [10] value, defined on a large space of games that includes all market games. Thus, the restriction of the Mertens value to MG, Ψ_M, satisfies the axioms. Additionally, Ψ_M is entirely determined by the core of a market game, and is itself an element of the core (construction of Ψ_M for markets is carried out in details in Mertens [11]). According to our result, this selection is uniquely determined by the axioms.

THEOREM 2.1. *If Ψ is a value on MG, then $\Psi = \Psi_M$.*

The next section is dedicated to the proof of the theorem.

3. THE PROOF

Before we begin, let us briefly describe the structure of the proof. We start by considering the unique linear extension of the given value Ψ onto the space spanned by market games;[6] this extension preserves value axioms. Then we observe that finite type market games are concave, non-decreasing and homogeneous functions (*market functions*) of finitely many mutually singular nonatomic probability measures (Proposition 3.1). These two facts are used to show that, for every vector $\mu = (\mu_1, ..., \mu_k)$ of mutually singular nonatomic probability measures, and every market function f in k variables,

$$\Psi(f \circ \mu) = \sum_{j=1}^{k} \psi_j^k \left(\frac{df}{dx_j}(x) \right) \mu_j, \tag{10}$$

where ψ_j^k is a positive linear functional, defined on the space of functions spanned by the (one-sided) partial derivatives of market functions in k variables. Each of the functional ψ_j^k is characterized as an integral of the function $\frac{df}{dx_j}(x)$ with respect to a measure λ_j^k on a subset of R_+^k (actually, its compactification in a certain topology), at least when $\frac{df}{dx_j}(x)$ is continuous in a certain sense. This measure is uniquely determined by Ψ (Corollary 3.9). We then describe the relations between measures λ_j^k for different k and j (Lemmas 3.10 and 3.11, and Corollary 3.13). By our Lemma 3.14, simultaneous existence of these relations characterizes the

[6] To be precise, we do it for every $MG(\eta)$.

measures λ_j^k in a unique way, independent of the value Ψ. This shows that there is a room for at most one value on MG. Since the Mertens value satisfies the axioms, it coincides with Ψ.

Denote by $\overline{MG}(\eta)$ ($\overline{MG}$) the linear spaces spanned by $MG(\eta)$ (respectively, MG). Observe that the given value Ψ possesses a well defined linear extension into $\overline{MG}(\eta)$, uniquely determined[7] by

$$\Psi(v_1 - v_2) = \Psi(v_1) - \Psi(v_2). \tag{11}$$

The extended Ψ is efficient (by the efficiency axiom), and it is continuous of norm 1,

$$\|\Psi(v)\| \leqslant \|v\|, \tag{12}$$

for any $v \in \overline{MG}(\eta)$ (by the contraction axiom). It implies that Ψ is actually a linear operator: from separability, $\Psi(rv) = r\Psi(v)$ for any rational r, and, from (12), the statement holds for any r. By the proof of Proposition 4.6 of [2], Ψ is positive, i.e., $\Psi(v)$ is a nonnegative measure whenever the game $v \in \overline{MG}(\eta)$ is monotone. To sum up, the extended Ψ is an efficient, linear, and positive operator on $\overline{MG}(\eta)$.

Now let $k \geqslant 2$. By $\bar{\bar{M}}_+^k$ denote the cone of market functions on R_+^k, i.e., those which are concave, continuous, nondecreasing, and homogenous of degree 1. Also let $\bar{\bar{M}}^k$ be the vector space of differences of functions in $\bar{\bar{M}}_+^k$. By $\bar{M}_+^k$ denote the subset of Lipschitz functions in $\bar{\bar{M}}_+^k$, and by $\bar{M}^k$—the linear space spanned by $\bar{M}_+^k$ in $\bar{\bar{M}}^k$.

For a k-tuple $\mu = (\mu_i)_{i=1}^k$ of mutually singular measures in NA^1 let $(S(\mu_i))_{i=1}^k$ be disjoint sets such that $\mu_j(S(\mu_i)) = \delta_{ij}$. Also denote $\bar{\mu} = \frac{1}{k}\sum_{i=1}^k \mu_i$. Let $MG^k(\mu)$ be the cone $\{f \circ \mu \mid f \in \bar{\bar{M}}_+^k\}$, and $\overline{MG}^k(\mu)$ be the space $\{f \circ \mu \mid f \in \bar{\bar{M}}^k\}$.

PROPOSITION 3.1. *For k, μ as above, $MG^k(\mu) \subset MG(\bar{\mu})$. Moreover, any $v \in MG$ is an element of $MG^k(\mu)$, for some $k \geqslant 2$ and mutually singular $\mu \in (NA^1)^k$.*

Proof. Let $f \in \bar{\bar{M}}_+^k$ and μ, $(S(\mu_i))_{i=1}^k$ as above. Consider an economy $E = (u, a, \bar{\mu}) \in \mathfrak{E}_F$, in which $u_t(x) \leqslant f(x)$ for every $t \in T$ and such that equality holds instead of inequality for all $x \in [0, k]^k$, $a_t = ke_j$ (e_j stands for the jth unit vector) if $t \in S(\mu_j)$, and $\bar{\mu} = \frac{1}{k}\sum_{i=1}^k \mu_i$. It is clear that $f \circ \mu = v_E$.

On the other hand, given $E = (u, a, \eta) \in \mathfrak{E}_F$, let $J = (T_1, ..., T_q)$ be a measurable partition of T that induces a finite σ-field H that satisfy (i) and

[7] Due to separability axiom.

(ii) in the definition of $E \in \mathfrak{E}_F$. As in the proof of Lemma 39.16 of Aumann and Shapley [2], there is $g \in \bar{\bar{M}}_+^{k+q}$($k$ is the number of commodities in the economy, i.e., a: $T \to R_+^k$) such that $v_E(S) = g(\eta^J(S), a_\eta(S))$ for any $S \in C$, where $\eta^J = (\eta_i^J)_{i=1}^q$, $\eta_i^J(S) = \eta(S \cap T_i)$ for any $S \in C$, and (recall) a_η is the vector measure given for each $S \in C$ by $a_\eta(S) = \int_S a_t \, d\eta$. By our assumption on a, each coordinate of $a_\eta(S)$ is a positive linear combination of η^J. Thus, after an appropriate normalization of η^J, there is an integer k', $f \in \bar{\bar{M}}_+^{k'}$, and mutually singular $\mu \in (NA^1)^{k'}$, such that $v_E = f \circ \mu$. Note that, without loss of generality, $k' \geqslant 2$ (otherwise v_E is a measure, which can always be represented as a sum of two mutually singular nonvanishing measures). ▮

Remark 3. Since, for mutually singular $\mu \in (NA^1)^k$, $\overline{MG}^k(\mu) \subset \overline{MG}(\bar{\mu})$, Ψ is an efficient, linear, and positive operator on $\overline{MG}^k(\mu)$.

