האוניברסיטה העברית בירושלים

THE HEBREW UNIVERSITY OF JERUSALEM

THE GAME FOR THE SPEED OF CONVERGENCE IN REPEATED GAMES OF INCOMPLETE INFORMATION

by

IRIT NOWIK and SHMUEL ZAMIR

Discussion Paper # 187 January 1999

מרכז לחקר הרציונליות
וההחלטות האינטראקטיביות

CENTER FOR RATIONALITY
AND INTERACTIVE DECISION THEORY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@sunset.ma.huji.ac.il
WWW: http://www.ma.huji.ac.il/~ranb

THE GAME FOR THE SPEED OF CONVERGENCE IN REPEATED GAMES OF INCOMPLETE INFORMATION

IRIT NOWIK AND SHMUEL ZAMIR

ABSTRACT. We consider an infinitely repeated zero-sum two-person game with incomplete information on one side, in which the maximizer is the (more) informed player. Such games have value $v_\infty(p)$ for all $0 \le p \le 1$. The informed player can guarantee that all along the game, the average payoff per stage will be greater or equal to $v_\infty(p)$, (and will converge from above to $v_\infty(p)$ if the minimizer plays optimally). Thus there is a conflict of interests between the two players regarding the speed of convergence of the average payoffs, to the value $v_\infty(p)$. In the context of such repeated games, we define a Game, denoted as $SG_\infty(p)$, for the speed of convergence, and a value for this game. We prove that the value exists for games with the highest error term, namely games in which $v_n(p) - v_\infty(p)$ is of the order of magnitude of $\frac{1}{\sqrt{n}}$. In that case the value of $SG_\infty(p)$ is also of the order of magnitude of $\frac{1}{\sqrt{n}}$. Then we show that in another class of games, the value does not exist.

For our first result we define for any infinite martingale $\mathfrak{X}^\infty = \{X_n\}_{n=1}^\infty$, the variation of it: $V_n(\mathfrak{X}^\infty) := E\sum_{k=1}^n |X_{k+1} - X_k|$, and prove that the variation of a *uniformly bounded,* infinite martingale $\mathfrak{X}^\infty$, can be of the order of magnitude of $n^{\frac{1}{2}-\varepsilon}$, for arbitrarily small $\varepsilon > 0$.

1. INTRODUCTION

In this paper we treat a two-person 0-sum repeated game, with incomplete information on one side (see e.g p.116 of [10]): Let A_1, A_2 be 2×2 matrices, each corresponding to the payoff of a two person zero-sum game, with elements: a_{ij}^k, where $k \in \{1,2\}$ represents the number of the matrix and $i \in I = \{T, B\}$ and $j \in J = \{L, R\}$ are the pure strategies of PI (the Maximizer) and PII (the minimizer) respectively. For each p, $0 \le p \le 1$, we consider the n-stage repeated game $G_n(p)$ defined as follows:

- At stage 0, chance chooses $k = 1$ with probability p, and $k = 2$ with probability $p' = 1 - p$. Both players know p, but (only) PI is informed also about the chosen value of k.
- At stage 1, PI chooses $i_1 \in I$, PII chooses $j_1 \in J$, and (i_1, j_1) is publicly announced.
- At stage m, $m = 2, 3..$ knowing $(i_1, j_1) \dots (i_{m-1}, j_{m-1})$, PI (resp.PII) chooses $i_m \in I$ (resp. $j_m \in J$,) and then (i_m, j_m) is announced.

Date: October 22, 1998.

- After stage n, PI receives from PII the following amount:

$$\frac{1}{n}\sum_{m=1}^{n} a^k_{i_m j_m}$$

(We divide by n in order to compare payoffs in $G_n(p)$ for different values of n.) When n is finite, $G_n(p)$ is a finite game and therefore has a (minmax) value (in mixed strategies.), which we denote by $v_n(p)$.

The strategies in $G_n(p)$:

Denote by h_m, the r.v that represents the history of announcements up to stage m: $(i_1, j_1)\dots(i_{m-1}, j_{m-1})$ and by $H_m = (I \times J)^{m-1}$, the set of all $m-$stage histories. ($H_1 = \varnothing$.)

A (behavioral) strategy for PI is: $\sigma_n = (\sigma_n^1, \sigma_n^2)$ where for each $k \in \{1,2\}$: $\sigma_n^k = (s_1^k, \dots s_n^k)$, and for all m, $1 \leq m \leq n$, s_m^k is a function from H_m into the set of probability distributions on I. The interpretation of σ_n is as follows: If k was chosen, then at stage m, given h_m, PI will choose T with probability $s_m^k(h_m)$.

A (behavioral) strategy for PII is: $\tau_n = (t_1 \dots t_n)$ where for all m, $1 \leq m \leq n$, t_m is a function from H_m into the set of probability distributions on J.

Remark 1.1. *The difference in the structure of the strategies of the two players is due to the fact that only PI knows the chosen value of* k, *and therefore can play differently in each of the two matrices.*

We define now the infinitely repeated game $G_\infty(p)$, as follows:

A strategy for PI in $G_\infty(p)$ is: $\sigma = (\sigma^1, \sigma^2)$, where for all $k \in \{1,2\}$, σ^k is an infinite sequence $\{s_n^k : n \geq 1\}$, and each s_n^k is a function from H_n into the set of probability distributions on I. A strategy for PII in $G_\infty(p)$, is: τ , where $\tau = \{t_n : n \geq 1\}$, and t_n is a function from H_n into the set of probability distributions on J.

$G_\infty(p)$ is a model for a game with a very large number of stages, in which the players do not know the exact number of stages that are going to be played.

In $G_\infty(p)$ the natural definition of payoff would be:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{n=1}^{n} a^k_{i_m j_m}\right)$$

but this limit may fail to exist for any pair of strategies. (see e.g Aumann and Maschler p.188 of [1].) Nevertheless the value $v_\infty(p)$, can be defined without defining the payoff: (see e.g p.187 in [1].)

For any pair of strategies, σ, τ, let $\gamma_n(\sigma,\tau)$ be the average expected payoff for the n first stages in $G_\infty(p)$ (or in any $G_l(p)$ $l \geq n$,) when σ and τ are played. that is:

$$\gamma_n(\sigma,\tau) = E_{p,\sigma,\tau}\left(\frac{1}{n}\sum_{m=1}^{n} a^k_{i_m j_m}\right).$$

($E_{p,\sigma,\tau}$ is the expectation with respect to the probability measure on H_{n+1} induced by p,σ,τ). From now on, we use σ and τ to denote strategies of PI and PII respectively, in the game $G_\infty(p)$.

Definition 1.2.

