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Abstract

We propose a model of gradual bargaining in the spirit of the Nash axiomatic theory. In this model the underlying set of payoff opportunities expands continuously with time. Unlike Nash's solution, that predicts a single agreement for each bargaining problem, our solution yields a continuous path of agreements – one for each point in time. It emerges from a simple and intuitive differential equation. We discuss the relationship between the gradual solution and the Nash solution, and characterize it axiomatically by using essentially one property, which is *Invariance with Respect to Increasing Transformations*. We interpret this property as an incentive compatibility requirement. By using the richer framework of gradual bargaining, our approach avoids some of the shortcomings of Nash's axiomatization. In particular we do not need the controversial axiom of IIA and the sets of payoff opportunities need not be convex. In the spirit of the *Nash Program* we propose several non-cooperative bargaining models that sustain our solution. Finally, we apply our model to discuss the allocation of physical (or monetary) assets when individuals' risk aversion changes over time.

1. Introduction

Almost 50 years after the first appearance of Nash's bargaining model, his theory remains one of the most important tools in economic analysis. The evolution of this theory over the last several decades has taken different directions. A significant corpus of the literature that stemmed form Nash's 1950 paper deals with reinterpreting the Nash solution, establishing alternative axiomatizations and developing non-cooperative games that sustain it. This latter objective has been particularly popular since the relationship between the Nash Solution and Rubinstein's (1982) pure non-cooperative bargaining model has become apparent (see Binmore, Rubinstein, Wolinsky (1986)). Another route was taken by works that have attempted to construct different solutions to the same problem addressed by Nash. The axiomatic approach that characterizes the Nash theory has allowed for a prolific discussion about the validity of the axioms and their interpretation. This has led to the emergence of alternative axioms and with them the construction of competing solution concepts. Excellent surveys of this vast literature can be found in Kalai (1985), Peters (1992) and Thomson (1996). It is fair to say, however, that much less has been done in the direction of expanding the domain of what constitutes a bargaining problem. It is precisely this route that we have chosen to embark on in this paper.

We view bargaining here as a gradual process in which consecutive agreements take place while players' underlying opportunities constantly change. Hence, in contrast to the conventional framework, which assumes the presence of a single and constant set of payoffs (i.e., the pie) and predicts a single agreement, in our setup the players' set of opportunities expands gradually and continuously and the solution specifies the whole path of agreements – one for each point in time. Our motivation in looking at this domain is twofold. Firstly, we believe that real life bargaining situations often involve a gradual process in which agreements are temporal and are subject to revision as the pie expands. Negotiations between management and workers concerning wages[1] or between managers and shareholders concerning the allocation of profits between bonuses and dividends have precisely this property: these amounts are not fixed and agreements are constantly revised.

[1] on which the Nash theory has been often applied.

Political negotiations have this feature as well. Seidmann and Winter (1997) present a non-cooperative model of gradual coalition formation and discuss a variety of such examples in a multilateral context.

Our model is relevant even in cases where the pie is acquired fully formed. Bargainers would often find it convenient to divide the underlying issue to several smaller ones and to negotiate the ultimate agreement gradually. This is often done in order to reduce the risk of a breakdown in the negotiation. Differences on smaller issues are resolved much easily. Indeed, this philosophy was introduced by Henry Kissinger during the seventies in the context of the Arab-Israeli conflict and, to a large extent, it still governs the approach to the peace process today. This intuition raises, of course, important questions regarding the agenda. The ultimate outcome of the bargaining depends on the way the large pie is broken into small pieces. In this paper, however, we will take the agenda to be exogenous.

Our second motivation for exploring gradual bargaining in the Nash context is the fact that our analysis of the new setup, and in particular our solution concept for the gradual bargaining problem, will provide a new insight on Nash's model and its solutions. By enriching the domain, in the way we show here, we solve some of the shortcomings of Nash's model that have been repeatedly discussed in the literature on axiomatic bargaining. Our model does not require the sets of payoff opportunities to be convex. To some extent they do not even need to be compact. This allows for a larger domain of applicability, which is usually not permitted in axiomatic bargaining[2]. Furthermore, our axiomatization does not use Nash's controversial axiom of Independence of Irrelevant Alternative[3] (IIA) and in fact it does not require any alternative axiom to replace it. Instead it imposes a stronger version of Nash's Invariance axiom. The characterization of our solution essentially builds on one property. Unlike Nash, who requires the invariance only with respect to affine transformations, our property imposes the invariance with respect to increasing transformations. We will interpret this property as an incentive compatibility

[2] A different approach for lifting this constraint was recently proposed by Conley and Wilkie (1995), Herrero (1989), and Zhou (1997).

[3] Other authors have replaced the IIA axiom in the Nash standard framework with other axioms (see for example Binmore (1987a) and Peters and van Damme (1993)).

requirement. It will be shown that this property jointly with the two standard axioms of efficiency and symmetry uniquely determines our solution.

As we have mentioned earlier, our solution yields a continuous path of agreements rather than a single agreement. Hence, for each problem it yields a function rather than a point. It is therefore defined by means of a differential equation. This equation displays a simple and intuitive principle. At each point in time players share the marginal growth of the pie in a way that the more needy person obtains more, where "neediness" is determined by the marginal rate of substitution between the welfare of the two players. To further support the solution, which we call the gradual Nash solution or GNS, we propose several "arbitration schemes" that give rise to it. In one of these schemes, players' divide each marginal "crumb" of the pie according to the Nash solution, where by "crumb" we refer to an infinitesimally small part. We show that if this scheme is applied on a discrete gradual bargaining problem it will generate a bargaining path that will converge to our GNS as the crumbs become smaller and smaller.

Our axiomatic treatment is presented in Section 5. In Section 6 we study another relationship between the GNS and the standard Nash solution. It turn out that if the set of opportunities expands in a homothetic way, then the bargaining path hits the Nash solution at each point in time. In Section 7 we establish the relationship between the GNS and the asymmetric Nash solution. We present asymmetric arbitration schemes that give rise to a generalized version of the GNS and with it to the asymmetric Nash solution as the ultimate agreement. Moreover, we show how the asymmetric Nash solution emerges from the symmetric GNS when the set of payoff opportunities expands in a quasi-homothetic way. This, we will argue, offers a new interpretation to the asymmetric Nash solution, which is quite different from the standard interpretation based on the idea that players have different bargaining powers (since Roth (1979)).

Our approach is both cooperative (axiomatic) and non-cooperative. Section 8 is devoted to the presentation of several bargaining games that support our solution. Again, the enrichment of the domain allows for more flexibility in terms of the choice of the bargaining model (compared with the standard Nash's framework). We present two types of bargaining procedures. The first is based on a sequence of ultimatum bargaining games over each crumb, i.e. the allocation of each crumb results from a “take it or leave it” proposal. The second game form assumes that each crumb is negotiated according to Rubinstein’s alternating offer procedure. We show that these two types of models virtually implement our solution. Specifically, when applied on discrete gradual bargaining problems they yield a subgame perfect equilibrium path of agreements that converges to our solution as we approach the continuum. In Section 8 we discuss the allocation of physical assets or money when players’ risk attitude changes over time. We show that this problem gives rise to a graduate bargaining problem and we establish a general equation that determines the allocation of the physical assets over time. We accompany this discussion with several examples in which the GNS can be derived analytically and we interpret the solution. We conclude in Section 10 with some final comments.

The approach of looking at continuous time processes and differential equations in the context of the standard Nash framework has been used by others in the past. Starting with Raifa (1953) who introduced an alternative solution to the Nash problem, and later Maschler, Owen, Peleg (1987), Binmore (1987c), Livne (1989) , Furth (1990), Bergman (1992) and Zhou (1997).

In contrast to these papers our main objective is to develop an axiomatic theory on a new domain of problems, namely, the domain of gradual bargaining problems, which we believe allows for a more elegant axiomatic treatment than those used for a variety of solution concepts in the standard Nash framework. Finally, our paper is also somewhat related to papers in the literature of non-cooperative bargaining theory, which look at discrete bargaining but with multiple pies. Examples are Winter (1997) and Seidmann and Winter (1997) in a multilateral context and Fershtman (1990) and John and Raith (1997) in a bilateral context.

2. The Standard Nash Problem

In the standard Nash paradigm a bargaining problem refers to a single pie represented by a compact and convex set S in R_+^2 that includes the origin (0,0)[4]. The Nash solution assigns to each set S a point N(S) which satisfies N(S) = arg $\max_{(x,y) \in S} x \cdot y$. Throughout this paper we will refer to N(S) as the standard Nash solution.