Remark 4. From the anonymity axiom,

$$\Psi(f \circ \mu) = \theta^{-1} \Psi(f \circ (\theta\mu)) \tag{13}$$

for all $f \in \bar{M}^k$, mutually singular $\mu \in (NA^1)^k$, and $\theta \in \Theta$. Reasoning as in Proposition 6.1 and Proposition 19.7 of Aumann and Shapley [2], we have

$$\Psi(f \circ \mu) = \sum_{i=1}^{k} \Psi(f \circ \mu)(S(\mu_i)) \cdot \mu_i. \tag{14}$$

Moreover, $\Psi(f \circ \mu)(S(\mu_i))$ do not depend on a particular k-tuple of mutually singular measures μ, or the sets $(S(\mu_i))_{i=1}^k$ chosen for such μ.

Remark 5. If $f \in \bar{M}^k$ is linear, i.e., $f(x) = a \cdot x$ for $a \in R^k$, then $\Psi(f \circ \mu) = a \cdot \mu$. This follows from the efficiency and linearity of Ψ on $\overline{MG}^k(\mu)$.

Denote by D_k the "diagonal" $\{x \in R^k \mid \forall i, j \ x_i = x_j\}$. An open cone N in R_+^k with $D_k \cap R_{++}^k \subset N$ will be called a conical diagonal neighborhood.

LEMMA 3.2. *Let N be a conical diagonal neighborhood, and let $1 \leqslant j \leqslant k$. There is a conical diagonal neighborhood $N' \subset N$ and a function h on R^k, possessing continuous second order derivatives on $R^k - \{0\}$, positively homogenous of degree* 1, *and vanishing on N', such that $\frac{d}{dx_j} h$ is nonnegative on R_+^k and $\frac{d}{dx_j} h \geqslant 1$ on $R_{++}^k - N$.*

Proof. For any $\varepsilon > 0$ it is easy to construct a nondecreasing twice differentiable function g^ε on R which vanishes on $[\frac{1}{k} - \varepsilon, \frac{1}{k} + \varepsilon]$ and coincides with the function $x - \frac{1}{k}$ on $R - [\frac{1}{k} - 2\varepsilon, \frac{1}{k} + 2\varepsilon]$. For any $i \neq j$ consider the function h_i^ε on R^k, given by

$$h_i^\varepsilon(x_1, ..., x_k) = \begin{cases} \left(x_i + \sum\limits_{l \neq j} x_l\right) g^\varepsilon\left(\dfrac{x_j}{x_i + \sum_{l \neq j} x_l}\right), \\ \qquad \text{if} \quad x \in R_+^k \text{ and } x_i + \sum\limits_{l \neq j} x_l > 0; \\ x_j - \dfrac{x_i + \sum_{l \neq j} x_l}{k}, \qquad \text{otherwise.} \end{cases} \tag{15}$$

Observe that h_i^ε possesses continuous second order derivatives on $R^k - \{0\}$ and that it is positively homogenous of degree 1, for ε small enough. Now define $h^\varepsilon = \sum_{i \neq j} h_i^\varepsilon$, and consider

$$N^\varepsilon = \left\{x \in R_+^k \;\middle|\; \forall i \neq j : \left|\frac{x_j}{x_i + \sum_{l \neq j} x_l} - \frac{1}{k}\right| < \varepsilon\right\},$$

a conical diagonal neighborhood. Note that h^ε vanishes on N^ε, $\frac{d}{dx_j} h$ is nonnegative on R_+^k and $\frac{d}{dx_j} h(x) \geqslant 1$ for $x \in R_{++}^k - N^{2\varepsilon}$. Since for ε small enough $N^{2\varepsilon} \subset N$, we can take $h = h^\varepsilon$ and $N' = N^\varepsilon$. ∎

Lemma 3.3. *If h: $R^k \to R$ possesses continuous second order derivatives on $R^k - \{0\}$ and is positively homogenous of degree* 1, *then* $h|_{R_+^k} \in \bar{M}^k$.

Proof. Choose any g on R^k which is strictly concave on $\{x \in R^k \mid \sum_{i=1}^k x_i = 1\}$, twice continuously differentiable on an open neighborhood of this set, and such that $f = g|_{R_+^k} \in \bar{M}_+^k$ is strictly increasing. Then for $K > 0$ large enough the function $Kf + h$ is in $\bar{M}_+^k$ (its Hessian is negatively definite for such K, hence the concavity), and so $h = (Kf + h) - Kf \in \bar{M}^k$. ∎

Lemma 3.4. *If μ is a k-tuple of mutually singular measures in NA^1 and $h \in \bar{\bar{M}}^k$ that vanishes on a conical diagonal neighborhood, then* $\Psi(h \circ \mu) = 0$.

Proof. By Lemma 5.5 of Dubey and Neyman [4] there is a $\bar{\mu}$-preserving $\theta \in \Theta$ ($\bar{\mu} = \frac{1}{k} \sum_{i=1}^k \mu_i$) such that

$$\sup \left\| \sum_{i \leqslant l} a_i \theta^i (h \circ \mu) \right\| \Big/ \sqrt{l} \xrightarrow[l \to \infty]{} 0, \tag{16}$$

where the supremum is taken over all sets of numbers a_i, $1 \leqslant i \leqslant l$, with $|a_i| \leqslant 1$. On the other hand, reasoning as in the proof of Proposition 5.12 in Dubey and Neyman [4], for every l there are $K > 0$ and $a_i^l \in \{0, 1\}$, $1 \leqslant i \leqslant l$, such that

$$\left\| \sum_{i \leqslant l} a_i^l \theta^i \Psi(h \circ \mu) \right\| \geqslant K \sqrt{l}\, \| \Psi(h \circ \mu) \|. \tag{17}$$

Since $\theta^i(h \circ \mu) = h \circ \theta^i \mu \in MG(\bar{\mu})$, it follows from (12), Remark 4, and linearity of Ψ on $MG(\bar{\mu})$ that (16) and (17) can be reconciled only if $\Psi(h \circ \mu) = 0$. ∎

For a positive integer $k \geqslant 2$, denote $\mathbf{1}_k = (1, ..., 1) \in R^k$. Let $S^k_\perp$ be the "equator" of the unit sphere in R^k perpendicular to $\mathbf{1}_k$, i.e., $\{x \in R^k \mid \|x\| = 1,\ x \cdot \mathbf{1}_k = 0\}$ for the Euclidean norm $\| \ \|$, and let Δ^k be the simplex $\{x \in R^k_+ \mid x \cdot \mathbf{1}_k = k\}$.