- *We say that PI can guarantee* $f(p)$ *in* $G_\infty(p)$, *if for any* $\varepsilon > 0$ *there is* σ_ε *and* N_ε, *such that:*

$$\gamma_n(\sigma_\varepsilon,\tau) - f(p) \geq -\varepsilon \quad \forall\tau, \qquad \forall n > N_\varepsilon.$$

- *We say that PII can guarantee* $g(p)$ *in* $G_\infty(p)$, *if for any* $\varepsilon > 0$ *there is* τ_ε *and* N_ε, *such that:*

$$\gamma_n(\sigma,\tau_\varepsilon) - g(p) \leq \varepsilon \quad \forall\sigma, \qquad \forall n > N_\varepsilon.$$

- *We say that* $G_\infty(p)$ *has a value* $v_\infty(p)$ *if both players can guarantee* $v_\infty(p)$.

Remark 1.3. *An alternative definition for the value of an infinitely repeated game would be the limit of the values of the n-stage games* $G_n(p)$. *namely:* $\hat{v}_\infty(p) := \lim_{n\to\infty} v_n(p)$ *(if this limit exists.) (see Zamir* [9]*).*

Denote: $D(p)$ as the game with payoff-matrix of $pA^1 + p'A^2$ (see e.g definition 3.10 p.123 of [10]). $D(p)$ can be interpreted as the 1-stage game, in which *both* players are not informed of the matrix chosen. Let $u(p) = valD(p)$, and $Cav\, u(p)$ be the smallest concave function , that is greater or equal to $u(p)$ on $[0,1]$.

Theorem(Aumann and Maschler): $v_\infty(p)$ and $\lim_{n\to\infty} v_n(p)$ both exist, and:

$$v_\infty(p) = \lim_{n\to\infty} v_n(p) = Cav\, u(p).$$

(see e.g proposition A, p.187 and Theorem C, p.191 of [1].)

In other words; In this model as long as we are interested only in the value of $G_\infty(p)$, both concepts: 'the value of the limit-game', and the 'limit of the values' (of the finite games), lead to the same result.

Aumann and Maschler constructed a strategy σ^* in $G_\infty(p)$ (namely the splitting strategy. see e.g p.126 of [10],) such that:

$$\inf_\tau \gamma_n(\sigma^*,\tau) \geq Cav\, u(p), \quad \forall n.$$

This implies: $v_n(p) \geq Cav\, u(p), \quad \forall n,$

The difference: $e_n(p) := v_n(p) - Cav\, u(p)$ is called the *error-term* of the game. The error-term is the extra gain that PI can guarantee (over $Cav\, u(p)$) in $G_n(p)$.

In analogy with the error-term, given σ in $G_\infty(p)$, let us look at:

$$\inf_\tau \gamma_n(\sigma,\tau) - Cav\, u(p).$$

This is the extra gain that PI can guarantee by using σ, if the game ends after n stages. (and similarly for PII: $\sup_\sigma \gamma_n(\sigma,\tau) - Cav\, u(p)$.)

Because as said before PI can guarantee to get at least $Cav\, u(p)$ for all n, then PI and PII have contradicting interests (as PI gets the payoff, and PII is the Payer): PI is interested in a slow speed of convergence of $\gamma_n(\sigma, \tau)$ to $Cav\, u(p)$, and PII is interested in a quick speed of convergence of $\gamma_n(\sigma, \tau)$ to $Cav\, u(p)$. It is therefore natural to define a new *0-sum* game denoted as $SG_\infty(p)$: *The Game for the speed of convergence of* $\gamma_n(\sigma,\tau) - Cav\, u(p)$ *to zero.* This is the topic of this note. We will now define formally the game $SG_\infty(p)$ and its value:

- The payoff matrices and the strategy-spaces for each player, are the same as in $G_\infty(p)$.
- As in $G_\infty(p)$ defining the payoff for every σ, τ is problematic and we will define a value for this game without it.

This value would have to represent the sequence of $\gamma_n(\sigma,\tau) - Cav\, u(p)$, In order to do that, we need to use zamir's following definition (see definition 2 of [11]):

Definition 1.4.

- *Two sequences of non-negative numbers* $f(n)$ *and* $g(n)$ *are said to be of the same order of magnitude, if there are constants* $c_1, c_2 > 0$, *and* N, *such that:*

$$c_2 g(n) \le f(n) \le c_1 g(n),$$

  $\forall n \ge N.$
  This will be denoted by $f(n) = O^*(g(n))$, *or* $g(n) = O^*(f(n))$.
- *If there is a* $c > 0$, *and* N, *such that:* $cg(n) \le f(n), \quad \forall n \ge N$, *we write that:* $g(n) \le O^*(f(n))$.

Definition 1.5.

- *We say that* PI can guarantee $f(n,p) \ge 0$ *in* $SG_\infty(p)$, *if for any* $\varepsilon > 0$, *there exists a sequence* $f_\varepsilon(n,p)$ *s.t:*
  - $n^\varepsilon f_\varepsilon(n,p) \ge f(n,p), \quad \forall n.$
  - *There is a strategy* $\sigma_f(\varepsilon, p)$ *and a const* $c_f(p) > 0$ *(independent of* ε*), such that for all* τ*:*

$$(\gamma_n(\sigma_f(\varepsilon,p),\tau) - Cav\, u(p)) \ \ge\ c_f(p) f_\varepsilon(n,p).$$

- *We say that* PII can guarantee $g(n,p) \ge 0$ *in* $SG_\infty(p)$, *if for any* $\varepsilon > 0$, *there exists a sequence* $g_\varepsilon(n,p)$ *s.t:*
  - $g_\varepsilon(n,p) \le g(n,p) n^\varepsilon, \quad \forall n.$
  - *There is a strategy* $\tau_g(\varepsilon,p)$ *and a const* $c_g(p) > 0$ *(independent of* ε*), such that for all* σ*:*

$$(\gamma_n(\sigma, \tau_g(\varepsilon,p)) - Cav\, u(p)) \ \le\ c_g(p) g_\varepsilon(n,p).$$

- *We say that the game $SG_\infty(p)$ has a* value, *if there are sequences such that PI and PII can guarantee $f(n,p)$ and $g(n,p)$ respectively in $SG_\infty(p)$ and $f(n) = O^*(g(n))$.*

  We then define: $v_s(p) := O^*(f(n)) = O^*(g(n))$.

Note that for $SG_\infty(0)$ (or $SG_\infty(1)$), there is always a value, and this value is $O^*(\mathbf{0})$, (where $\mathbf{0}$ is the constant zero sequence,) since this is a game with complete information.

2. Definitions and Preliminary Results

Proposition 2.1. *For all σ, τ, strategies in $G_\infty(p)$:*

$$\inf_\tau \gamma_n(\sigma,\tau) \leq v_n(p) \leq \sup_\sigma \gamma_n(\sigma,\tau).$$

This proposition expresses the intuition that both, PI and PII, have more freedom in their choices of strategies in $G_n(p)$ than in $G_\infty(p)$, since a strategy in $G_n(p)$ can depend on n, and a strategy in $G_\infty(p)$ (where the end of the game is not known to the players) can not depend on n. In other words, any strategy available to any of the players in $G_\infty(p)$, is available to them also in $G_n(p)$ (as its n -truncation), hence they can not 'do better' in $G_\infty(p)$, than in $G_n(p)$.