Note that if the Pareto frontier of S is smooth and is given by the equation H(x,y) = c, then, at the Nash point, the curve $x \cdot y$= const and the curve H(x,y) = c have a common tangential line. This means that the gradients of these two curves are parallel at this point. The gradient of the H function is $(H_x(x,y), H_y(x,y))$ and the gradient of $x \cdot y$ is simply the vector (y,x). Hence, at the Nash point we must have:

$$\frac{H_x(x,y)}{H_y(x,y)} = \frac{y}{x}.$$

We will later extend this property in the definition of our gradual solution.

3. The Gradual Bargaining Problem

In our gradual bargaining framework two players are engaged in a bargaining process over a pie which grows over time[5]. At each point of time an agreement has to be reached concerning the allocation of the pie currently available. Our graduate solution will attempt to predict the complete path of agreements from the initial point to the ultimate pie depending on the way the players' underlying opportunities grow over time.

We start with some preliminary definitions and notations:

A Gradual Bargaining Problem (GBP) is a function H from R_+^2 to [0,1]. The available pie at time t is denoted by $D(t) = \{(x,y);\ t \geq H(x,y)\}$.

We impose the following conditions on H:

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[4] For the sake of simplified notations we will assume a zero disagreement point all through the paper although none of our results depend on this assumption.

[5] One should not confuse the sense in which our pies change over time with Rubinstein's story of "shrinking pies". The difference is not merely the fact that our pies expand while
the latter, there is essentially a single pie and the model predicts a single agreement. In our framework the pie grows and the model predicts a path i.e., an agreement for each crumb.

1. H is smooth and strictly increasing in both x and y[6]. We will denote by $\partial D(t)$ the Pareto frontier of D(t), i.e. ,

$\partial D(t) = \{(x,y) \in D(t)$, s.t. $(x',y') > (x,y)$ implies that $(x',y') \notin \partial D(t)\}$. We will often refer to $\partial D(t)$ as the level curve of the function H at time t. Note that condition 1 implies that D(t) satisfies free disposal, i.e. if $(x,y) \in D(t)$ and $(x,y) \geq (x',y')$[7] then $(x',y') \in D(t)$.

2. If $t > t'$ then $D(t') \subset D(t)$, and $D(0) = \{(0,0)\}$.

Condition 2 requires that the underlying opportunities for the bargainers never shrink, i.e. during each infinitesimal time a new crumb of the pie is being acquired, and there are no re-negotiations over crumbs previously allocated. The necessary and sufficient conditions on the function H that guarantee conditions 1 and 2 can be found in Kannai (1977). Note that our conditions on H do not imply that the sets of payoff opportunities D(t) are convex or even compact[8]. Hence our framework allows sets which are forbidden in the standard Nash theory. Figure 1 depicts a gradual bargaining problem and its level curves.

Figure 1. A Gradual Bargaining Problem

A Gradual Solution (GS) is a function that assigns to each GBP H a continuos path s_H, which depends only on the level curves of H. In its parametric form s_H is given by a pair of functions $s(t) = (x(t),y(t))$ satisfying $s(t) \in D(t)$ for all t. The functions x(t) and y(t) represent the total payoff acquired up to time t by players 1 and 2 respectively. Note that when x(t) and y(t) are monotonically increasing, s(t) can be alternatively represented by a function y(x). We will often refer to y(x) or s(t) as the bargaining path of the GBP H. We are now ready to introduce our specific solution for GBPs.

The Gradual Nash Solution (GNS) assigns to each GBP H the unique solution to the following differential equation:

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[6] One can get practically the same results when H is only Lipschitz and does not have critical points (i.e. H_x and H_y cannot be both zero).

[7] The inequalities $\geq$ and $>$ on vectors should be taken coordination-wise.

[8] Take for example $H(x,y) = xy - \frac{1}{1+x} - \frac{1}{1+y}$.

$$\frac{dy(x)}{dx} = \frac{H_x(x,y)}{H_y(x,y)} \qquad (1)$$

Equation (1) generates a vector field that determines the direction of the bargaining path from each point (x,y) in D(1). Note that the right hand side of equation (1) is the marginal rate of substitution (MRS) between the welfare of the two players. It determines the number of utils player 2 has to forgo in order to increase player 1's welfare by one util, while remaining on the same pie (level curve). Specifically, if the MRS is, say 3, then the solution imposes that the subsequent crumb is allocated so that player 2 obtains three times more than player 1. Thus at each point in time the players move from one agreement to another by allowing the more "needy" person to obtain more. Note the relationship between equation (1) and the one defining the Nash solution. In the latter case the relation to the MRS is required only once (globally) while in our solution this relation has to hold for each point in time.

Geometrically, the direction in which the players choose to move according to the solution is orthogonal to the flipped gradient[9]. This is depicted in figure 2.

Figure 2. The Bargaining Direction

Two more technical remarks about our solution:

Remark 1. Equation (1) above determines a unique bargaining curve y(x). This follows from our condition 1 on H. However, it does not determine the specific parameterization of this curve x(t), y(t). Whenever we refer to the solution in its parametric form we will assume that the parameterization is such that H(x(t),y(t)) = t. This means that at time t the players arrive at the frontier of D(t). Remark 2 below argues that this can be done without loss of generality.

Remark 2. Equation (1), in fact, depends on H only through its level curves. To demonstrate this consider a new function given by h(x,y) = G(H(x,y)), where G is smooth.

[9] The flipped gradient is the vector (H_x,-H_y). Its orthogonal vector is (H_y,H_x), which points at H_x/H_y. Note that this direction is identical to the one obtained by flipping the tangent (at the point (x,y)) which is orthogonal to the gradient. Note that the same direction is obtained by flipping the tangent at the point on ∂D(t).

Then $H(x_1, y_1) = H(x_2, y_2)$ implies that $G(H(x_1, y_1)) = G(H(x_2, y_2))$. Thus the level curves are preserved, and we can verify that equation (1) for h defines exactly the same curve:

$$\frac{dy}{dx} = \frac{h_x(x,y)}{h_y(x,y)} = \frac{G'(H(x,y))\,H_x(x,y)}{G'(H(x,y))\,H_y(x,y)} = \frac{H_x(x,y)}{H_y(x,y)}$$

4. Discrete Time and the Gradual Nash Solution

We use the discrete version of the gradual bargaining problem to motivate the GNS (equation 1). We will show how equation (1) emerges from a variety of different arbitration schemes that determine the allocation of crumbs in the expanding pie. We will start with a scheme that is based on the idea that at each point of time players allocate the next available crumb by using the standard Nash solution. Then we will argue that the same equation emerges from the rival solution to Nash's namely, the Kalai Smorodinsky solution. In our second type of scheme players take alternate turns in acquiring the whole crumb at each period, i.e. player 1 starts by taking the first crumb, then player 2 takes the second, etc. In a third scheme that will be discussed at Appendix A.1, the property rights on crumbs at each point of time are allocated randomly with respect to equal probabilities.

We note here that although all our arbitration schemes are based on the idea that each crumb is handled separately, a less "myopic" scheme cannot lead to a Pareto dominating outcome. This is because a point x(t),y(t) on the frontier of D(t) represents the players' utility from the whole stream of payoffs up to time t. Hence any arbitration scheme that yields a point on $\partial D(t)$ at time t is Pareto efficient.

A Discrete Time Gradual Bargaining Problem is a nested collection of standard bargaining problems in R^2, given by sets the $D(t_1) \subset D(t_2) \subset \ldots \subset D(t_k)$, which satisfy conditions 1 and 2 in the definition of GBP. We will now describe several arbitration schemes that result in our solution. We use the term arbitration scheme to distinguish from formal non-cooperative games. These schemes should be thought of as procedures or algorithms by which an arbitrator determines the allocation of each crumb in the gradual problem. We will later turn some of these schemes into formal game forms when we discuss the non-cooperative implementation of our solution.

Arbitration scheme 1: Allocating each crumb by means of the Nash solution:

Take a GBP H and establish a discrete problem given by D(δ),D(2δ),…,D(kδ). Assume that players are moving from a settlement on pie j to a settlement on pie j+1 by dividing the j+1st crumb using the Nash solution, (i.e. by using the settlement on pie j as the disagreement point of pie j+1). We argue that this scheme lead to our equation when crumbs become smaller and smaller.

Specifically, suppose that at stage j the settlement is at some point (x,y) on ∂D(jδ) satisfying H(x,y) = jδ. Then the new settlement will be at a point (x+Δx, y+Δy) such that H(x+Δx, y+Δy)=(j+1)δ. If the new settlement is established by allocating the additional crumb according to the standard Nash solution then it has to satisfy the following maximization problem: max Δx Δy subject to:

$$\begin{cases} H(x,y) = jd \\ H(x+\Delta x, y+\Delta y) = (j+1)d \end{cases}$$

The process is depicted geometrically in figure 3.