Endow the set $\Lambda^k = (\Delta^k - \{\mathbf{1}_k\}) \cup S^k_\perp$ with the following sequential topology: if $(x_n)_{n=1}^\infty$ is a sequence in $S^k_\perp$, or in $(\Delta^k - \{\mathbf{1}_k\})$ and it does not converge to $\mathbf{1}_k$, then $x_n \xrightarrow[n \to \infty]{} x$ if it does so in the Euclidean topology on the corresponding set. If $(x_n)_{n=1}^\infty \subset \Delta^k - \{\mathbf{1}_k\}$ and it converges to $\mathbf{1}_k$ from some direction $y \in S^k_\perp$ (which means $\frac{x_n - \mathbf{1}_k}{\|x_n - \mathbf{1}_k\|} \xrightarrow[n \to \infty]{} y$ in the Euclidean topology), then $x_n \to y$. In this topology the set Λ^k is homeomorphic to a closed $k-1$-dimensional ring $\{x \in R^{k-1} \mid 1 \leqslant \|x\| \leqslant 2\}$, and so it is metrizable and compact.

Given a concave function f: $R^k_+ \to R$ and $x \in R^k_+$, $y \in R^k$, the directional derivative of f at x in the direction[8] y is

$$df(x, y) = \lim_{\varepsilon \downarrow 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}. \tag{18}$$

The directional derivative $df(x, y)$ always exists and the function $df(x, \cdot)$ is concave and finite on all R^k [12, Theorem 23.1]. By concavity, all directional derivatives of $df(x, \cdot)$,

$$df(x, y, z) = \lim_{\varepsilon \downarrow 0} \frac{df(x, y + \varepsilon z) - df(x, y)}{\varepsilon}. \tag{19}$$

also exist and are finite.

Denote by $B(\Lambda^k)$ the Banach space of bounded and measurable functions on Λ^k. Consider the subspace V of $[B(\Lambda^k)]^k$, of all k-tuples $g(f)$, for $f \in \bar{M}^k$, defined by

$$g(f)(x) = \begin{cases} (df(\mathbf{1}_k, y, e_j))_{j=1}^k, & x \in S^k_\perp; \\ (df(x, e_j))_{j=1}^k, & x \in \Delta^k - \mathbf{1}_k. \end{cases} \tag{20}$$

Every $f \in \bar{M}^k$ is uniquely determined by the knowledge of its partial its derivatives $df(x, e_j)$, $j = 1, ..., k$, on $\Delta^k - \{\mathbf{1}_k\}$. Therefore a linear functional ψ^k_j, given by

$$\psi^k_j(g(f)) = \Psi(f \circ \mu)(S(\mu_j)), \tag{21}$$

[8] If $x \in R^k_+ - R^k_{++}$, we only consider $y \in R^k_+$.

is well defined on V. It does not depend on a particular k-tuple of mutually singular measures μ, or the sets $(S(\mu_i))_{i=1}^k$ chosen for such μ, by Remark 4. If $g(f)\geqslant 0$ then $f\circ\mu$ is a monotonic game, and so the positivity of Ψ implies $\psi_j^k(g(f))\geqslant 0$. It follows that the functional ψ_j^k is positive. Finally, if a k-tuple of constant functions in V is identified in an obvious manner with a vector in R^k, then for $c=(c_1, ..., c_k)\in R^k$

$$\psi_j^k(c)=c_j. \tag{22}$$

To see this, apply Remark 5 to the game $f(\mu)=c\cdot\mu$.

Since $[B(\Lambda^k)]^k$ can be viewed as $B(\bigsqcup_{i=1}^k \Lambda^k)$ (bounded measurable functions on the union of k disjoint copies of Λ^k), V can be regarded as its subspace, and ψ_j^k—as a positive projection (i.e., nonnegative functions are mapped by ψ_j^k into R_+, and constant functions are mapped into their values). By the Kantorovitch Theorem (e.g., [15, Theorem 83.15]), ψ_j^k can be extended to the entire $B(\bigsqcup_{i=1}^k \Lambda^k)$ as a positive projection. It is then a sum of positive linear functionals $(\psi_k^j)_i$, each defined on the corresponding coordinate of $[B(\Lambda^k)]^k$. Property (22) together with positivity now imply that $(\psi_j^k)_i$ are zero unless $i=j$. Therefore ψ_j^k is a function of jth coordinate alone, belonging to the space $B(\Lambda^k)$.

If we restrict our attention to the subspace of continuous functions $C(\Lambda^k)$, then by Riez representation theorem

$$\psi_j^k(g)=\int g(x)\, d\lambda_j^k(x) \tag{23}$$

for some positive measure λ_j^k on Λ^k and all $g\in C(\Lambda^k)$. Choosing $c=e_i$ (the ith unit vector) in (22), for $i=1, ..., k$, we obtain $\psi_k^j(e_i)=\delta_{ij}$, and so λ_j^k is a probability measure. Taking g to be h as in Lemma 3.2 for any diagonal neighborhood N, and using the fact that $h\in \bar{M}^k$ (by Lemma 3.3), Lemma 3.4 yields $\psi_j^k(g(h)_j)=0$. Since $g(h)_j\in C(\Lambda^k)_+$ and $dh(x, e_j)\geqslant 1$ for $x\in R_+^k-N$, $\lambda_j^k(\Delta^k-N)=0$. It follows that λ_j^k is supported on $S_\perp^k$, and in the sequel we will also view it as a measure on $S_\perp^k$. Accordingly, for any $f\in\bar{M}^k$ such that $g(f)_j\in C(\Lambda^k)$,

$$\psi_j^k(g(f)_j)=\int_{S_\perp^k} df(\mathbf{1}_k, y, e_j)\, d\lambda_j^k(y). \tag{24}$$

LEMMA 3.5. *The representation* (24) *holds for any $f\in\bar{M}^k$ such that $g(f)_j$ is continuous λ_j^k-a.e.*

Proof. Given $f\in\bar{M}^k$ such that $g(f)_j$ is continuous λ_j^k-a.e., there is clearly a lower semi-continuous function g_1, and an upper semi-continuous function g_2, both of which are bounded, such that $g_1\leqslant g(f)_j\leqslant g_2$ and the inequalities hold as equalities at the continuity points of $g(f)_j$ (which are λ_j^k-a.e.). From [3, Theorem 12.7.8] there are bounded sequences $\{g_1^n\}_{n\geqslant 1}$, $\{g_2^n\}_{n\geqslant 1} \subset C(\Lambda^k)$ such that $g_1^n \leqslant g_1$, $g_2 \leqslant g_2^n$ for every n, and $g_1^n \to g_1$, $g_2^n \to g_2$ pointwise. By the positivity of ψ_j^k

$$\psi_j^k(g_1^n) \leqslant \psi_j^k(g(f)_j) \leqslant \psi_j^k(g_2^n). \tag{25}$$

Applying (23) to g_1^n, g_2^n and using the bounded convergence theorem, we have

$$\psi_j^k(g(f)_j) = \int g(f)_j \, d\lambda_j^k(x). \quad \blacksquare \tag{26}$$

Our next step is to show that the measure λ_j^k is determined entirely by ψ_j^k.