Proof. Let: σ_n^*, τ_n^* be respectively, optimal strategies of PI and PII in $G_n(p)$, and for any pair of strategies σ and τ of PI and PII in $G_\infty(p)$, we denote by σ_n, τ_n, the n -truncation of σ, τ respectively.

$$\inf_\tau \gamma_n(\sigma,\tau) = \inf_{\tau_n} \gamma_n(\sigma,\tau_n) \leq \gamma_n(\sigma,\tau_n^*) \leq \gamma_n(\sigma_n^*,\tau_n^*) = v_n(p).$$

For the first inequality, note that: τ_n^* can not do better than be the best reply to σ. Similarly: $\sup_\sigma \gamma_n(\sigma,\tau) \geq v_n(p)$. □

Proposition 2.2. *If there is a value $v_s(p)$ for $SG_\infty(p)$, then it satisfies:*

$$v_s(p) = O^*(e_n(p)).$$

Proof. from proposition 2.1 we get that for any σ, τ:

$$\inf_\tau \gamma_n(\sigma,\tau) - Cav\, u(p) \leq e_n(p) \leq \sup_\sigma \gamma_n(\sigma,\tau) - Cav\, u(p), \qquad \forall n.$$

Now if there is a value $v_s(p)$ for $SG_\infty(p)$, then by definition 1.5, there are sequences $f(n,p)$, $g(n,p)$, such that:

1. $v_s(p) = O^*(f(n,p))$.

   and for all $\varepsilon > 0$, there are strategies: $\sigma_f(\varepsilon,p)$, $\tau_g(\varepsilon,p)$, and sequences: $f_\varepsilon(n,p)$, $g_\varepsilon(n,p)$ such that:
2. $n^\varepsilon f_\varepsilon(n,p) \geq f(n,p)$, $\qquad \forall n$.
3. $g_\varepsilon(n,p) \leq g(n,p) n^\varepsilon$, $\qquad \forall n$.

   and there are consts $c_f(p) > 0$, $c_g(p) > 0$ such that:

4.

$$c_f(p)f_\varepsilon(n,p) \le \inf_\tau \gamma_n(\sigma_f(\varepsilon,p),\tau) - Cav\, u(p) \le e_n(p)$$

$$\le \sup_\sigma \gamma_n(\sigma,\tau_g(\varepsilon,p)) - Cav\, u(p) \le c_g(p)g_\varepsilon(n,p).$$

Letting ε go to 0, we get that: $c_f(p)f(n,p) \le e_n(p) \le c_g(p)g(n,p)$. Hence: $e_n(p) = O^*(f(n)) = O^*(g(n))$, and therefore: $v_s(p) = O^*(e_n(p))$. □

The main result of this note is that although $v_s(p)$ (unlike $e_n(p)$), does not always exist, it does exist for the special class of games, in which: $e_n(p) = O^*\left(\frac{1}{\sqrt{n}}\right) \quad \forall p \in (0,1)$, and hence equals to $O^*\left(\frac{1}{\sqrt{n}}\right)$ by proposition 2.2.

The games for which $e_n(p) = O^*\left(\frac{1}{\sqrt{n}}\right) \quad \forall p \in (0,1)$, were characterized by Mertens and Zamir as the games for which: $\sqrt{n}(v_n(p) - Cav\, u(p)) \longrightarrow \Phi(p)$, where $\Phi(p)$ is an appropriately scaled Normal density function (see Mertens and Zamir [6]). We shall refer to this class as the class of *"Normal Games"*.

For any strategy σ of PI in $G_\infty(p)$, (or in $G_n(p)$) let us define a sequence of r.v (see e.g p.189 of [4] and p.122 of [10]):

$$P_1 \equiv p$$

$$n > 1 \qquad P_n := P_{p,\sigma,\tau}(K = 1 \mid h_n).$$

That is P_n is the conditional probability, given h_n and given σ , τ, that at stage 0 chance chose $k = 1$. Although P_n is not known to PII since he does not necessarily know σ, it plays a central role in the analysis. Basically this is because, by the minmax theorem any optimal (minmax) strategy σ of PI, guarantees the value even when it is known to PII, who can then compute P_n.

It is easily seen that the distribution P_n, is conditionally independent of τ (since τ is independent of K.) and using Bayes' law we get that:

$$P_{n+1} = \begin{cases} \frac{s_n{}^1(h_n)}{\bar{s}_n(h_n)} P_n, & \text{if } i_n = T \\ \\ \frac{s_n^{1\prime}(h_n)}{\bar{s}'_n(h_n)} P_n, & \text{if } i_n = B \end{cases}$$

where $\bar{s}_n(h_n) = P_n s_n^1(h_n) + P'_n s_n^2(h_n)$

and $s_n^{1\prime}(h_n) = 1 - s_n^1(h_n)$. (see in the Introduction.)

It is easy to see that $\mathcal{P}^\infty = \{P_n\}_{n=1}^\infty$, is a Martingale.

Hence any σ, a strategy of PI in $G_\infty(p)$, yields an infinite Martingale $\mathcal{P}^\infty$ of probabilities, namely of random variables satisfying: $0 \le P_n \le 1, \qquad \forall n$.

For all n and all $m \ge n$ let:

$$V_n(\mathcal{P}^m) = E\sum_{k=1}^{n} |P_{k+1} - P_k|$$

be the *variation* of $\mathcal{P}^m = \{P_k\}_{k=1}^m$. The variation is a measure for the expected amount of information revealed by PI up to (and including) stage n, when using the strategy σ. Note that the definition for $V_n(\mathcal{P}^m)$ holds also for $m = \infty$.

The variation $V_n(\mathcal{P}^m)$ serves as a key role in the analysis, since the extra gain of PI (beyond $Cav\, u(p)$) is constrained by the amount of information he reveals. More precisely: it is proven (see p.224 of [3]), that there is $c > 0$, such that:

$$\inf_\tau \gamma_n(\sigma,\tau) \le \frac{1}{n}\sum_{k=1}^{n} Eu(P_k) + \frac{c}{n}V_n(\mathcal{P}^m) \le Cav\, u(p) + \frac{c}{n}V_n(\mathcal{P}^m) \tag{1}$$

for all n and all σ (a strategy in $G_m(p)$, $m \ge n$.)

Using Cauchy-Schwarts inequality, and the fact that $\mathcal{P}^\infty$ is a uniformly bounded Martingale, it is shown (see e.g proposition 3.8 p.122 of [10],) that:

$$\text{for all } n,\quad m \ge n:\quad V_n(\mathcal{P}^m) \le a(p)\sqrt{n} \qquad \text{for some } a(p) > 0 \tag{2}$$

Remark 2.3. *It follows from* (1) *and* (2), *that:* $e_n(p) \le O^*\left(\frac{1}{\sqrt{n}}\right)$.