Fig 3 (Nash on each crumb)

Using the Taylor expansion we obtain
$$\begin{cases} H_x \Delta x + H_y \Delta y = d \\ \max(\Delta x \Delta y) \end{cases}$$

The above maximization problem is equivalent to
$$\max\left(\frac{d - H_y \Delta y}{H_x} \Delta y \right)$$

Differentiating with respect to Δy we get $d - 2H_y \Delta y = 0$, which implies

$$\Delta y = \frac{d}{2H_y},\ \Delta x = \frac{d}{2H_x}$$

leading to the to the differential equation (1) as δ tends to zero.

Surprisingly, if we would have based our arbitration scheme on the Kalai & Smorodinsky (1975) solution instead of the Nash, we would have arrived at the same differential equation. This will be argued at the end of Appendix A.1. We have also verified that other solution concepts for the standard Nash problem, such as the (continuous) Raiffa solution and the Perles-Maschler solution lead to the very same equation when applied as arbitration schemes for allocating crumbs. Our solution to the gradual problem is, thus, universal in the sense that it can be generated by a variety of solution concepts for the standard problem, which possess completely different properties and interpretations. We note here, however, that not every well-known solution will qualify for this matter. The Egalitarian Solution (see Kalai 1977b) for example, leads to a different differential equation when used as an arbitration scheme. This will all become more apparent later when we discuss the axiomatic approach.

We now turn to a different type of scheme. Arbitration scheme 2 below is a version of the structure developed in Bergman (1992). Our equation (1) is hence equivalent to his equation (3.3). Bergman has derived the equation by looking at the Rubinstein alternating offer game when the cake shrinks in a general way and involves a single agreement (see also Binmore 1987b). He has concluded that the Rubinstein game only rarely leads to the Nash solution. Although the mathematical structure of the alternating offers is the same, our approach is different in terms of its economics and in some sense diagonal. We look at an axiomatic framework where the cake expands and an interim agreement is reached at each point in time.

Arbitration scheme 2: Each player gets the whole crumb in his turn:
The players alternate turns. At each stage one of the players obtains the whole crumb. Let (x,y) be the settlement reached on pie j and assume H(x,y) = c. Suppose that at this point it is player 1's turn to obtain a crumb, then the new settlement is at the point (x+Δx,y) on a higher level curve satisfying:

$$H(x+\Delta x, y) = H(x, y) + d\ .$$

On the subsequent stage player 2 acquires the whole crumb, which leads to the point ($x+\Delta x, y+\Delta y$) lying on yet another level curve (as shown in figure 4):

$$H(x+\Delta x, y+\Delta y) = H(x,y) + 2d \ .$$

Fig 4 (Alternating turns)

Now using the Taylor expansion, the above two equations can be written as:

$$\begin{cases} H(x,y) + H_x(x,y)\Delta x = H(x,y) + d \\ H(x,y) + H_x(x,y)\Delta x + H_y(x,y)\Delta y = H(x,y) + 2d \end{cases}$$

or

$$\begin{cases} H_x(x,y)\Delta x = d \\ H_y(x,y)\Delta y = d \end{cases}$$

which yields:

$$\Delta x H_x(x,y) = \Delta y H_y(x,y)$$

So as we go to the limit allowing smaller and smaller crumbs, we arrive at equation (1) again. Note that we would have established the same derivation if we assumed that player 2 is the first to obtain a crumb. In Appendix A.1 we show that if we revise Arbitration Scheme 2 by assuming that the players toss a fair coin at each stage to determine who will get the crumb, then we would get the same solution. This appendix also presents formally the sense in which the discrete bargaining path converges to the continuous path determined by our solution. This convergence result will be used later when we refer to the non-cooperative models that support our solution.

We will now introduce the axiomatic treatment to our solution for the gradual bargaining problem.

5. The Axiomatic Approach

Earlier we have demonstrated the relationship between our solution to the GBP and the standard Nash solution. However the Nash solution itself inherits much of its attractiveness from its elegant axiomatic characterization. In this section we will develop an independent axiomatization for the gradual Nash solution and discuss its relation to the one supporting the standard Nash solution as well as other solutions. The advantage of our characterization is that it is based on three very simple properties and does nor require Nash's controversial IIA axiom.

The main feature of our characterization is the requirement that the solution is invariant with respect to all increasing transformations, not necessarily affine. Another desirable property of the axiomatization is the fact that it requires very little structure on the shape of the pies. Unlike the conventional Nash framework the pies don't need to be convex nor even compact.

The first axiom requires that at each point of time the acquired pie is fully allocated between the players. In fact this requirement is merely a choice of parametrization for the solution:

Axiom 1 (Efficiency): At any time t $(x(t),y(t)) \in \partial D(t)$.

Our second axiom is a straightforward extension of Nash's symmetry axiom. It asserts that in symmetric problems the bargaining path forms a 45° straight line.

Axiom 2 (Symmetry): If $H(x,y) = H(y,x)$ for all (x,y), then $x(t) = y(t)$ for all t.

The most significant axiom of our characterization requires that the solution is invariant with respect to all *increasing* transformations. It is a strong version of Nash's axiom of invariance with respect to affine transformations. Specifically,

Axiom 3 (Invariance with Respect to Increasing Transformations, IIT): Let A and B be two increasing and smooth transformations from R_+ to R_+, $A(0) = B(0) = 0$. Let H and H* be two bargaining problems such that $H(x,y) = H^*(A(x),B(y))$. Set $s(H) = (x(t),y(t))$, and $s(H^*) = (x^*(t),y^*(t))$. Then $x^*(t) = A(x(t))$ and $y^*(t) = B(y(t))$, i.e. the solution of H* is the transformed solution of H.

We view axiom 3 as a property of incentive compatibility. This axiom actually requires that players cannot benefit (nor lose) by misrepresenting their utility function[10], e.g. when reporting falsely about their risk postulates to the arbitrator. If a player misrepresents his utility function the solution shifts according to the same transformation so that in terms of the physical allocation nothing would change, yielding the player the same utility level with respect to his true preferences. We will refer to this intuition later when we discuss the allocation of divisible assets in Section 9. Note also the relation between our axiom and Nash's axiom of invariance with respect to affine transformation. We argue that although our axiom is stronger[11], it can be more easily interpreted and justified[12]. The standard interpretation of Nash's axiom of invariance implicitly assumes that players are expected utility maximizes. This assumption is not necessary in our framework as our axiom requires invariance with respect to increasing transformations[13]. Indeed, when presenting the Nash solution in class we have been repeatedly faced with the question: "Why does Nash require invariance only with respect to affine, and not with respect to monotonic transformations?" Our usual response to this question is to note that in the Nash framework if we replace Nash's invariance axiom with its stronger version, then together with the other three axioms of Nash we obtain an *impossibility result*. There exists no solution that will satisfy the 4 axioms[14].

This should not be viewed as a drawback of the stronger version of the axiom, but rather as a shortcoming of the domain used by Nash's conventional framework. Indeed,

[10] As a function of the received physical asset.

[11] Note that the sense in which this axiom is stronger than Nash's is somewhat informal. This is because our axioms operate on a domain of problems which is different than Nash's.

[12] We would argue that it is not necessarily true that the weaker the axiom the "better". This is because there are two sides to this coin. A weak axiom would make the uniqueness result more interesting, but may make the solution itself less interesting.

[13] A somewhat polar approach was taken by Rubinstein, Safra, and Thomson (1992) who have disposed of the invariance axiom completely by developing a Nash solution which is based on ordinal preferences.

[14] Moreover, any subset of the four will allow existence. Specifically, the argument is the following: if the four axioms are imposed, then the only possible candidate is the Nash solution. But the Nash solution does not satisfy the Invariance with respect to increasing transformations. Hence, there exists no solution satisfying the 4 axioms. Now drop the IIA and you get multiple solutions, i.e. any GBP embedded in the original problem will give rise to a solution on the standard problem satisfying the 3 axioms. Drop the invariance axiom, and the set of solutions includes Nash and the Egalitarian solution. Drop the symmetry

our framework that views bargaining as inter-temporal interaction permits the use of this axiom for a complete characterization of the solution. Moreover it allows us to dispose of the more controversial axiom of IIA.

Theorem 1 characterizes the Gradual Nash Solution. The proof is relegated to Appendix A.2, which also shows that the three axioms are independent (claim 1).

Theorem 1: There exists a unique gradual solution satisfying axioms 1 – 3. This solution is the Gradual Nash Solution (GNS).