For a concave function f: $R_+^k \to R$ and $x \in R_{++}^k$, denote by $\partial f(x)$ the set of supergradients of f at x—all $p \in R^k$ that satisfy $f(x) - f(y) \geqslant p \cdot (x-y)$ for every $y \in R_+^k$. Then, for y, $z \in R^k$ [12, Theorem 23.4]

$$df(x, y) = \min_{p \in \partial f(x)} y \cdot p \tag{27}$$

and

$$df(x, y, z) = \min_{p \in \partial f(x)_y} z \cdot p, \tag{28}$$

where $\partial f(x)_y = \{p \in \partial f(x) \mid \min_{p' \in \partial f(x)} y \cdot p' = y \cdot p\}$ [12, Corollary 23.5.3].

LEMMA 3.6. *Let $f \in \bar{M}^k$. Then*:

(i) *for every $z \in R^k$ the function $df(\mathbf{1}_k, \cdot, z)$ is constant on any half-plane in R^k that has D_k as its boundary*;

and, for almost every $y \in S_\perp^k$ (with respect to the Haar measure on $S_\perp^k$):

(ii) *the functions $df(\mathbf{1}_k, \cdot, z)$ are continuous at y for all z*;

(iii) *if $x_n \xrightarrow[n\to\infty]{} \mathbf{1}_k$ from direction y, then $df(x_n, z) \xrightarrow[n\to\infty]{} df(\mathbf{1}_k, y, z)$ for all z.*

Proof. It suffices to consider only $f \in \bar{M}_+^k$. Since $df(\mathbf{1}_k, \mathbf{1}_k) = df(\mathbf{1}_k, -\mathbf{1}_k) = f(\mathbf{1}_k)$, $\mathbf{1}_k \cdot p = f(\mathbf{1}_k)$ for all $p \in \partial f(\mathbf{1}_k)$ by (27). Therefore (i) follows from (28).

For almost every $y \in R^k$ $\min_{p \in \partial f(\mathbf{1}_k)} y \cdot p$ is attained at a unique point, i.e., $\partial f(\mathbf{1}_k)_y$ contains a single point. Since $\mathbf{1}_k \cdot p$ is constant on $\partial f(\mathbf{1}_k)$, $\min_{p \in \partial f(\mathbf{1}_k)} y \cdot p$ is attained at a unique point for almost every $y \in S_\perp^k$. Fix one such $y = y_0$. Then $\partial f(\mathbf{1}_k)_{y_0}$ contains only one element. By [12, Theorem 24.5], applied to the function $g(y) = df(\mathbf{1}_k, y)$ at the point y_0,

$\partial g(y) = \partial f(\mathbf{1}_k)_y$ approaches $\partial f(\mathbf{1}_k)_{y_0}$ as y tends to y_0, and so, by (28), the functions $df(\mathbf{1}_k, y, z)$ are continuous at y_0 for all z. Since it happens for almost every y_0, (ii) is established. Point (iii) follows similarly, using [12, Theorem 24.6]. ∎

Given a convex and compact subset C of R, denote by f_C the function on R^k_+ defined by $f_C(x) = \min_{c \in C} x \cdot c$. By the envelope theorem, $C(y) \equiv \{c \in C \mid y \cdot c = f_C(y)\}$ is the set of supergradients to f_C at y:

$$C(y) = \partial f_C(y). \tag{29}$$

Let M^k be the linear space spanned by all functions f_C with compact and strictly convex[9] $C \subset \mathbf{\Delta}^k$. Observe that $M^k \subset \bar{M}^k$, and that $f_C \in M^k$ for all compact and strictly convex $C \subset \{x \in R^k \mid x \cdot \mathbf{1}_k = a\}$, for any $a \in R$.

LEMMA 3.7. *For all $f \in M^k$ $g(f)_j \in C(\Lambda^k)$.*

Proof. For a strictly convex and compact $C \subset \mathbf{\Delta}^k$ the function f_C is continuously differentiable on $R^k - D^k$. By the strict convexity of C, for $y \notin D^k$ the set $C(y)$ is a singleton. By (29), $\partial f_C(y)$ is also a singleton, and so f_C is differentiable on $\mathbf{\Delta}^k - \{\mathbf{1}_k\}$; being a concave function, it is continuously differentiable there as well. This also implies that $df_C(\mathbf{1}_k, \cdot, z) = df_C(\cdot, z)$ are continuous at $y \notin D^k$. Thus $g(f)_j$ is continuous on $S^k_\perp$ and on the interior of $\mathbf{\Delta}^k - \{\mathbf{1}_k\}$.

Now, let $x_n \to y$ in the topology on Λ^k. Then $df_C(x_n, z) = df_C(\mathbf{1}_k, x_n, z) = df_C(\mathbf{1}_k, \frac{x_n - \mathbf{1}_k}{\|x_n - \mathbf{1}_k\|}, z)$, where the last equality holds by (i) of Lemma 3.6. This expression converges to $df_C(\mathbf{1}_k, y, z)$ by continuity of the function $df_C(\mathbf{1}_k, \cdot, z)$ that we established above. Thus $g(f)_j \in C(\Lambda^k)$. ∎

PROPOSITION 3.8. *The space $\{df(\mathbf{1}_k, y, e_j) \mid f \in M^k\}$ is dense in $C(S^k_\perp)$.*

Proof. Let $\underline{K}$ stand for the set of all closed cones C in $(\mathbf{1}_k)^\perp = \{x \in R^k \mid x \cdot \mathbf{1}_k = 0\}$, based at 0, such that for any nonzero $x \in C$ $x_j > 0$. For $C \in \underline{K}$ denote $D(C) = \{y \in (\mathbf{1}_k)^\perp \mid \forall x \in C : y \cdot x \geqslant 0\}$. For a given $C \in \underline{K}$ denote $\hat{C} = C \cap S^k_\perp$ and $C_1 = \{x \in C \mid x_j \leqslant 1\}$. The set C_1 is compact and convex.

We claim that for any $C \in \underline{K}$:

(i) if $y \in int(\widehat{D(C)})$ (the relative interior of $\widehat{D(C)}$), then $C_1(y) = \{0\}$;

(ii) if $y \in S^k_\perp - \widehat{D(C)}$, then $x_j = 1$ for any $x \in C_1(y)$.

Before giving the proof, note that (i) and (ii) tell us that for any $y \in S^k_\perp - \partial\widehat{D(C)}$ (the relative boundary of $\widehat{D(C)}$), all elements of $C_1(y)$ have

[9] That is, the extreme points of C coincide with its relative boundary in $\mathbf{\Delta}^k$.

the same jth coordinate, and so, $C_1(y)$ being the set of supergradients of f_{C_1} at y, this coordinate is $df_{C_1}(\mathbf{1}_k, y, e_j)$. By (28), (i), and (ii),

(i′) if $y \in int(\widehat{D(C)})$, then $df_{C_1}(\mathbf{1}_k, y, e_j) = 0$;

(ii′) if $y \in S^k_\perp - \widehat{D(C)}$, then $df_{C_1}(\mathbf{1}_k, y, e_j) = 1$.