It was shown (see Zamir [8] ,) that $O^*\left(\frac{1}{\sqrt{n}}\right)$ is the least upper bound for the order of magnitude of $e_n(p)$. Namely there exists a game in which: $e_n(p) \ge \frac{pp'}{\sqrt{n}}$, $\forall n$ $\forall p \in (0,1)$.

Hence from (1) it follows, that for each n, there is a uniformly bounded martingale $\mathcal{P}^n$ s.t:

$$V_n(\mathcal{P}^n) \ge cpp'\sqrt{n}. \qquad \text{for some } c > 0.$$

Mertens and Zamir also showed (see Theorem 2.4 p. 255 of [5]) that for the *infinite* uniformly bounded martingale $\mathcal{P}^\infty$, $\lim_{n\to\infty}\left\{\frac{V_n(\mathcal{P}^\infty)}{\sqrt{n}}\right\} = 0$.

From this point we proceed as follows:

- Although there isn't any *infinite* uniformly bounded martingale $\mathfrak{X}^\infty$, satisfying: $V_n(\mathfrak{X}^\infty) \ge a\sqrt{n}$, $\forall n$ for some $a > o$, we prove in part 3 that $\sqrt{n}$ can be reached asymptotically, that is: For every $\varepsilon > 0$, we will construct an infinite martingale $\mathfrak{X}_\varepsilon^\infty$, satisfying:

  $$V_n(\mathfrak{X}_\varepsilon^\infty) \ge cn^{\frac{1}{2}-\varepsilon}, \qquad \forall n$$

  for some $c > 0$.
- In Part 4 we construct for any $\varepsilon > 0$, a strategy σ in $G_\infty(p)$, for PI, that yields an infinite $\mathcal{P}_\varepsilon^\infty$, that coincides with $\mathfrak{X}_\varepsilon^\infty$ in some interval (l,u) in $[0,1]$.
- In Part 5 : We prove that in the *Normal Games*, there is a value $v_s(p)$ for $SG_\infty(p)$ $\forall p \in (0,1)$, and that: $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$.
- We conclude by showing in Part 6 a class of games that for all p, do not have a value for $SG_\infty(p)$.

3. The Variation of Uniformly Bounded Infinite Martingales

Our first result states that although the n-stage variation of a uniformly bounded infinite martingale is smaller than $O^*(\sqrt{n})$, it can be of $O^*\left(n^{\frac{1}{2}-\varepsilon}\right)$, for arbitrarily small $\varepsilon > 0$.

Theorem 3.1. *For any $\varepsilon > 0$, $\eta > 0$ and $0 \le l < p < u \le 1$, there is $c > 0$ and a Martingale $\mathfrak{X}_\varepsilon^\infty = \{X_n\}_{n=1}^\infty$, with: $EX_1 = p \quad \forall n$, that satisfies:*

(a) $P(l < X_n < u, \quad \forall n) > 1 - \eta$

(b) For all n: $\quad V_n(\mathfrak{X}_\varepsilon^\infty) \ge cn^{\frac{1}{2}-\varepsilon}$.

Furthermore, we will prove that:

$$P\left\{\sum_{k=1}^{n} |X_{k+1} - X_k| \ge cn^{\frac{1}{2}-\varepsilon}, \qquad \forall n\right\} = 1.$$

Proof. We construct a Martingale that satisfies (a) and (b).

For a given $0 < \theta < 1$, let $Y_k, \quad k = 1, 2 \ldots,$ be i.i.d. random variables, defined by:

$$P(Y_k = \theta) = \theta' \qquad \text{and} \qquad P(Y_k = \theta') = \theta$$

(where $\theta' = 1 - \theta$.)

The required martingale $\mathfrak{X}_\varepsilon^\infty$ is now defined as follows:

$X_1 \equiv p$ and for all $n > 1$:

$$X_n := X_{n-1} + \frac{Y_n}{(n_0 + n)^{\frac{1}{2}+\varepsilon}},$$

where $n_0 = n(\varepsilon, p, l, u)$ is a constant that we choose so that $\mathfrak{X}_\varepsilon^\infty$ satisfies (a) and (b).

To choose n_0 so that $\mathfrak{X}_\varepsilon^\infty$ satisfies (a), we proceed as follows:

$$Var\,(X_n) = Var\left\{\sum_{k=n_0+2}^{n_0+n} \frac{Y_k}{k^{\frac{1}{2}+\varepsilon}}\right\} = \sum_{k=n_0+2}^{n_0+n} \frac{\theta\theta'}{k^{1+2\varepsilon}} \le M_\varepsilon,$$

where $M_\varepsilon = \sum_{k=1}^\infty \frac{\theta\theta'}{K^{1+2\varepsilon}}$. Note that $E|X_n| \le Var\,(X_n) + 1 \le M_\varepsilon + 1$.

By the Martingale Convergence Theorem (see e.g p.244 of [7]), we get that (since $E|X_n| \le M_\varepsilon + 1, \quad \forall n$): $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite, and: $EX_\infty = p$.

Using Egoroff's theorem (see e.g p.88 of [2]), we get that the convergence of X_n to X_∞ is even stronger, namely almost uniformly. that is: For all $\eta, \delta > 0$, there is an N, such that:

$$P\left(|X_n - X_\infty| < \delta, \quad \forall n > N\right) \ge 1 - \eta.$$

Lemma 3.2. *For any $\delta, \eta > 0$, there is an $N = N(\delta, \eta)$, such that for any $n_0 > N$, the process $\mathfrak{X}_\varepsilon^\infty$ defined with n_0, will satisfy:*

$$P\left(|X_n - X_\infty| < \delta, \qquad \forall n\right) \ge 1 - \eta.$$

Proof. Consider the above defined process with $n_0 = 0$, that is for each $n > 1$:

$$\widetilde{X}_n = p + \sum_{k=2}^{n} \frac{Y_k}{k^{\frac{1}{2}+\varepsilon}} \tag{3}$$

$$\widetilde{X}_\infty = p + \sum_{k=2}^{\infty} \frac{Y_k}{k^{\frac{1}{2}+\varepsilon}}. \tag{4}$$

By Egoroff's theorem we have, that for any $\delta > 0$ and $\eta > 0$, there is an $N = N(\delta, \eta)$, such that:

$$P\left(|\widetilde{X}_n - \widetilde{X}_\infty| < \delta, \quad \forall n > N\right) \geq 1 - \eta. \tag{5}$$

Now for any n_0: $X_\infty - X_n \equiv \widetilde{X}_\infty - \widetilde{X}_{n+n_0}$ so we have:

$$P\left(|X_\infty - X_n| < \delta, \quad \forall n\right) = P\left(|\widetilde{X}_\infty - \widetilde{X}_{n+n_0}| < \delta, \quad \forall n\right) = P\left(|\widetilde{X}_\infty - \widetilde{X}_n| < \delta, \quad \forall n > n_0\right).$$

and so for any such process defined with $n_0 > N$, where N satisfies (5), we get:

$$P\left(|X_\infty - X_n| < \delta, \quad \forall n\right) = P\left(|\widetilde{X}_\infty - \widetilde{X}_n| < \delta, \quad \forall n > n_0\right) \geq P\left(|\widetilde{X}_\infty - \widetilde{X}_n| < \delta, \quad \forall n > N\right)$$

$$\geq 1 - \eta,$$

and with that we proved lemma 3.2. □

To complete the definition of the martingale $\mathfrak{X}_\varepsilon^\infty$, we now define n_0 as follows:

Given ε, p, l and u, s.t: $0 \leq l < p < u \leq 1$, define $\delta^* = min\left\{\frac{p-l}{2}, \frac{u-p}{2}\right\}$. For fixed $\eta > 0$, let n_0 be the minimal $N(\delta^*, \eta)$, that satisfies the inequality of lemma (3.2) for $\delta = \delta^*$.