As we have argued earlier Nash's IIA axiom is not required for the characterization of the NGS nor is it necessary to replace it with another axiom, as is the case with the various solution concepts that emerged from challenging the IIA axiom (e.g. the Kalai Smorodinsky solution or the Perles Maschler solution). Nevertheless, it is interesting to note that natural versions of both the IIA and the Kalai & Smorodinsky Monotonicity axioms are satisfied by our GNS. These versions are discussed in Appendix A.2 claims 2 and 3. We find this fact to be particularly stunning as these two axioms are incompatible in the standard Nash framework (when imposed jointly with efficiency and symmetry). It is precisely the richness of our framework that allows for the compatibility of these properties. This observation is, in fact, reminiscent to the phenomenon that much of the impossiblity results in social choice theory are resolved by enriching the domain of social choice functions. Finally, we note that in contrast to the axioms discussed above one property, which cannot be reconciled with the NGS, is Kalai's (1977b) axiom of *step-by step negotiation*. This property imposes that for two bargaining problems (in the standard Nash framework) S and T with $S \subset T$, solving T in two stages (i.e. by first solving S, and then using the solution as a disagreement point for the remaining pie) would yield the same outcome as in solving T in one stage, by completely ignoring S. Kalai has used this property to characterize the Egalitarian solution. In our context this property would mean that the bargaining path is independent of the way the pie is expanded in time (i.e

axiom and the set includes the solution in which player 1 gets it all. Finally, drop the efficiency axiom and the set includes a solution which constantly gives (0,0) to the two players.

independent of the level curves of H) which is against the very essence of our solution concept. This is not much a surprise. The Egalitarian solution used as an arbitration scheme to allocate crumbs yields the differential equation $dy/dx = 1$, which is different than ours.

6. Homothetic Problems

In the definition of gradual bargaining problems we have allowed players' underlying opportunities to expand in a rather general way. In particular the expansion may favor one player more than the other. However, one interesting special case is when the expansion is homothetic, i.e. where all feasible outcomes are increased with respect to a fixed percentage. Specifically, a gradual bargaining problem H is said to be *homothetic* if for each $t \in [0,1]$ there exists $1 \geq \lambda \geq 0$ such that $D(t) = \lambda D(1) = \{ (\lambda x, \lambda y); (x,y) \in D(1)\}$, where $D(1)$ represents the final pie, i.e. the one established at $t = 1$.

Proposition 1 below asserts that in a homothetic GBP our solution yields the standard Nash solution on the final pie. In fact the same assertion holds for any point in time t.

Proposition 1: Let H be a homothetic gradual bargaining problem and let $(x(t),y(t))$ be its gradual Nash solution, then $(x(t),y(t))$ is the standard Nash solution of the problem given by $D(t)$.

The proof of Proposition 1 is presented in Appendix A.3. We remark that in general GBPs, the path of the Gradual Nash Solution may not terminate at the Nash solution of the final pie. It may not even go through any of the Nash solutions of the intermediate pies. This will be made apparent later when we discuss examples. Figure 5 depicts the gradual solution in the homothetic case.

Fig 5. (Homothetic level curves)

7. Asymmetry

Our solution, as well as the standard Nash solution, assumes that the two bargainers differ only in terms of the sets of payoff opportunities, so that if these sets are symmetric the players are allocated the same payoff in the solution. Kalai (1977a) has introduced the asymmetric version of the Nash solution (see also Roth 1979). The concern of these works was with cases in which players may possess different bargaining powers or abilities, so that even if the set of feasible payoff vectors is symmetric, players may be allocated different payoffs. These authors, and may who followed them, interpreted this general version as a solution for bargaining situations in which asymmetry is built into the bargaining procedure, which gives rise to unequal bargaining power. This interpretation has been supported later by a variety of formal non-cooperative models (most notably Binmore (1987b)). These models have sustained the asymmetric Nash solutions as an equilibrium outcome. Interestingly, in terms of applications in economics, the asymmetric Nash solutions has been arguably more popular than the original symmetric version. This is partly because the assumption of symmetry is very stringent and may not be adequate for many of the underlying stories in which the bargaining model is applied.

In this Section we discuss the relationship between our gradual solution and the asymmetric Nash solution. We will approach this problem by looking at asymmetry from two rather polar angles. In the first one we will refer to non-symmetric arbitration schemes of the sort we have used previously to establish the symmetric gradual solution. In this schemes crumbs are either being allocated randomly to the players with respect to unequal probabilities, or players alternate turns in acquiring crumbs but the sizes of the crumbs are unequal, so that one player always obtain a larger crumb. We will show how these two non-symmetric arbitration schemes lead to an asymmetric Nash solution on the final pie.

In the second approach, which leads to a more surprising result, we show how the asymmetric Nash solution can arise from a *symmetric* arbitration scheme, when the opportunity sets expand in a quasi-homothetic way. This result, we will argue, offers a new interpretation to the asymmetric Nash solutions, one that is quite different from the procedural interpretation.

7.1 Asymmetric arbitration schemes

Arbitration Scheme I

Consider a homothetic discrete time gradual bargaining problem $B = \{D(t_1), D(t_2),\ldots, D(t_k)\}$ with unequal crumbs. Specifically, assume that the crumb is of size $p\delta$ at odd stages and is of size $q\delta$ at even stages[15]. In arbitration scheme I the players alternate turns in acquiring crumbs, starting with player 1 (who acquires crumbs of size $p\delta$).

Arbitration Scheme II

Consider a homothetic discrete time gradual bargaining problem $B = \{D(t_1), D(t_2),\ldots, D(t_k)\}$ with equal crumbs. According to arbitration scheme II at each stage an unfair coin is tossed to determine the owner of the current crumb, i.e. player 1 acquire the crumb with probability p (constant over time) and player 2 acquires it with probability (1-p).

Figures 6 and 7 depict the two arbitration schemes I and II.

Figure 6 (Arbitration Scheme I)

Figure 7 (Arbitration Scheme II)

In Propositions 2 and 3 we will show that the arbitration schemes I and II give rise to the gradual solution determined by the following differential equations

$$\frac{dy}{dx} = \frac{q}{p}\frac{H_x(x,y)}{H_y(x,y)} \tag{2}$$

We will use this result to establish the relationship between these schemes and the Asymmetric Nash solution in the standard framework: specifically the p-asymmetric Nash solution is defined to be the function that assigns to each bargaining problem D the maximizer of $x^p \cdot y^{(1-p)}$ over the set D.

The proofs of the following Propositions appear in Appendix A.3.

[15] we always can set q=1-p.

Proposition 2: Let $\{B^j\}$ be an infinite sequence of discrete time GBPs of the form described in *arbitration scheme I*, and let $s^j(t)$ be the bargaining path generated by that scheme. If $\{B^j\}$ converges to the continuous GBP H then $s^j(t)$ will converge to the GS s(H) given by the solution to equation (2). Moreover, this solution yields the p-asymmetric Nash solution on the final pie D(1).

Proposition 3: Let $\{B^j\}$ be an infinite sequence of discrete time GBPs with equal crumbs. Let $s^j(t)$ be the bargaining path generated by the expected payoff that players obtain at each stage, with respect to *arbitration scheme II*. If $\{B^j\}$ converges to the continuous GBP H, then $s^j(t)$ will converge to the GS s(H) given by the solution to equation (2), and s(H) yields the p-asymmetric Nash solution on the final pie D(1).

7.2 Symmetric Arbitration Schemes on Quasi-Homothetic Problems

As we have mentioned earlier the conventional interpretation of the Asymmetric Nash solution attributes the asymmetry to the unmodeled bargaining protocol. Selten, among others, has argued against this interpretation, by suggesting that if the protocol is asymmetric to the extent that it is payoff relevant, then we should expect the protocol to be negotiated as well in a preceding bargaining phase. Proposition 4 bellow proposes an interpretation of a different nature. If we view a bargaining situation as a gradual process, then the asymmetric Nash outcome can arise from the fact that along the process the pie has expanded in a way that consistently favors one party. Since the gradual process involves many interim agreements, which are binding, the final allocation depends on the whole process by which the pie has been expanded to its eventual size.

A GBP H is said to be quasi-homothetic if for some $p\in[0,1]$ and some $\lambda\in[0,1]$ we have $D(t) = (\lambda^{1-p}, \lambda^{p})\cdot D(1) = \{ (\lambda^{1-p}x, \lambda^{p}y); (x,y) \in D(1)\}$.

Proposition 4: Let H be a quasi-homothetic GBP with parameter p, then the GNS yields the p-asymmetric Nash solution on the final pie D(1).

The proof of Proposition 4 is presented in Appendix A.3. Figure 8 displays the GNS for quasi-homogeneous GBP.