We now prove (i) and (ii). Suppose that $y \in int(\widehat{D(C)})$ and there is $x \in C_1(y)$ which is different from 0. Form the definition of $\widehat{D(C)}$, $y \cdot x = 0$. Then $y' \cdot x < 0$ for some $y' \in \widehat{D(C)}$ in a neighborhood of y, contradicting the definition of $\widehat{D(C)}$. If $y \in S^k_\perp - \widehat{D(C)}$ then there is $x \in C$, such that $y \cdot x < 0$. Therefore, for any $x \in C_1(y)$ $y \cdot x < 0$, and, in particular, $x \neq 0$. It follows then that $x_j = 1$, since otherwise there is $t > 1$ such that $tx \in C$ and certainly $y \cdot (tx) < y \cdot x$, in contradiction to the choice of x.

Suppose that $\{df(\mathbf{1}_k, y, e_j) \mid f \in M^k\}$ is not dense in $C(S^k_\perp)$. By Hahn–Banach theorem, there is then a nonzero measure λ on $S^k_\perp$ such that

$$\int df(\mathbf{1}_k, y, e_j)\, d\lambda(y) = 0 \tag{30}$$

for all $f \in M^k$. Denote $\underline{K}^\lambda = \{C \in \underline{K} \mid |\lambda|\, (\partial(\widehat{D(C)})) = 0\}$, where $|\lambda|$ is the variation of the measure λ. For any $C \in \underline{K}^\lambda$ there is a sequence $(C_{n,1})_{n=1}^\infty$ of strictly convex and compact sets in $(\mathbf{1}_k)^\perp$ such that $f_{C_{n,1}} \to f_{C_1}$ pointwise (e.g., [6]). By [12, Theorem 24.5]

$$df_{C_{n,1}}(\mathbf{1}_k, y, e_j) \xrightarrow[n\to\infty]{} df_{C_1}(\mathbf{1}_k, y, e_j) \tag{31}$$

for any y in which all elements of $\partial f_{C_1}(\mathbf{1}_k)_y = C_1(y)$ have the same jth coordinate. After noting that $f_{C_{n,1}}$ are all in M^k, (i′), (ii′), (31), (30), and the bounded convergence theorem yield

$$\lambda(\widehat{D(C)}^c) = \lim_{n\to\infty} \int df_{C_{n,1}}(\mathbf{1}_k, y, e_j)\, d\lambda(y) = 0. \tag{32}$$

As $\lambda(S^k_\perp) = 0(df(\mathbf{1}_k, y, e_j)$ can be a nonzero constant), λ is zero on any sets of the form $\widehat{D(C)}$ and $\widehat{D(C)}^c$ for $C \in \underline{K}^\lambda$.

As $\{\widehat{D(C)} \mid C \in \underline{K}\}$ is closed under intersections $(\widehat{D(C_1)} \cap \widehat{D(C_2)} = \widehat{D(C)}$, where C is a convex hull of C_1 and C_2), λ is actually determined on the algebra A generated by $\{\widehat{D(C)} \mid C \in \underline{K}^\lambda\}$ as the zero measure. It can be easily seen that A is countably generated and that it separates points, and so the σ-field generated by it is the Borel σ-field on $S^k_\perp$. Therefore λ is zero on all Borel subsets of $S^k_\perp$, i.e., it is the zero measure, contrary to our assumption. This shows that $\{df(\mathbf{1}_k, \cdot, e_j) \mid f \in M^k\}$ is indeed dense in $C(S^k_\perp)$. ∎

The following is a corollary of Lemma 3.7 and Proposition 3.8.

COROLLARY 3.9. *The measure λ^k_j is the unique measure such that*

$$\psi^k_j(g(f)_j) = \int df(\mathbf{1}_k, y, e_j)\, d\lambda^k_j(y) \tag{33}$$

holds for all $f \in M^k$.

Remark 6. For a vector $\mu \in (NA^1)^k$ of mutually singular measures and $a_1, a_2 \in \Delta^k$ let C_{a_1, a_2} be the linear segment $[a_1 \cdot \mu, a_2 \cdot \mu]$. Then

$$f_C(\mu) = \min(a_1 \cdot \mu, a_2 \cdot \mu) = \tfrac{1}{2}((a_1 + a_2) \cdot \mu - |(a_1 - a_2) \cdot \mu|). \tag{34}$$

Since $(a_1 - a_2) \cdot \mu = \eta_1 - \eta_2$ for two mutually singular non-atomic non-negative measures η_1 and η_2 with the same total mass, $\Psi(|(a_1 - a_2) \cdot \mu|) = \Psi(|\eta_1 - \eta_2|) = 0$, from Remark 4. By Remark 5 and linearity of Ψ,

$$\Psi(f_{C_{a_1, a_2}} \circ \mu) = \tfrac{1}{2}(a_1 + a_2) \cdot \mu. \tag{35}$$

Denoting $w = a_1 - a_2$, this and Lemma 3.5 imply that whenever $\lambda^k_j(\{x \mid w \cdot x = 0\}) = 0$,

$$\lambda^k_j(\{x \mid w \cdot x \geqslant 0\}) = \lambda^k_j(\{x \mid w \cdot x \leqslant 0\}) = \tfrac{1}{2}. \tag{36}$$

Since $\lambda^k_j(\{x \mid w \cdot x\}) = 0$ for almost every $w \in S^k_\perp$, (36) also holds almost everywhere.

Remark 7. Equality (36) in Remark 6 for almost every w is well-known to imply that the measure λ^k_j is symmetric with respect to reflections, i.e., for any measurable $S \subset S^k_\perp$

$$\lambda^k_j(S) = \lambda^k_j(-S) \tag{37}$$

(e.g., [13, Lemma 2.3]).

Given $x \in R^k - D_k$, denote by $\Upsilon^k(x)$ the point of intersection of $S^k_\perp$ with the half-plane that contains the point x and has D_k as its boundary. For $\alpha \in [0, 1]$, and $0 \leqslant j \leqslant k$, let $\Gamma^\alpha_j \colon S^{k+1}_\perp \to R^k$ be the map given for $x = (x_1, ..., x_{k+1})$ by $\Gamma^\alpha_j(x)_l = x_l$ if $l \neq j$, and $\Gamma^\alpha_j(x)_j = \alpha x_{k+1} + (1 - \alpha)\, x_j$. Also denote $\Upsilon^\alpha_j = \Upsilon^k \circ \Gamma^\alpha_j$, whenever it is defined.