- We now prove that the martingale $\mathfrak{X}_\varepsilon^\infty$, satisfies (a):

$$P\left(l < X_n < u, \quad \forall n\right) \geq P\left(|X_n - p| < 2\delta^*, \quad \forall n\right)$$

$$\geq P\left(|X_n - X_\infty| + |X_\infty - p| < 2\delta^*, \quad \forall n\right)$$

$$\geq P\left(|X_n - X_\infty| < \delta^* \quad and \quad |X_\infty - X_1| < \delta^*, \quad \forall n\right)$$

$$= P\left(|X_n - X_\infty| < \delta^*, \quad \forall n\right) \geq 1 - \eta.$$

- To prove that $\mathfrak{X}_\varepsilon^\infty$satisfies (b):

  Denote: $\alpha = min(\theta, \theta')$.

$$\sum_{k=1}^{n} |X_{k+1} - X_k| = \sum_{k=n_0+2}^{n_0+n} \frac{|Y_k|}{k^{\frac{1}{2}+\varepsilon}} \geq \sum_{k=n_0+2}^{n_0+n} \frac{\alpha}{k^{\frac{1}{2}+\varepsilon}}$$

$$\geq cn^{\frac{1}{2}-\varepsilon}$$

for some $c > 0$, hence:

$$P\left\{\sum_{k=1}^{n}|X_{k+1}-X_k| \geq cn^{\frac{1}{2}-\varepsilon}, \qquad \forall n\right\} = 1.$$

In particular, $V_n(\mathfrak{X}_\varepsilon^\infty) \geq cn^{\frac{1}{2}-\varepsilon}$, $\forall n$, concluding the proof of Theorem 3.1.

□

4. Constructing for PI a Strategy with Maximal Variation

As mentioned earlier:

$$\inf_\tau \gamma_n(\sigma,\tau) \leq Cav\, u(p) + \frac{c}{n}V_n(\mathcal{P}_\varepsilon^\infty), \qquad \forall n.$$

Therefore a strategy for PI, that will give him highest $O^*(\inf_\tau \gamma_n(\sigma,\tau))$, must have maximal $O^*(V_n(\mathcal{P}_\varepsilon^\infty))$, where $\mathcal{P}_\varepsilon^\infty$ is the Martingale of the conditional probabilities derived from σ.

We shall use the Martingale $\mathfrak{X}_\varepsilon^\infty$ constructed in the proof of Theorem 3.1, in the following way:

For any $\varepsilon > 0$, we will construct a strategy σ_ε for PI, such that inside an interval (l,u), the sequence of conditional probabilities will be the same as $\mathfrak{X}_\varepsilon^\infty$ for that ε. The only information about the history that PI will use at stage n, is the conditional probability P_n, and so we can denote: $s_n^k(P_n)$ as the probability that PI will choose T at stage n, given $K=k$, and given P_n. In part 3, we defined the constant $n_0 = n(\varepsilon,p,l,u)$. Since ε and p are fixed, we abbreviate this by $n_0(l,u)$. We now define the following sequence: $\varphi(n) = (n_0(l',u')+n)^{\frac{1}{2}+\varepsilon}$, where $l' = \frac{1}{\sqrt{n_0(0,1)}}$ and $u' = 1 - \frac{1}{\sqrt{n_0(0,1)}}$.

Definition of σ_ε: given $0 < \theta < 1$, for any stage $n = 1,2\cdots$:

- If $l' < P_n < u'$, then:

$$s_n^1(P_n) = \theta + \frac{\theta\theta'}{\varphi(n)P_n} \qquad s_n^2(P_n) = \theta - \frac{\theta\theta'}{\varphi(n)P_n'}$$

- Otherwise: $s_n^k(P_n) = s(P_n)$ for $k=1,2$. where $s(P_n)$ is an optimal strategy of PI in $D(P_n) = P_nA_1 + (1-P_n)A_2$.

Remark 4.1. *The reason for choosing l', u' as above is for $s_n^k(P_n)$ to be well defined. that is, for $k=1,2$ and n, the P_n will satisfy: $0 \leq s_n^k(P_n) \leq 1$.*

Remark 4.2. *Note that for any two intervals s.t. $(a,b) \subseteq (c,d)$, we have: $n_0(a,b) \geq n_0(c,d)$, since by the definition of δ^*, we get that: $\delta^*(a,b) \leq \delta^*(c,d)$, (where $\delta^*(x,y)$ is δ^* that correspond to the case of interval (x,y).) Hence any N that satisfies (5) for $\delta^*(a,b)$, satisfies (5) for $\delta^*(c,d)$. Since we defined $n_0(c,d)$ to be the minimal N that satisfies (5) for $\delta^*(c,d)$, we get that: $n_0(a,b) \geq n_0(c,d)$.*

Lemma 4.3. *The σ_ε defined above satisfies:*

1. *For all k, n, the P_n satisfies:* $0 \leq s_n^k(P_n) \leq 1$.
2. *The strategy σ_ε yields a Martingale* $\mathcal{P}_\varepsilon^\infty = \{P_n\}_{n=1}^\infty$, *which coincides with* $\mathfrak{X}_\varepsilon^\infty$ *(defined in part 3) inside* (l', u'), *and is absorbed outside* (l', u').

Proof.

1. • If $l' < P_n < u'$ then:

$$s_n^1(P_n) = \theta + \frac{\theta\theta'}{\varphi(n)P_n}.$$

We have to prove that: $\theta + \frac{\theta\theta'}{\varphi(n)P_n} \leq 1$. this is equivalent to: $\frac{\theta\theta'}{\varphi(n)P_n} \leq \theta'$ which is equivalent to: $\theta \leq \varphi(n)P_n$. Now $\varphi(n) = (n_0(l', u') + n)^{\frac{1}{2}+\varepsilon}$ so:

$$\varphi(n)P_n \geq (n_0(l', u') + n)^{\frac{1}{2}+\varepsilon} l' > \left(\frac{n_0(l', u')}{n_0(0,1)}\right)^{\frac{1}{2}}.$$

By Remark 4.2: $\frac{n_0(l',u')}{n_0(0,1)} \geq 1$, so we get:

$$\varphi(n)P_n \geq 1 > \theta.$$

In the same way we prove that $0 \leq s_n^2(P_n) \leq 1$.