Figure 8 (The GNS on quasi-homogeneous GBPs)

8. The Nash Program for the Gradual Solution (Non-Cooperative Implementation)

One of the ultimate criteria for judging the attractiveness of a cooperative solution concept like the Nash bargaining solution is the availability of plausible and simple non-cooperative games that supports it. Nash himself who was aware of the importance of models proposed the "demand game" to support his solution. This agenda has been known ever since as the Nash program. Later Binmore (1987b) and Binmore, Rubinstein and Wolinsky (1986) have shown how the solution can be sustained by much simpler non-cooperative bargaining games, which build on Rubinstein's (1982) model of alternating offers. In this Section we will utilize our discussion on arbitration schemes to develop a similar approach to our gradual solution. We will describe three formal bargaining procedures in which crumbs are bargained sequentially. In the first two models crumbs are negotiated via "take it or leave it" offers. In one model the property rights (on crumbs) are acquired randomly with respect to equal probabilities, and in the second one players alternate the property rights at each stage in which a crumb is established.

The third model has the feature that crumbs are negotiated by means of an alternating offer procedure with an exogenous probability of breakdown. To avoid discussing extensive games with continuous time[16] we will analyze these games on discrete gradual bargaining problems. We will show that the subgame perfect equilibrium behavior

[16] See Perry and Reny (1993) for such an approach.

in these games[17] induce a bargaining path which converges to our gradual solution as crumbs become smaller and smaller.

Consider a discrete generalized Nash problem with stages of time $t_1,\ldots,t_k$. In the models we will describe below, the bargaining on crumb j at stage t_j takes place after an agreement has been reached on all of the earlier crumbs. A feasible proposal at stage t_j constitutes a point $(x(t_j),y(t_j))$ on the boundary of $D(t_j)$ such that $x(t_j) \geq x^*(t_{j-1})$ and $y(t_j) \geq y^*(t_{j-1})$, where $(x^*(t_{j-1}), y^*(t_{j-1}))$ is the agreement at stage t_{j-1} (on $\partial D(t_{j-1})$). We now describe the procedures in detail:

1. Ultimatum games on Crumbs.

1.1. Alternating the Ultimatum Power

We begin with a model in which the rules of the game represent a situation in which players' property rights are alternated at each stage in which a crumb of the pie is being acquired. At odd stages player 1 proposes a division of the crumb, and at even stages player 2 makes the proposal. Each proposal is a "take it or leave it" offer regarding the allocation of the crumb that has been acquired at that stage. If the offer is accepted then the crumb will be divided accordingly. However, if the offer is rejected then the current crumb is removed and the bargaining continues at the next stage on a new crumb with a feasible proposal by the other player.

1.2 A Random Selection of the Proposing Player

This game is identical to the one described in 1.1, except that instead of alternating the right to make a proposal, the proposer is determined at each stage by a random draw with the probability of ½ for each player.

[17] Our extensive form games are all of perfect information, which means that the way the pies are expanding is assumed to be commonly known at time t=0. However, as we will show later this assumption has no strategic consequences. This is because at each point in time t, a player is better off receiving a larger share from the current crumb regardless of future expansion.

2. Alternating Offer Bargaining on each Crumb.

In this game we will assume that the level curves induce convex sets[18]:

Players use a Rubinstein type bargaining method on each crumb with a probability p of breakdown after a rejection. Specifically, stage t_j starts with a player, say 1, making an offer. If player 2 accepts, then an agreement has been reached on stage t_j's crumb and the bargaining continues on the next crumb at stage t_{j+1}. If 2 rejects the offer, then with probability p the current crumb is removed and the bargaining proceeds on to the next crumb and with probability (1-p) stage t_j continues with an offer by 2. The bargaining in stage t_j continues in this manner until either an agreement is reached regarding the allocation of the current crumb or the crumb has been removed. Note that it is not necessary for the initial proposal on each crumb to be made by the same player.

The proofs of the three propositions below appear in Appendix A.4.

Proposition 5: Let B^i be an infinite sequence of discrete gradual bargaining problems (GBP) which converges to the continuous GBP B. Consider the models 1.1 and let $s^i = \{s^i(t_j)\}_j$ be the agreement path resulting from a subgame perfect equilibrium of the corresponding game with respect to B^i. Then s^i converges to the gradual Nash solution (x(t),y(t)) of the continuous GBP B.

Proposition 6: The same statement as Proposition 5 but with respect to the model 1.2.

Proposition 7: Let B^i be an infinite sequence of discrete GBPs which converges to the continuous GBP B. Consider model 2 and let $s^i = \{s^i(t_j)\}_j$ be the agreement path on B^i resulting from a subgame perfect equilibrium in the limit as p (the probability of breakdown) goes to zero. Then as i goes to infinity s^i converges to the gradual Nash solution (x(t),y(t)) of the continuous GBP B.

[18] We conjecture that the result based on this model will hold true with a weaker version of this assumption.

9. Allocating Assets when Preferences Change over Time

We have previously interpreted our notion of the gradual bargaining problem as representing situations in which players' underlying payoff opportunities expand gradually and non-uniformly. However, an alternative interpretation might view the problem as representing an environment in which the actual physical benefits expand uniformly but players' preferences change over time.

Consider two individuals who are involved in allocating some divisible asset (for example money) that is acquired gradually over the time horizon [0,1]. At time t precisely a portion t of the asset has been acquired. Although both players always prefer more to less of the asset, their utility functions (e.g. risk postulates) *change over time*. We denote by U(t,·) and V(t,·) the utility functions at time t of players 1 and 2 respectively (U(t,w) stands for player 1's time t utility of owning w units of the asset). At time t players available pie in the utility space is given by:

$$R(t) = \{(u,v) \text{ s.t. } u = U(t,w),\ v = V(t,t-w)\ \ 0 \le w \le 1\}$$

We will say that U and V induce the gradual bargaining problem H if at each time t the level curve ∂ D(t) of H coincides with R(t). If U and V induce a gradual bargaining problem, then we can use our solution to predict the way the asset is allocated over time. In the utility space this would simply mean solving the differential equation (1), which defines the GNS. But we will be interested in more than that. For specific applications we would like our solution to predict the actual share of the asset w(t) that player 1 receives at time t. Player 2's share at that point in time will be t-w(t).

Proposition 8 specifies the way w(t) is determined by the utility functions of the players through a differential equation.

Proposition 8: Let U and V be utility functions for players 1 and 2 that induce a GBP. Let $\tilde{V}(t,\cdot)$ denote player 2's utility as a function of player 1's share, i.e. $\tilde{V}(t,w) = V(t,t-w)$. Then at the corresponding gradual Nash solution we have:

$$\frac{dw}{dt} = -\frac{1}{2}\left[\frac{U_t}{U_w} + \frac{\tilde{V}_t}{\tilde{V}_w}\right] \qquad (3)$$

where $U_t, U_w, \tilde{V}_t, \tilde{V}_w$ denote the partial derivatives of U and $\tilde{V}$ with respect to t and w. The proof of Proposition 8 is presented in Appendix A.5

We will refer to equation (3) as the dual equation of the GNS as it determines the way the allocation of the assets depends on the way preferences change over time. Note that equation (3) can be also be written as:

$$\frac{dw}{dt} = \frac{1}{2}\left[1 + \frac{V_t}{V_w} - \frac{U_t}{U_w}\right]$$

Hence player 1 gets a larger share of the crumb in time t if and only if his "MRS" (marginal rate of substitution) for t vs. w is smaller than that of the other player, i.e., $\frac{V_t}{V_w} > \frac{U_t}{U_w}$. Note also that if one of the players' utility function is constant with respect to t (i.e., $V_t = 0$), then, according to the solution, the allocation of the asset depends only on the other player's preferences. This property follows directly from the invariance with respect to increasing transformations, which is satisfied by the GNS in its general form. We will come back to these properties soon when we discuss examples.

Examples

Example 1.

Two risk averse players are facing a stream of assets that have to be allocated. At each infinitesimal period they have to allocate an asset of size ε. Although both players are risk averse, their attitude to risk is different. Player 1's risk aversion changes over time: as time elapses he becomes more risk averse. Specifically, player 1's utility function as a function of time is given by $U(t,w) = -e^{-tw}+1$. Player 2 has a constant risk aversion as a function of t, which is $V(t,w) = -e^{-w/2}+1$. Hence at t = ½ both players have the same risk aversion but player 1 is less risk averse than player 2 prior to time t = ½ and more risk averse later[19]. How should they allocate the stream of payoffs according to our solution?

[19] One can interpret the underlying preferences as representing an interaction between a worker and a firm. The worker's risk aversion grows as he gets closer to retirement. But the firm has a constant risk

Using the dual equation in Proposition 8 we get:

$$\frac{dw}{dt} = \frac{t - w}{2t}$$

Solving the differential equation and setting w(0) = 0, we obtain that the share allocated to player 1 is $w(t) = t/3$. Hence, player 1 always obtains a third of the crumb. Figure 9 below depicts the level curves in this case and the corresponding bargaining path.