Remark 8. Let $\mu=(\mu_1, \mu_2, ..., \mu_{k+1}) \in (NA^1)^{k+1}$ be mutually singular, $f \in M^k$, $\alpha \in [0, 1]$ and $1 \leqslant j \leqslant k$. Define $f_j^\alpha \in \bar{M}^{k+1}$ by

$$f_j^\alpha(x) = f(\Gamma_j^\alpha(x)). \tag{38}$$

By (14) of Remark 4, applied to the two representations $f_j^\alpha(\mu_1, \mu_2, ..., \mu_{k+1})$ and $f(\Gamma_j^\alpha(\mu_1, \mu_2, ..., \mu_{k+1}))$ of the same game,

$$\Psi(f_j^\alpha(\mu))(S(\mu_{k+1})) \cdot \mu_{k+1} + \Psi(f_j^\alpha(\mu))(S(\mu_j)) \cdot \mu_j \tag{39}$$

$$= \Psi(f(\Gamma_j^\alpha(\mu)))(S(\mu_j) \cup S(\mu_{k+1})) \cdot (\alpha\mu_{k+1} + (1-\alpha)\, \mu_j). \tag{40}$$

This equals

$$\Psi(f \circ (\mu_1, ..., \mu_k))(S(\mu_j)) \cdot (\alpha\mu_{k+1} + (1-\alpha)\, \mu_j), \tag{41}$$

again by Remark 4. It follows that

$$\Psi(f \circ \Gamma_j^\alpha(\mu))(S(\mu_{k+1})) = \alpha \Psi(f \circ (\mu_1, ..., \mu_k))(S(\mu_j)), \tag{42}$$

and therefore

$$\psi_{k+1}^{k+1}(g(f_j^\alpha)_{k+1}) = \alpha \psi_j^k(g(f)_j). \tag{43}$$

LEMMA 3.10. *For any $k \geqslant 2$ and $1 \leqslant j \leqslant k$*

$$\lambda_j^k = \Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1} \tag{44}$$

(*i.e., for any Borel set $K \subset S_\perp^k$, $\lambda_j^k(K) = \lambda_{k+1}^{k+1}((\Upsilon_j^\alpha)^{-1}(K)))$ for almost every $\alpha \in [0, 1]$.*

Proof. Observe that for almost every $\alpha \in (0, 1]$

$$\lambda_{k+1}^{k+1}(\{ y \in S_\perp^{k+1} \mid \Gamma_j^\alpha(y) \in D_k \}) = 0, \tag{45}$$

since the sets $(\Gamma_j^\alpha)^{-1}(D_k)$ are disjoint for different α. If $f \in M^k$ then $\{ y \in S_\perp^{k+1} \mid \Gamma_j^\alpha(y) \in D_k \}$ contains the set of discontinuity points of $df_j^\alpha(\mathbf{1}_{k+1}, \cdot, e_{k+1})$.

Given $f \in M^k$ and α that satisfies (45), Remark 8 yields

$$\psi_j^k(g(f)_j) = \frac{1}{\alpha} \psi_{k+1}^{k+1}(g(f_j^\alpha)_{k+1}). \tag{46}$$

By applying Lemma 3.5 to f_j^α, this is equal to

$$\frac{1}{\alpha}\int df_j^\alpha(\mathbf{1}_{k+1}, (y_1, ..., y_{k+1}), e_{k+1})\, d\lambda_{k+1}^{k+1}(y_1, ..., y_{k+1}) \tag{47}$$

$$=\int df(\mathbf{1}_k, \Gamma_j^\alpha(y_1, ..., y_{k+1}), e_j)\, d\lambda_{k+1}^{k+1}(y_1, ..., y_{k+1}). \tag{48}$$

By (i) of Lemma 3.6 the last integral equals

$$\int df(\mathbf{1}_k, \Upsilon_j^\alpha(y_1, ..., y_{k+1}), e_j)\, d\lambda_{k+1}^{k+1}(y_1, ..., y_{k+1})$$

$$=\int df(\mathbf{1}_k, (y_1, ..., y_k), e_j)\, d(\Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1})(y_1, ..., y_k). \tag{49}$$

We have shown that for all $f \in M^k$

$$\Psi_j^k(g(f)_j) = \int df(\mathbf{1}_k, y, e_j)\, d(\Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1})(y). \tag{50}$$

Therefore, by Corollary 3.9,

$$\Upsilon_j^\alpha \circ \lambda_{k+1}^{k+1} = \lambda_j^k. \quad \blacksquare \tag{51}$$

LEMMA 3.11. *The assertion of Lemma* 3.10 *holds for* $\alpha = 0$, *provided* $k \geqslant 3$.

Proof. There is a sequence $\alpha_n > 0$ satisfying (45) in Lemma 3.10, such that $\alpha_n \to 0$. Let $1 \leqslant j \leqslant k$ and $f \in M^k$. As in the proof of Lemma 3.10, for every n

$$\psi_j^k(g(f)_j) = \int df(\mathbf{1}_k, \Gamma_j^{\alpha_n}(y_1, ..., y_{k+1}), e_j)\, d\lambda_{k+1}^{k+1}(y_1, ..., y_{k+1}). \tag{52}$$

A limit as $n \to \infty$ of the right-hand side can be taken. By continuity of $df(\mathbf{1}_k, \cdot, e_j)$ in all $y \in R_+^k - D_k$, (1) of Lemma 3.6 and the bounded convergence theorem, this limit is equal to

$$\int_{(y_1, ..., y_k) \notin D_k} df(\mathbf{1}_k, (y_1, ..., y_k), e_j)\, d\lambda_{k+1}^{k+1}(y_1, ..., y_{k+1})$$

$$+ df(\mathbf{1}_k, e_j, e_j)\, \lambda_{k+1}^{k+1}(\{a_{k+1}\}) + df(\mathbf{1}_k, -e_j, e_j)\, \lambda_{k+1}^{k+1}(\{-a_{k+1}\}), \tag{53}$$

where

$$a_{k+1} \in S_{\perp}^{k+1}, \qquad (a_{k+1})_l = \begin{cases} \dfrac{k}{\sqrt{k^2+k}}, & l = k+1 \\ -\dfrac{1}{\sqrt{k^2+k}}, & \text{otherwise.} \end{cases}$$

By the efficiency of Ψ,

$$\sum_{j=1}^{k} \psi_j^k(g(f)_j) = f(\mathbf{1}_k). \tag{54}$$

Also, as $f \in M^k$,

$$\sum_{j=1}^{k} df(\mathbf{1}_k, y, e_j) = f(\mathbf{1}_k) \tag{55}$$

for any $y \in R^k$. Therefore, summing the right hand and the left-hand sides of the equality (53) over $j = 1, ..., k$, we obtain

$$f(\mathbf{1}_k) = f(\mathbf{1}_k)\, \lambda_{k+1}^{k+1}(\{ y \mid \Gamma_j^0(y) \notin D_k \}) + \sum_{j=1}^{k} [df(\mathbf{1}_k, e_j, e_j)\, \lambda_{k+1}^{k+1}(\{a_{k+1}\}) \\ + df(\mathbf{1}_k, -e_j, e_j)\, \lambda_{k+1}^{k+1}(\{-a_{k+1}\})]. \tag{56}$$

If $k \geqslant 3$ one can easily find closed and strictly convex $C_1, C_2 \subset \Delta^k$ such that $C_1(e_j) = C_2(e_j)$ for $j \neq 1$, $C_1(-e_1) = C_2(-e_1)$, and $C_1(e_1)$ differs from $C_2(e_1)$ in the first coordinate (e.g., take a strictly convex C_1 and modify it slightly at a small neighborhood of the singleton $C_1(e_1)$, preserving strict convexity, and use the resulting set as C_2). By (29) and (28) the function $f = f_{C_1} - f_{C_2}$ violates the equality (56) if $\lambda_{k+1}^{k+1}(\{a_{k+1}\}) > 0$, and so $\lambda_{k+1}^{k+1}(\{a_{k+1}\}) = 0$. Similarly it can be shown that $\lambda_{k+1}^{k+1}(\{-a_{k+1}\}) = 0$.