• If $P_n \notin (l', u')$, so by definition: for $k = 1, 2$, $s_n^k(P_n) = s(P_n)$ which is an optimal strategy of PI in $D(P_n)$, and then clearly: $0 \leq s(P_n) \leq 1$.

2. $\{P_n\}_{n=1}^\infty$ is a Martingale that satisfies:

$$P_{n+1} = \begin{cases} \frac{s_n^1(P_n)}{\bar{s}_n(P_n)} P_n, & \text{if } i_n = T \\ \\ \frac{s_n^1{}'(P_n)}{\bar{s}_n'(P_n)} P_n, & \text{if } i_n = B \end{cases} \tag{6}$$

(recall $\bar{s}_n(P_n) = P_n s_n^1(P_n) + P_n{}' s_n^2(P_n),$) so: $P(i_n = T) = \bar{s}_n(P_n)$ and $P(i_n = B) = \bar{s}_n'(P_n)$ and hence:

• If $P_n \in (l', u')$ then by definition:

$$\bar{s}_n(P_n) = P_n\left(\theta + \frac{\theta\theta'}{\varphi(n)P_n}\right) + P_n'\left(\theta - \frac{\theta\theta'}{\varphi(n)P_n'}\right) = \theta.$$

And so by (6), if $P_n \in (l', u')$, then:

$$P_{n+1} = \begin{cases} \frac{\left(\theta + \frac{\theta\theta'}{\varphi(n)P_n}\right)P_n}{\theta}, & \text{If } i_n = T \\ \\ \frac{\left(\theta' - \frac{\theta\theta'}{\varphi(n)P_n}\right)P_n}{\theta'}, & \text{If } i_n = B. \end{cases}$$

That is: $P\left(P_{n+1} = P_n + \frac{\theta'}{\varphi(n)}\right) = P(i_n = T) = \theta$ and: $P\left(P_{n+1} = P_n - \frac{\theta}{\varphi(n)}\right) = P(i_n = B) = \theta'$.

In other words, if $P_n \in (l', u')$, then:

$$P_{n+1} = P_n + \frac{Y_n}{(n_0(l', u') + n)^{\frac{1}{2}+\varepsilon}}.$$

- If $P_n \notin (l', u')$ then $s_n^k(P_n) = s(P_n)$, $k = 1, 2$, that is σ_ε is then non-revealing, and therefore: $P_{n+1} = P_n$ with probability 1, i.e. P_n is absorbed outside (l', u'), concluding the proof of lemma 4.3.

□

Lemma 4.4. *For the Martingale $\mathcal{P}_\varepsilon^\infty$ derived from σ_ε: $V_n(\mathcal{P}_\varepsilon^n) \geq cn^{\frac{1}{2}-\varepsilon}$, $\forall n$.*

Proof. Because in (l', u') $\mathcal{P}_\varepsilon^\infty = \{P_n\}_{n=1}^\infty$ coincides with $\mathfrak{X}_\varepsilon^\infty = \{X_n\}_1^\infty$ defined in part 3, then:

$$P\left(l' < P_n < u', \qquad \forall n\right) = P\left(l' < X_n < u', \qquad \forall n\right) \geq 1 - \eta.$$

In particular, for all n: $P\left(l' < P_m < u', \qquad \forall m \leq n\right) \geq 1 - \eta$.
And so:

$$V_n(\mathcal{P}_\varepsilon^\infty) = E\sum_{k=1}^{n} |P_{k+1} - P_k| \geq$$

$$E\left\{\sum_{k=1}^{n} |P_{k+1} - P_k| \quad \middle| \quad l' < P_m < u', \qquad \forall m \leq n\right\} P\left(l' < P_m < u', \quad \forall m \leq n\right)$$

$$\geq E\sum_{k=n_0+2}^{n_0+n} \frac{|Y_k|}{(n_0(l', u') + k)^{\frac{1}{2}+\varepsilon}} \cdot (1 - \eta) \geq \sum_{k=n_0+2}^{n_0+n} \frac{\alpha(1-\eta)}{(n_0(l', u') + k)^{\frac{1}{2}+\varepsilon}} \geq cn^{\frac{1}{2}-\varepsilon},$$

for some $c > 0$. □

5. The Speed of Convergence in the Normal Games

We will now use σ_ε that we constructed in section 4, for the Normal Games. We will show that by using it, PI guarantees maximal speed of convergence. That is there is a $c > 0$, such that for all τ: $\quad \gamma_n(\sigma_\varepsilon, \tau) - Cav\, u(p) \geq \frac{c}{n^{\frac{1}{2}+\varepsilon}}, \qquad \forall n.$

Theorem 5.1. *In the Normal Games there is a value $v_s(p)$ for $SG_\infty(p)$, for all $0 < p < 1$, and $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$.*

The Normal games were characterized by Mertens and Zamir who showed that a Normal game has a presentation of (see p.289 of [3]):

$$A_1 = \begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix} \qquad A_2 = \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix}$$

$0 < \theta, a, b < 1$ and without loss of generality $a > b$.

We will prove the Theorem by proving two Lemmas:

Lemma 5.2. *In the Normal Games: PI can guarantee $O^*\left(\frac{1}{\sqrt{n}}\right)$ in $SG_\infty(p)$, for all* $0 < p < 1$.

Proof. For $\varepsilon > 0$, use σ_ε which was defined in part 4.

At stage n:

- if $P_n = p_n$, such that $l' < p_n < u'$, and PII is using (t, t'), then the payoff for this stage is:

$$\left[p_n\left(s_n^1(p_n),\ \ s_n^{1\prime}(p_n)\right)\begin{pmatrix}\theta' a & -\theta' a' \\ -\theta a & \theta a'\end{pmatrix} + p_n'\left(s_n^2(p_n),\ \ s_n^{2\prime}(p_n)\right)\begin{pmatrix}\theta' b & -\theta' b' \\ -\theta b & \theta b'\end{pmatrix}\right]\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$= \left[\left(p_n\theta + \tfrac{\theta\theta'}{\varphi(n)},\ \ p_n\theta' - \tfrac{\theta\theta'}{\varphi(n)}\right)\begin{pmatrix}\theta' a & -\theta' a' \\ -\theta a & \theta a'\end{pmatrix} + \left(p_n'\theta - \tfrac{\theta\theta'}{\varphi(n)},\ \ p_n'\theta' + \tfrac{\theta\theta'}{\varphi(n)}\right)\begin{pmatrix}\theta' b & -\theta' b' \\ -\theta b & \theta b'\end{pmatrix}\right]\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$= p_n\left(\theta,\ \ \theta'\right)\begin{pmatrix}\theta' a & -\theta' a' \\ -\theta a & \theta a'\end{pmatrix}\begin{pmatrix}t \\ t'\end{pmatrix} + p_n'\left(\theta,\ \ \theta'\right)\begin{pmatrix}\theta' b & -\theta' b' \\ -\theta b & \theta b'\end{pmatrix}\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$+\frac{\theta\theta'}{\varphi(n)}\left[\left(1,\ \ -1\right)\begin{pmatrix}\theta' a & -\theta' a' \\ -\theta a & \theta a'\end{pmatrix} + \left(-1,\ \ 1\right)\begin{pmatrix}\theta' b & -\theta' b' \\ -\theta b & \theta b'\end{pmatrix}\right]\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$= \frac{\theta\theta'}{\varphi(n)}\left[\left(\theta' a + \theta a,\ \ -\theta' a' - \theta a'\right) + \left(-\theta' b - \theta b,\ \ \theta' b' + \theta b'\right)\right]\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$= \frac{\theta\theta'}{\varphi(n)}\left(a - b,\ \ -a' + b'\right)\begin{pmatrix}t \\ t'\end{pmatrix}$$