Figure 9 (Example A)

Note that this solution does not depend on player 2's risk aversion as long as it is constant in time. Even if player 2 is risk neutral they still share the asset in the same manner. This is the essence of our Invariance axiom. If it were known that a player has a constant risk aversion, then misrepresenting his preferences would never result in a larger share of the asset. The bargaining path (at the utility space) will of course depend on player 2's (constant) risk aversion but not the actual allocation of the asset. Note also that player 1 suffers from the fact that his risk aversion increases in time. This is because his share is always smaller than the one player 2 receives. This will reverse itself when player 1's risk aversion decreases in time.

Example 2.

Player 1's utility function is given by $U(t,w) = -e^{-w/(t+1)}+1$, and player 2's utility function is given by $V(t,w) = -e^{-w/2}+1$. Note that player 1's risk aversion decreases in t. By Proposition 8 we have:

$$\frac{dw}{dt} = \frac{1 + t + w}{2 + 2t}$$

and the solution is $w(t)= 1+t-(1+t)^{1/2}$, yielding player 1 a larger share of the crumb at every point in time. Our observation in the two examples above can be summarized in the following claim:

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aversion. The two have to bargain a scheme that will determine the worker's share in the stream of profits

Claim: Suppose that player 1's utility function is given by $1-e^{-g(t)w}$ and that player 2 has a constant utility function over time. If $g'(t) > 0$ (i.e. growing risk aversion at time t), then $w(t) < 1/2$. However if $g'(t) < 0$ then $w(t) > 1/2$.

The proof of the claim follows directly form the dual equation. We will omit the details here.

We now move to an example in which both players' risk aversion change over time.

Example 3.

$U(t,w) = -e^{-tw}+1$ and $V(t,w) = e^{-wt^a} + 1$, a>0. The resulting differential equation is given by :

$$\frac{dw}{dt} = \frac{(1+a)(t-w)}{2t}$$

and its solution is $w(t) = (1+a)t/(3+a)$. The parameter a indicates the pace by which player 2 become more risk averse as time elapses. The faster his risk aversion grows in time (i.e. the larger a) the smaller his share is. Figure 10 depicts the corresponding level curves and the bargaining curve for a = 0.5.

Figure 10 (Example B)

Example 4.

We now conclude by demonstrating that the bargaining curve in the utility space may be neither concave nor convex:

Take $U(t,w) = -e^{-(t+1)w} + 1$, and $V(t,w) = -e^{-w(t+1)^2} + 1$. We then obtain the equation:

$$\frac{dw}{dt} = \frac{(1+3t-3w)}{(2+2t)}$$

whose solution is $w(t) = (9t-1)/15 + (1+t)^{-3/2}/15$. The bargaining path is depicted in figure 11 below.

Figure 11 (Example C)

earned by the company.

10. Discussion

We have introduced a model of gradual bargaining based on Nash's axiomatic theory. We have constructed the Gradual Nash Solution and supported it by a variety of approaches, including the axiomatic and the non-cooperative approach. We believe that the model can be applied on a variety of topics, which are characterized by the fact that the underlying set of potential agreements changes over time and when players reach temporal agreements that are constantly revised. We will mention here one potential application which we find particularly attractive.

The standard marking to market procedure used in futures contracts has a nature similar to our gradual bargaining problem. This procedure requires the involving parties to rebalance their accounts on a daily basis according the change in the expected gains or losses, rather then waiting for the end of the term. This is done in order to prevent either party to default on its obligations. We suggest that a model of the sort we are analyzing here can be used to address the question of how future contracts can be renegotiated gradually according to the marking to market principle. Specifically, the per period crumb in this story is the change in the future market value of the asset[20].

To broaden the scope of such applications, further extensions of our model are desirable. One which is relatively easy is the construction of an n-person version of our (bilateral) bargaining model. The definition of the solution will be more complex. However it will still involve an ordinary (not partial) differential equation (known as a Pfaffian) which determines a vector field in the n-dimensional space. A different direction of extension is allowing the pies to change over time in a more general way. We would like to allow the pie to expand and to shrink[21] within the same gradual bargaining problem. More ideally we would have liked to allow the level curves $\partial D(t)$ to cut each other, i.e. to allow the set of opportunities to change in an arbitrary way that neither corresponds to expansion nor to

[20] Note that the application of our model to this problem is valid only under the assumption that the futures price of the asset grows.

[21] Note again that the meaning of "shrinkage" here is different than that of Rubinstein's shrinking pie.

shrinkage. These extensions will allow the model to be applied on practically every situation in which the bargaining environment changes over time. At the moment these extensions present some technical difficulties, which we hope can be resolved in future research.

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Appendixes

Appendix A.1

We begin with the definition of convergence of discrete GBP to continuous GBP:

A sequence of discrete gradual bargaining problems $B^j = \{D^j(t^j_1), D^j(t^j_2),\ldots, D^j(t^j_k)\}$ $1\leq j<\infty$, is said to be embedded if $\{D^j(t^j_i)\}\subset\{ D^{j+1}(t^{j+1}_i)\}$ for all j, which means that the problem B^{j+1} refines the set of pies defining B^j.

An embedded sequence of discrete gradual bargaining problems $B^j = \{D^j(t^j_1), D^j(t^j_2),\ldots, D^j(t^j_k)\}$ $1\leq j<\infty$, is said to converge to a continuous bargaining problem H if and only if for any number $\delta > 0$ there exists N, such that for all $j > N$ between any two level curves ∂D_i and ∂D_{i+1} (defined by $H(x,y) = i\cdot\delta$ and $H(x,y) = (i+1)\delta$) there exists at least one curve $\partial D^j(t^j_n)$ of B^j.

The convergence of discrete bargaining paths $s^j = \{s^j(t_1), s^j(t_2),\ldots, s^j(t_k)\}$ to a continuous bargaining path s(t) is taken to be the standard convergence in the Hausdorff topology (of sets in R^2).

Lemmas 1, 2 and 3 assert that the three arbitration schemes discussed in Section 3 yield bargaining paths that converge to our GNS as the discrete gradual bargaining problems converge to a continuous GBP.

Lemma 1: Let $B^j=\{D^j(t_1), D^j(t_2),\ldots, D^j(t_k)\}$ be an infinite sequence of embedded discrete GBPs that converges to the continuous GBP H. Let $s^j(\cdot)$ be the paths generated by arbitration scheme 1 with respect to B^j and $s(\cdot)$ be the Gradual Nash solution of H. Then the paths s^j converge to s as j goes to infinity.

Proof: Consider two consecutive pies of B^j, say $D^j(t_i)$ and $D^j(t_{i+1})$. Because of the convergence property their boundaries $\partial D^j(t_i)$ and $\partial D^j(t_{i+1})$ are arbitrarily close to two level curves of H. Say $\partial D^j(t_i)\approx\{H(x,y) = c\}$ and $\partial D^j(t_{i+1})\approx\{H(x,y) = c+\delta\}$. By construction of the Arbitration scheme 1 the two consecutive agreement points $s^j(t_i)=(x^j(t_i), y^j(t_i))$ and $s^j(t_{i+1})=(x^j(t_{i+1}), y^j(t_{i+1}))$ must satisfy the Nash condition

$$\max((x - x^j(t_i)) (y - y^j(t_i)))$$

among all (x,y) on $\partial D^j(t_{i+1})$. Denote $\Delta x = x^j(t_{i+1}) - x^j(t_i)$, $\Delta y = y^j(t_{i+1}) - y^j(t_i)$. Then the Nash condition can be approximated by the following restricted maximization problem $\max(\Delta x \cdot \Delta y)$ under the following constraints: $H(x,y) = c$, $H(x+\Delta x, y+\Delta y) = c+\delta$, see figure 3. Using the Taylor series and keeping only first order terms we obtain

$$\begin{cases} \max(\Delta x \Delta y) \\ s.t.\ H_x \Delta x + H_y \Delta y = d \end{cases}$$

which can also be written as an unconstrained maximization problem

$$\max(\Delta y \cdot (\delta - H_y \Delta y)/H_x).$$

This quadratic maximization can be solved by: $\Delta y = \delta/(2H_y)$, $\Delta x = \delta/(2H_x)$. This implies that the increments Δx and Δy satisfy the equation $\Delta y/\Delta x = H_x/H_y$ which in the limit is the equation (1). Because of the uniqueness of the solution of equation (1) (the initial bargaining point is fixed at the origin) the limit of solutions of discrete GBPs is the continuous solution of the limit GBP. **Q.E.D.**

Lemma 2: Let B^j be an infinite sequence of discrete GBPs that converges to the continuous GBP H. Let $s^j(\cdot)$ be the paths generated by arbitration scheme 2 with respect to B^j and $s(\cdot)$ be the Gradual Nash solution of B. Then the paths s^j converge to s as j goes to infinity.