The last two summands in (53) are consequently equal to zero, and so it can be rewritten as

$$\Psi_j^k(g(f)_j) = \int df(\mathbf{1}_k, x, e_j)\, d\Upsilon_j^0 \circ \lambda_{k+1}^{k+1}(x). \tag{57}$$

Since it holds for all $f \in M^k$, $\lambda_j^k = \Upsilon_j^0 \circ \lambda_{k+1}^{k+1}$ by Corollary 3.9. ∎

Corollary 3.12. *The measure λ_j^k in Corollary 3.9 is independent of j.*

Proof. Since clearly $\Upsilon_1^0 = \Upsilon_j^0$, for any $j = 1, ..., k$

$$\lambda_1^k = \Upsilon_1^0 \circ \lambda_{k+1}^{k+1} = \Upsilon_j^0 \circ \lambda_{k+1}^{k+1} = \lambda_j^k$$

by Lemma 3.11. ∎

The measure λ_j^k will be denoted by λ^k from now on.

Given a permutation θ of $\{1, ..., k\}$, denote $\theta x = (x_{\theta(1)}, ..., x_{\theta(k)})$ for any $x \in R^k$. The following lemma is implied by the anonymity of value.

Lemma 3.13. *The measure λ^k is symmetric with respect to permutations of coordinates, i.e., it satisfies $d\lambda^k(x) = d\lambda^k(\theta x)$ for any permutation θ of $\{1, ..., k\}$.*

Proof. From (13) in Remark 4, for every $f \in M^k$

$$\psi_j^k(g(f)_j) = \psi_{\theta(j)}^k(g(\theta f)_{\theta(j)}), \tag{58}$$

where $\theta f(x) = f(\theta x)$. The right hand term is equal to

$$\int g(\theta f)_{\theta(j)}(x)\, d\lambda^k(x) = \int g(f)_j(x)\, d\lambda^k(\theta^{-1} x). \tag{59}$$

By Corollary 3.9, $d\lambda^k(x) = d\lambda^k(\theta^{-1} x)$. ∎

The following lemma uses a trick in the spirit of (B) in [10, pp. 29–30]. Let $(U_i)_{i=1}^\infty$ be a sequence of identically independently distributed (i.i.d.) random variables, each with the standard Cauchy distribution, and denote by κ^k the distribution of $\Upsilon^k(U_1, ..., U_k)$.

Lemma 3.14. *Suppose that for each $k \geqslant 3$ λ^k is a probability measure on $S_\perp^k$ such that:*

(i) $\lambda^k = \Upsilon_k^\alpha \circ \lambda^{k+1}$ *for every k and almost every α in $[0, 1]$;*

(ii) *specifically, it holds for $\alpha = 0$;*

(iii) λ^k *is symmetric with respect to permutations of coordinates;*

(iv) λ^k *is symmetric with respect to reflections.*

Then $\lambda^k = \kappa^k$ for all $k \geqslant 3$.

Proof. Take $w \in S^3_\perp$ be such that $w \cdot (x_1, x_2, x_3) \neq 0$ almost everywhere for a λ^3-distributed (x_1, x_2, x_3). By successive applications of property (ii), $w \cdot (x_1, x_2, x_3) \neq 0$ almost everywhere for a λ^n-distributed $(x_1, ..., x_n)$.

Note that for any $(x_1, ..., x_n) \in R^n - D_n$ such that $w \cdot (x_1, x_2, x_3) \neq 0$,

$$\frac{x_i - x_4}{w \cdot (x_1, x_2, x_3)} = \frac{\Upsilon^k(x)_i - \Upsilon^k(x)_4}{w \cdot (\Upsilon^k(x)_1, \Upsilon^k(x)_2, \Upsilon^k(x)_3)} \tag{60}$$

for every $i = 5, ..., n$. Consider now random variables $z_i = \frac{x_i - x_4}{w \cdot (x_1, x_2, x_2)}$, $i = 5, ..., n$, for λ^n-distributed $(x_1, ..., x_n)$. By property (ii) and the equalities (60), $(z_i)_{i=5}^n$ have the same distribution if they are induced by λ^n-distributed $(x_1, ..., x_n)$ or λ^{n+1}-distributed $(x_1, ..., x_{n+1})$. Thus the distribution of a sequence $(z_i)_{i=5}^k$ is independent of λ^n-distributed $(x_1, ..., x_n)$, for any $n \geqslant k$. It also follows that the distribution of $(z_i)_{i=5}^{k+1}$ extends that of $(z_i)_{i=5}^k$, and so, by Kolmogorov extension theorem, there is a unique limiting distribution of an infinite sequence $(z_i)_{i=5}^\infty$, which extends the distribution of $(z_i)_{i=5}^k$ for each $k \geqslant 5$. The sequence $(z_i)_{i=5}^\infty$ of random variables is finitely exchangeable by property (iii), i.e., $(z_i)_{i=5}^\infty$ and $(z_{\theta(i)})_{i=5}^\infty$ have the same distribution, for a permutation θ on $\{5, 6, ...\}$ with $\theta(i) = i$ for all but finitely many i. By de-Finetti exchangeability principle (e.g., [9]), $(z_i)_{i=5}^\infty$ are, conditionally on the σ-algebra of tail events F_∞, independent and identically distributed (i.i.d.). By equality (60) and property (i) the sequence $(z_5, z_6, ..., z_{k-1}, \alpha z_{k+1} + (1-\alpha) z_k)$ has the same distribution as $(z_i)_{i=5}^k$ for almost every (and therefore every) $\alpha \in [0, 1]$, for any $k \geqslant 5$. By property (iii), the sequence $(\alpha z_5 + (1-\alpha) z_6, z_7, ..., z_{k+1})$ has the same distribution as $(z_i)_{i=5}^k$. Since it holds for every $k \geqslant 5$, the infinite sequence $(\alpha z_5 + (1-\alpha) z_6, z_7, z_8, ...)$ has the same distribution as $(z_i)_{i=5}^\infty$. It means that for every $\alpha \in [0, 1]$, conditionally on F_∞, the random variables $\alpha z_5 + (1-\alpha) z_6$, z_5, z_6 have the same distribution, and so, z_5 and z_6 being i.i.d., they must have a stable of degree 1 distribution. Therefore, conditionally on F_∞, $(z_i)_{i=5}^\infty$ are distributed as $(m + \sigma U_i)_{i=5}^\infty$, where m, σ are measurable with respect to F_∞, and $(U_i)_{i=5}^\infty$ are i.i.d. with the standard Cauchy distribution.