$$= \frac{\theta\theta'}{\varphi(n)}(a-b)(t+t') = \frac{\theta\theta'(a-b)}{\varphi(n)}$$

This is true for all t, and for all $l' < p_n < u'$, so when σ_ε is used by PI, the payoff g_n for stage n satisfies:

$$E(g_n \mid l' < P_n < u') = \frac{\theta\theta'(a-b)}{\varphi(n)}, \qquad \forall t, \tag{7}$$

- Now, if $P_n \notin (l', u')$ then $s_n^k(P_n) = s(P_n) = \theta$, so:

$$E(g_n \mid P_n \notin (l', u')) = 0 \qquad \forall t.$$

Hence for all n, and all τ:

$$\gamma_n(\sigma_\varepsilon, \tau) = \frac{1}{n}\sum_{k=1}^{n} E(g_k(\sigma_\varepsilon, \tau)) \geq \frac{1}{n}E\left\{\sum_{k=1}^{n}(g_k(\sigma_\varepsilon, \tau)) \middle| \quad l' < P_m < u', \qquad \forall m \leq n\right\}(1-\eta)$$

and so by equation (7):

$$\gamma_n(\sigma_s,\tau) \geq \left[\frac{1}{n}\sum_{k=1}^{n}\frac{\theta\theta'(a-b)}{\varphi(k)}\right](1-\eta) = \frac{(1-\eta)\theta\theta'(a-b)}{n}\sum_{k=1}^{n}\frac{1}{\varphi(k)} \geq \frac{c}{n^{\frac{1}{2}+\varepsilon}},$$

for some $c > 0$.

Hence by definition 1.5, this concludes the proof of lemma 5.2.

□

Lemma 5.3. *PII can guarantee* $O^*\left(\frac{1}{\sqrt{n}}\right)$ *in* $SG_\infty(p)$ *for all* $0 < p < 1$.

Proof. PII has a strategy τ_B, based on Blackwell's approachability theorem, (see e.g Aumann and Maschler p.225 of [3]) , which guarantees him in any $G_\infty(p)$, (not just in the Normal-Games,) not to pay more than: $Cav\, u(p) + \frac{a(p)}{\sqrt{n}}$, for all $0 < p < 1$, for some $0 < a(p) < \infty$. □

Proof of Theorem 5.1: By the definition of $v_s(p)$, Lemmas 5.2 and 5.3 imply that there is a value $v_s(p)$ for $SG_\infty(p)$ in the Normal Games, and that: $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$ $\forall p \in (0,1)$. □

6. A Case in which the Game $SG_\infty(p)$, does not Have a Value

Since for all p, $v_n(p)$ is a non-decreasing function in n, (see proposition 3.19 in [10]) and $v_n(p) \geq Cav\, u(p)$ $\forall n$, then $v_1(p) = Cav\, u(p)$, implies $v_n(p) = Cav\, u(p)$ $\forall n$, and thus $e_n(p) = 0$, $\forall n$. Hence in this special case PI cannot gain any benifit from his extra knowledge and therefore we consider this game to be trivial:

Definition 6.1. *If* $v_1(p) = Cav\, u(p)$, *then we say that* $G_\infty(p)$ *is a* trivial Game.

Note that $G_\infty(0)$ and $G_\infty(1)$ are alwayes trivial. It is easy to see that if $G_\infty(p)$is trivial, then $v_s(p)$ exists and $v_s(p) = O^*(0)$.

Theorem 6.2. *If* $u(p)$ *is strictly concave on* $[0,1]$ *and* $G_\infty(p)$ *is not trivial, then the game* $SG_\infty(p)$ *does not have a value.*

This Theorem expresses the following intuition: Games with strictly Concave $u(p)$ represent cases in which PI prefers the situation that none of players PI and PII knows which is the game played, rather than the situation that both of them *do* know. To see that, note that when none of the players know which game is being played then they play $D(p)$ and the value is $u(p)$. If both players know which game is played, then both can play optimal in *that* game, so the value is:

$$pv_1 + p'v_2.$$

28 MAR '00 14:47 CENTER RATIONALITY 972 2 6513681 P.15
Received: 28.Mar.00 06:54 AM From: 97226513681 To: 4258719567 Get faxes by email. Free. eFax.com Page: 15 of 18

where v_1, v_2 are the values of A_1, A_2 respectively. (Note that: $u(1) = v_1$ and $u(0) = v_2$.) By the strict concavity of $u(p)$, we have:

$$u(p) > pu(1) + p'u(0) = pv_1 + p'v_2, \qquad \text{for all } 0 < p < 1.$$

So in such games we would expect PI to be conservative in his use of information, in order not to reveal it to PII. It turns out that in $G_\infty(p)$, PI should *never* use his information:

Lemma 6.3. *If $u(p)$ is strictly concave on $[0,1]$, then if PI guarantees $f(n,p)$ in $SG_\infty(p)$, then $f(n,p) = O^*((0))$.*

Proof. Since $u(p)$ is strictly concave, then for all p:

$$Cav\, u(p) = u(p)$$

1. If σ is a non-revealing (NR) strategy, that is for all n: $s_n^1(h_n) \equiv s_n^2(h_n)$ and then for all n:

$$\inf_\tau \gamma_n(\sigma, \tau) \leq u(p) = Cav\, u(p).$$

   Thus the NR optimal strategy in $G_\infty(p)$, consisting of playing repeatedly an optimal strategy in $D(p)$, guarantees $O^*())$ in $SG_\infty(p)$.
2. We claim that any other strategy σ, is not (even) optimal in $G_\infty(p)$. We do that by proving that there is an N, such that: $\inf_\tau \gamma_n(\sigma,\tau) < Cav\, u(p)$. $\quad \forall n > N$.
   Let $\hat{n}$ be the first stage such that: $s_{\hat{n}}^1(h_n) \not\equiv s_{\hat{n}}^2(h_n)$.
   For all n:

$$\inf_\tau \gamma_n(\sigma, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} Eu(P_i) + \frac{c}{\sqrt{n}}$$