Proof: Consider three consecutive pies of B^j, say $D^j(t_i)$, $D^j(t_{i+1})$, and $D^j(t_{i+2})$. Because of the convergence property their boundaries $\partial D^j(t_i)$, $\partial D^j(t_{i+1})$ and $\partial D^j(t_{i+2})$ are very close to three level curves of H. Say $\partial D^j(t_i) \approx \{H(x,y) = c\}$, $\partial D^j(t_{i+1}) \approx \{H(x,y) = c+\delta\}$ and $\partial D^j(t_{i+2}) \approx \{H(x,y) = c+2\delta\}$. By construction of the Arbitration scheme 2 the three consecutive agreement points $s^j(t_i)=(x^j(t_i), y^j(t_i))$, $s^j(t_{i+1})=(x^j(t_{i+1}), y^j(t_{i+1}))$ and $s^j(t_{i+2})=(x^j(t_{i+2}), y^j(t_{i+2}))$ must satisfy the following conditions (see figure 4)

$$x^j(t_{i+1}) = x^j(t_i)+\Delta x,\ y^j(t_{i+1}) = y^j(t_i)$$

$$x^j(t_{i+2}) = x^j(t_{i+1}),\ y^j(t_{i+2}) = y^j(t_{i+1})+\Delta y$$

At the first step the first player takes the whole crumb and at the second step the second player takes the next crumb (the crumbs are of equal size δ). Using the level curves of H instead of $\partial D^j(t)$ we can rewrite these conditions as

$$H(x^j(t_{i+1}), y^j(t_{i+1})) = H(x^j(t_i), y^j(t_i)) + \delta$$
$$H(x^j(t_{i+2}), y^j(t_{i+2})) = H(x^j(t_{i+1}), y^j(t_{i+1})) + \delta$$

By Arbitration scheme 2 this can also be written as

$$H(x^j(t_i)+\Delta x, y^j(t_i)) = H(x^j(t_i), y^j(t_i)) + \delta$$
$$H(x^j(t_i)+\Delta x, y^j(t_i)\ )+\Delta y) = H(x^j(t_i)+\Delta x, y^j(t_i)) + \delta$$

Using the Taylor series and keeping only first order terms we obtain

$$H(x^j(t_i), y^j(t_i)) + H_x(x^j(t_i), y^j(t_i))\Delta x = H(x^j(t_i), y^j(t_i)) + \delta$$
$$H(x^j(t_i), y^j(t_i)) + H_x(x^j(t_i), y^j(t_i))\Delta x + H_y(x^j(t_i), y^j(t_i))\Delta y = H(x^j(t_i), y^j(t_i)) + 2\delta$$

Or after simplification

$$H_x(x^j(t_i), y^j(t_i))\Delta x = \delta$$
$$H_y(x^j(t_i), y^j(t_i))\Delta y = \delta,$$

which is equivalent to the differential equation (1) when δ tends to zero. Because of the uniqueness of the solution of equation (1) (the initial bargaining point is fixed at the origin) the limit of solutions of discrete GBPs is the continuous solution of the limit GBP. **Q.E.D**.

Lemma 3: Consider the arbitration scheme on a discrete GBP in which at each stage the next crumb is given to a player who is randomly chosen with equal probabilities. Let B^j be an infinite sequence of discrete GBPs that converges to the continuous GBP H. Let $s^j(\cdot)$ be the paths generated by the arbitration scheme described above with respect to B^j and $s(\cdot)$ be the Gradual Nash solution of B. Then the paths s^j converge to s as j goes to infinity.

Proof: Consider two consecutive pies of B^j, say $D^j(t_i)$ and $D^j(t_{i+1})$. Because of the convergence property their boundaries $\partial D^j(t_i)$ and $\partial D^j(t_{i+1})$ are arbitrarily close to two level curves of H. Say $\partial D^j(t_i) \approx \{H(x,y) = c\}$ and $\partial D^j(t_{i+1}) \approx \{H(x,y) = c+\delta\}$. In this arbitration scheme the next agreement point $s^j(t_{i+1}) = (x^j(t_{i+1}), y^j(t_{i+1}))$ is not known in advance but is a result of a random draw. However one can easily calculate the expected position of this point. With equal probabilities the next agreement point will be at

$(x^j(t_i)+\Delta^*x, y^j(t_i))$ or at $(x^j(t_i), y^j(t_i)+\Delta^*y)$, such that $H(s^j(t_{i+1})) = c + \delta$. Hence, the expected position of the next intermediate agreement point is

$$(x^j(t_i)+\Delta^*x/2,\ y^j(t_i)+\Delta^*y/2)$$

where

$$\begin{cases} H(x^j(t_i)+\Delta^* x, y^j(t_i)) = c + d \\ H(x^j(t_i), y^j(t_i)+\Delta^* y) = c + d \end{cases}$$

using the Taylor series decomposition we obtain

$$\begin{cases} H_x \Delta^* x = d \\ H_y \Delta^* y = d \end{cases}$$

and thus the increments Δx and Δy satisfy $\dfrac{\Delta y}{\Delta x} = \dfrac{H_x}{H_y}$

which in the limit becomes the equation (1). Because of the uniqueness of a solution of equation (1) (the initial bargaining point is fixed at the origin) the limit of solutions of discrete GBPs is the continuous solution of the limit GBP**. Q.E.D.**

We conclude this section by arguing that an arbitration scheme based on the Kalai-Smorodinsky (K-S) solution will lead to the same GBS given by equation 1:

The K-S solution of a standard bargaining problem S is given by the intersection between the Pareto frontier ∂S and the line connecting the zero point and the utopian point (m_1,m_2) (where $m_1 = \max \{x; (x,y) \in S\}$)

Starting with an agreement (x,y) on some level curve we obtain the following equations:

$$\begin{cases} H(x,y) = c \\ H(x+\Delta^* x, y) = H(x,y) + d \\ H(x, y+\Delta^* y) = H(x,y) + d \end{cases}$$

Where Δ^*x and Δ^*y are the utopian levels for players 1 and 2 respectively. Since locally, the level curves are virtually linear the new agreement is at $(x+\Delta x, y+\Delta y)$, where $(\Delta x, \Delta y)=(\Delta^*x/2, \Delta^*y/2)$. Now use the standard Taylor expansion to establish the rest as we have done before.

Appendix A.2

The Proof of Theorem 1 (axiomatization):

We start by showing that our solution satisfies the 4 axioms.

1. Efficiency. As argued in Remark 1, equation (1) determines the bargaining curve but not its parameterization. In its parametric form the solution was defined to satisfy $H(x(t),y(t)) = t$ which means $(x(t),y(t)) \in \partial D(t)$.

2. Symmetry. Note that if $H(x,y) = H(y,x)$, then $H_x(x,y) = H_y(x,y)$ and thus $dx/dy = 1$, which means that the solution is simply the straight line $x = y$.

4. IIT: Let $a = A(x)$ and $b=B(y)$ be two smooth and increasing transformations from R_+ to R_+, such that $A(0)=0$ and $B(0)=0$. Let h and H be two GBPs such that

$$h(a, b) = h(A(x), B(x)) = H(x, y).$$

Differentiating H with respect to x yields $H_x = h_a A'(x)$. Differentiating H with respect to y yields $H_y = h_b B'(y)$. Note also that differentiating a and b with respect to x and y respectively we get: $da = A'(x)\, dx$, and $db = B'(y)\, dy$

Since A and B are increasing A' and B' are positive, equation (1) can be written as

$$\frac{db}{B'(y)}\frac{A'(x)}{da} = \frac{h_a A'(x)}{h_b B'(y)},$$

or

$$\frac{db}{da} = \frac{h_a(a,b)}{h_b(a,b)},$$

which is the same equation defined on the new variables a, b for the GBP h.

We now move to the proof of the uniqueness of the solution. We first argue that a monotonic transformation of utilities preserves the property $H_x > 0$, $H_y > 0$. To see this write $h(U(x), V(y)) = H(x,y)$ and differentiate with respect to x to obtain: $h_u U_x = H_x$. So $H_x > 0$, implies immediately $h_u > 0$.

Note that the IIT axiom partitions the set of all bargaining problems into equivalence classes. Two functions (problems) are said to belong to the same class if and only if one can be obtained from the other through increasing transformations U and V.