Observe that $\sigma \neq 0$ with probability 1, since we would otherwise get $x_5 = x_6 = \cdots = x_k$, for any k, with some positive probability, in contradiction to property (ii). Conditionally on F_∞, $\frac{z_k - z_5}{z_6 - z_5} = \frac{(m + \sigma U_k) - (m + \sigma U_5)}{(m + \sigma U_6) - (m + \sigma U_5)} = \frac{U_k - U_5}{U_6 - U_5}$. Therefore, for λ^n-distributed $(x_1, x_2, ..., x_n)$ $(n \geqslant 7)$ the sequence $(\frac{x_k - x_5}{x_6 - x_5})_{i=7}^n = (\frac{z_k - z_5}{z_6 - z_5})_{i=7}^n$ is distributed as $(\frac{U_k - U_5}{U_6 - U_5})_{i=7}^\infty$. By property (iii), $(\frac{x_k - x_1}{x_2 - x_1})_{i=3}^{n-5}$ is distributed as $(\frac{U_k - U_1}{U_2 - U_1})_{i=3}^{n-5}$. Bearing in mind property (iv), the distribution of the sequence $(\frac{x_k - x_1}{x_2 - x_1})_{i=3}$ determines in a unique way the distribution of $\Upsilon^{n-5}(x_1, ..., x_{n-5})$ as $\Upsilon^{n-5}(U_1, ..., U_{n-5})$. On the other hand, by successive applications of property (ii), $\Upsilon^{n-5}(x_1, ..., x_{n-5})$ is λ^{n-5} distributed, if $n \geqslant 8$. Therefore, $\lambda^k = \kappa^k$ for $k \geqslant 3$. ∎

By Lemmas 3.10, 3.11 and 3.13, and Remark 7, the sequence $(\lambda^k)_{k=3}^{\infty}$ in our proof satisfies the conditions of Lemma 3.14, and therefore $\lambda^k = \kappa^k$ for $k \geqslant 3$. The measures κ^k are nonatomic, and so the proof of Lemma 3.11 shows that λ^2 is also determined uniquely, by $\lambda^3 = \kappa^3$.

Since κ^k is absolutely continuous with respect to the Haar measure on $S_\perp^k$, parts (ii) and (iii) of Lemma 3.6 imply that for every $f \in \bar{M}^k$ the function $g(f)$ is continuous λ^k-almost everywhere on Λ^k. Therefore, by Lemma 3.5, (24) holds for every $f \in \bar{M}^k$. Since the measure λ^k does not depend on a particular value satisfying the axioms in Theorem 2.1,

$$\Psi(f \circ \mu) = \Psi_M(f \circ \mu) \tag{61}$$

for any $f \in \bar{M}^k$ and mutually singular $(\mu_i)_{i=1}^k \in (NA^1)^k$, by Remark 4.

Given $f \in \bar{\bar{M}}_+^k$, consider homogeneous of degree 1 functions f^n, given on Δ^k by

$$f^n(x) = \min(f(x), f_{C_n}(x)),$$

where C_n is the convex set $\{x \in R^k \mid x \cdot \mathbf{1}_k = f(\mathbf{1}_k) + 1, \ \|x\| \leqslant n\}$. Being concave and positively homogeneous of degree 1, the function f is Lipschitz on any conical diagonal neighborhood whose boundary does not intersect the boundary of R_+^k (with the exception of zero). For large enough n, f^n coincides with f on one such neighborhood, and with a Lipschitz function on its complement in R_+^k. Therefore f^n is Lipschitz on R_+^k, and, being concave as a minimum of concave functions, it is an element of $\bar{M}^k$. Thus, by (61),

$$\Psi(f^n \circ \mu) = \Psi_M(f^n \circ \mu) \tag{62}$$

On the other hand, since f coincides with f^n on a conical diagonal neighborhood, by Lemma 3.4

$$\Psi(f \circ \mu - f^n \circ \mu) = \Psi(f \circ \mu) - \Psi(f^n \circ \mu) = 0, \tag{63}$$

and the same equality holds for Ψ_M. Hence (62) and (63) show that

$$\Psi(f \circ \mu) = \Psi_M(f \circ \mu) \tag{64}$$

for any $f \in \bar{\bar{M}}^k$ and mutually singular $\mu \in (NA^1)^k$.

By Lemma 3.1 every market game is of the form $f \circ \mu$, for some k, $f \in \bar{\bar{M}}_+^k$ and mutually singular $\mu \in (NA^1)^k$, and so Ψ coincides with the Mertens value on MG. This concludes our proof.

REFERENCES

1. R. J. Aumann, Values of markets with a continuum of traders, *Econometrica* **32** (1964), 39–50.
2. R. J. Aumann and L. S. Shapley, "Values of Non Atomic Games," Princeton Univ. Press, Princeton, NJ, 1974.
3. J. Dieudonne, "Treatise on Analysis," Vol. II, Academic Press, New York, 1976.
4. P. Dubey and A. Neyman, Payoffs in nonatomic economies: An axiomatic approach, *Econometrica* **52** (1984), 1129–1150.
5. P. Dubey and A. Neyman, An equivalence principle for perfectly competitive economies, *J. Econ. Theory* **75** (1997), 314–344.
6. P. M. Gruber, Approximations of convex bodies, *in* "Handbook of Convex Geometry," North-Holland, Amsterdam, 1993.
7. J. C. Harsanyi and R. Selten, "A General Theory of Equilibrium Selection in Games," MIT press, Cambridge, MA, 1988.
8. S. Hart, Measure-based values of market games, *Math. Oper. Res* **5** (1980), 197–228.
9. M. Loeve, "Probability Theory," second ed., Van Nostrand, New York, 1960.
10. J. F. Mertens, The Shapley value in the non-differentiable case, *Internat. J. Game Theory* **17** (1988), 1–65.
11. J. F. Mertens, Nondifferentiable TU markets: The value, *in* "The Shapley Value: Essays in Honor of Lloyd S. Shapley" (A. E. Roth, Ed.), Chap. 15, Cambridge Univ. Press, Cambridge, UK, 1988.
12. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
13. B. Rubin, Inversion and characterization of hemispherical transform in spherical harmonics, *J. Anal. Math.* **77** (1999), 105–128.
14. L. S. Shapley, A value for *n*-person games, *in* "Contributions to the Theory of Games" (H. W. Kuhn and A. W. Tucker, Eds.), Vol. II, Princeton Univ. Press, Princeton, NJ, 1953.
15. A. C. Zaanen, "Riesz Spaces II," North-Holland Mathematical Library, North-Holland, Amsterdam, 1983.