   (see (1) and (2) in part 2).
   - For all $n \leq \hat{n}$: $\quad P_i \equiv p$
   - For $n = \hat{n}$: $\quad P_{\hat{n}+1} \not\equiv P_{\hat{n}}$.
     So by the strict concavity of $u(p)$, given $P_{\hat{n}}$:

$$E\left(u(P_{\hat{n}+1} \mid P_{\hat{n}})\right) < u(E\left(P_{\hat{n}+1} \mid P_{\hat{n}}\right)) = u(P_{\hat{n}}) = u(p)$$

   - denote: $-\delta = Eu(P_{\hat{n}+1}) - u(p)$.
     $\{u(P_n)\}$ is a *Super-Martingale*, since using Jensen's inequality:

$$E\left(u(P_n) \mid u(P_{n-1})\right) \leq u\left((EP_n \mid P_{n-1})\right) = u(P_{n-1}).$$

So: For all $n > \hat{n}$: $Eu(P_n) - u(p) \leq -\delta$. Hence:

$$\inf_\tau \gamma_n(\sigma,\tau) \leq \frac{1}{n}\left[\sum_{i=1}^{\hat{n}} Eu(P_i) + \sum_{\hat{n}+1}^{n} Eu(P_i)\right] + \frac{c}{\sqrt{n}}$$

$$\inf_\tau \gamma_n(\sigma,\tau) \leq \frac{1}{n}\left[\sum_{i=1}^{\hat{n}} u(p) + \sum_{\hat{n}+1}^{n}(u(p) - \delta))\right] + \frac{c}{\sqrt{n}}$$

$$\inf_\tau \gamma_n(\sigma,\tau) \leq u(p) - \delta \cdot \frac{n-\hat{n}}{n} + \frac{c}{\sqrt{n}}$$

Thus: $$\inf_\tau \gamma_n(\sigma,\tau) - u(p) \leq -\delta\left(1 - \frac{\hat{n}}{n}\right) + \frac{c}{\sqrt{n}}$$

For n large enough, the left side of the last inequality is strictly negative, so σ is not an optimal strategy in $G_\infty(p)$, concluding the proof of Lemma 6.3.

□

Lemma 6.4. *If $u(p)$ is concave on $[0,1]$ and $G_\infty(p)$ is not trivial, then if PII guarantees $g(n,p)$ in $SG_\infty(p)$, then $g(n,p) \geq O^*\left(\frac{1}{n}\right)$.*

Proof. If the game is not trivial, then PI can play σ_n^* in $G_n(p)$ defined as follows: For first $n-1$ stages play for every realization of h_m, $m \leq n$, optimal in D(p). So up to stage $(n-1)$:

$$\inf_\tau \gamma_{n-1}(\sigma_n^*,\tau) = u(p).$$

At stage n, play optimal in $G_1(p)$, to guarantee in this stage: $v_1(p) > u(p)$. denote: $c(p) = v_1(p) - u(p)$, then PI can guarantee:

$$\inf_\tau \gamma_n(\sigma_n^*,\tau) = u(p) + \frac{c(p)}{n}.$$

So then for any strategy τ_n of PII in $G_n(p)$:

$$\gamma_n(\sigma_n^*,\tau_n) \geq u(p) + \frac{c(p)}{n}.$$

Since $u(p)$ is concave, then for all p: $u(p) = Cav\,u(p)$, and so:

$$\gamma_n(\sigma_n^*,\tau_n) \geq Cav\,u(p) + \frac{c(p)}{n}.$$

Now PII cannot play better in $G_\infty(p)$ than in any $G_n(p)$ (see proposition 2.1 in part 2), which implies that if PII guarantees $g(n,p)$ in $SG_\infty(p)$ then:

$$[g(n,p)] \geq \left[\frac{1}{n}\right].$$

□

Proof of Theorem 6.2: By Lemmas 6.3 and 6.4 and by the definition of $v_s(p)$, we get that the game $SG_\infty(p)$ does not have a value. □

We conclude with an example of a game in which for all $0 < p < 1$, $v_s(p)$ does not exist and the gap between any $f(n,p), g(n,p)$ that PI and PII can guarantee respectively in $SG_\infty(p)$, is bounded away from zero by $\frac{\ln n}{n}$.

Let:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This game was presented by Aumann and Maschler and it was proved by Zamir, (see Theorem 4 of [8]) that $e_n(p) = O^*\left(\frac{\ln n}{n}\right)$, $\forall p \in (0,1)$. For this game $u(p) = p(1-p)$, which is a strict concave function and hence by lemma 6.3, if PI can guarantee $f(n,p)$ in $SG_\infty(p)$, then: $f(n,p) = O^*())$.

On the other hand PII can not do better in $G_\infty(p)$ than in any $G_n(p)$, hence: if PII can guarantee $g(n,p)$ in $SG_\infty(p)$then: $g(n,p) \geq O^*\left(\frac{\ln n}{n}\right)$.

References

[1] Aumann, R.J. and Maschler M. (1995) 'Repeated games with incomplete information' *The MIT Press, Cambridge*

[2] Halmos, P.R. (1950): 'Measure theory' *D.Van Nostrand Company Inc.*

[3] Mertens, J.F., Sorin, S and Zamir, S. (1994) 'Repeated games:Part B', CORE Discussion Paper 21, *Université Catholique de Louvain, Louvain-la-Neuve, Belgium.*

[4] Mertens, J.F. and Zamir, S. (1976) 'The normal distribution and repeated games' *International Journal of Game Theory,* 5:187-197.

[5] Mertens, J.F. and Zamir, S. (1977) 'The maximal variation of a bounded martingale' *Israel Journal of Mathematics,* 27. No 3-4.

[6] Mertens, J.F. and Zamir, S. (1995) 'Incomplete information games and the normal distribution' Discussion Paper 70, *Center for Rationality and Interactive decision theory, the Hebrew Universiy of Jerusalem.*

[7] Ross, S.M. (1983) 'Stochastic Processes' *John Wiley and Sons.* p.244.

[8] Zamir, S. (1971-72): 'On the relation between finitely and infinitely repeated games with incomplete information' *International Journal of Game Theory,* 1:179-198.

[9] Zamir, S. (1973) 'On the notion of value for games with infinitely many stages' *The Annals of Statistics* 1.No 4:791-796.

[10] Zamir, S. (1992) 'Repeated games of incomplete information' *Hand Book of Game Theory, Edited by R.J. Aumann and S.Hart,* 1:ch.5.

[11] Zamir, S. (1973) 'On repeated games with general information function' *International Journal of Game Theory* Vol 2 Issue 4.

CENTER FOR RATIONALITY, AND INTERACTIVE DECISION THEORY, THE HEBREW UNIVERSITY OF JERUSALEM