We first demonstrate that there is at least one symmetric bargaining problem in each equivalence class. To show this we construct explicitly the transformation which turns any problem into a symmetric one. Take any bargaining problem H(x,y) and find a transformation U(x), V(y) or its inverse X(u), Y(v) such that the transformed problem is $H(X(u),Y(v)) = h(u,v)$ and $h(v,u) = h(u,v)$ for every u and v. Then

$$H(X(u),Y(v)) = h(u,v) = h(v,u) = H(X(v),Y(u)) .$$

Differentiating this expression with respect to u and v we get:

$$H_x(X(u),Y(v))X'(u) = H_y(X(v),Y(u))Y'(u)$$
$$H_y(X(u),Y(v))Y'(v) = H_x(X(v),Y(u))X'(v)$$

This defines the transformation uniquely (subject to the initial condition X(0) = Y(0) = 0) by the differential equation:

$$\frac{dY}{dX} = \frac{H_x(X,Y)}{H_y(X,Y)}$$

This equation defines a change of variables that transforms the bargaining problem H into a symmetric bargaining problem h. Note that this transformation is precisely the GNS. Since the solution of the symmetric bargaining problem is known, the original problem H has a unique solution as well. We next demonstrate that every two symmetric problems belonging to the same equivalence class have the same solution.

Consider any two symmetric bargaining problems belonging to the same class of equivalence. Denote them by h and H and note that they can be transformed into each other by some transformations U and V. This means that

$$h(u,v) = h(v,u)$$
$$H(x,y) = H(y,x)$$
$$h(U(x),V(y)) = H(x,y)$$
$$h(u,v) = H(X(u),V(y))$$

Due to the symmetry we get

$$h(U(x),V(y)) = h(U(y),V(x)) = h(V(x),U(y)) = h(V(y),U(x)) ,$$

and differentiating with respect to x gives

$$h_u(U(x),V(y))U'(x) = h_v(U(y),V(x))V'(x) = h_u(V(x),U(y))V'(x) = h_v(V(y),U(x))V'(x) ,$$

which immediately implies U’(x) = V’(x). Because U(0) = V(0) we immediately obtain V(x) = U(x) for all x. Thus H and h induce the same level curves (see Remark 1), and therefore have the same solution. We have shown that all symmetric problems of the same class yield the same solution. To show the uniqueness we now argue as follows: take a problem H and a symmetric problem h in its equivalence class. By the Efficiency and the Symmetry axioms the solution is uniquely determined on h. Because all symmetric problems in that class have the same solution and since H is transformed to h in a unique fashion we obtain by the ITT axiom that the solution is uniquely determined on H. This completes the proof of Theorem 1. **Q.E.D.**

Claim 1: The three axioms Symmetry, Efficiency and IIT are independent.

Proof: If we drop the Symmetry axiom then every asymmetric GNS (resulting from equation (2) defined in Section 7.1) satisfies the two remaining axioms. If we drop the Efficiency axiom, then the solution (x(t),y(t)) = (0,0) for all t and all GBPs satisfies the two remaining axioms. Finally if we drop the IIT, then the solution that yields the standard Nash point on every pie satisfies the two remaining axioms. **Q.E.D.**

Claim 2: The GNS satisfies the following version of the IIA Axiom: Let H and H* be two GBPs such that $D(t) \subseteq D^*(t)$ for all t. If s(H*) is feasible with respect to H[22], then s(H) = s(H*).

Proof: IIA: Let H and H* be as specified in the statement of the IIA. Let (x(t),y(t)) be the solution for H and (x*(t),y*(t)) the solution for H*, satisfying the following equation

$$\frac{dy^*}{dx^*} = \frac{H_x^*(x^*, y^*)}{H_y^*(x^*, y^*)} \tag{1*}$$

Note that H* lies below H, since each level curve of H* is further to the right from the corresponding level curve of H. Each function H and H* generates its own bargaining vector field. To define this vector field we can rewrite equation (1) in an equivalent form

$$H_y dy = H_x dx$$

or

$$(dx, dy) \perp (H_x, -H_y).$$

However any vector orthogonal to $(H_x, -H_y)$ is also parallel to the vector (H_y, H_x), thus equation (1) is equivalent to the condition that the bargaining direction is always parallel

[22] which means $s(H^*)(t) \in D(t)$.

to the rotated gradient $(dx, dy) \| (H_y, H_x)$. The vector field determining the bargaining dynamic has the property that at each point (x, y) the bargaining direction is parallel to (H_y, H_x). The two vector fields defined by H and H* are different. However both functions coincide along the curve s(H)(t)=(x(t), y(t)). Moreover, they have a common (parallel) gradient along this curve, since the level curves are tangential along s(H) and cannot pass through each other. Otherwise the condition D(t) $\subseteq$ D*(t) is violated. Since both functions have parallel gradients along s(H), the two vector fields generated by H and H* are also parallel along this curve. However, s(H) is parallel to the vector field generated by H, hence it is also parallel to the vector field generated by H*. Therefore the curve s(H) must also satisfy equation (1*).

Since both H and H* are smooth and do not have critical points there exists a unique solution to the differential equation specified by the vector field generated by H*. However we have demonstrated that s(H*) is the unique solution of the modified problem, hence s(H) = s(H*).

Formally, if H(x,y) ≥ H*(x,y), H(x(t),y(t))=H*(x(t),y(t)) and (H_x(x(t),y(t)), H_y(x(t),y(t)) | | (H^*_x(x(t),y(t)), H^*_y(x(t),y(t)), then the curve (x(t),y(t)) with the initial condition x(0) = y(0) = 0 satisfies equation (1). It also satisfies equation (1*), since the two vector fields coincide on this curve, and this is its unique solution starting at the origin. **Q.E.D.**

We conclude this appendix by showing that our solution also satisfies a version of the Kalai-Smorodinsky (1975) monotonicity axiom. We will assume that the pies D(t) are compact for this matter.

Claim 3: The GNS satisfies the following property: Let H and H* be two GBP such that for any t, and t' ∂D*(t) and ∂D(t') intersect not more than once. Without loss of generality, assume that ∂D*(t) is always steeper. Then the GNS of H* is always above the GNS of H.

Proof: The condition that the level curves of one problem are always steeper than those of the other implies that the vector field generated by equation (1) for H* has a higher slope at any point than the one for H (see figure 12). The result follows immediately from the uniqueness of the solution starting at the origin. **Q.E.D.**

Figure 12. The Kalai-Smorodinsky property

The Kalai Smorodinsky axiom in the standard framework imposes that if we take two problems B and B* such that the maximal payoff for player 1 is the same in the two problems and such that $B \subset B^*$, then player 2's payoff with respect to B* is greater than his payoff with respect to b. Not e that if we generate two GBP H and H* by shifting the frontiers of B and B* homothetically (i.e. D(t) = tB and D*(t) = tB*), then by claim 3 the NGS yields player 2 a higher payoff in H* at each point in time. It is this sense in which claim 3 generalizes the Kalai-Smorodinsky axiom.

Appendix A.3

Proof of Proposition 1. Let Y(x) denote the outside boundary of D(1). Y(x) is a well defined function due to the conditions $H_x > 0$, $H_y > 0$. This means that for any time t and for any point (x,y) on the frontier of D(t) there exists some λ between 0 and 1 for which (x/λ,y/λ) lies on the Y curve, or alternatively satisfies the following equation:

$$\frac{y}{l} = Y\left(\frac{x}{l}\right). \qquad (4)$$

Consider the family of curves parameterized by λ as $Y_l(x) = l\,Y\left(\frac{x}{l}\right)$. The frontier of D(1) corresponds to Y_1, and all the curves Y_λ are homothetic.

The standard Nash solution for the bargaining problem given by D(1) is obtained by maximizing x·y over the set of points that satisfy equation (4) for λ = 1. Hence by the first order condition we get:

$$\frac{d\left(xY(x)\right)}{dx} = 0.$$

Alternatively $$Y(x_N) + x_N Y'(x_N) = 0\,. \tag{5}$$

Where $(x_N, Y(x_N))$ is the standard Nash solution of the problem D(1).

Set a function H such that
$H(x,y) = l\,(x,y)$,
where $\lambda(x,y)$ is the coefficient such that the point $(x/\lambda, y/\lambda)$ lies on the final frontier Y. This assumption is without loss of generality since the GNS depends on the H function only through its level curves. Using equation (4) we now obtain
$$\frac{y}{l\,(x,y)} = Y\left(\frac{x}{l\,(x,y)}\right). \tag{6}$$
Differentiating (6) with respect to x gives:
$$\frac{-y\,l_x}{l^2} = Y'\left(\frac{x}{l}\right)\left(\frac{l - x\,l_x}{l^2}\right),$$
or $$-y\,l_x = Y'\left(\frac{x}{l}\right)\left(l - x\,l_x\right).$$

Differentiating (6) with respect to y gives:
$$\frac{l - y\,l_y}{l^2} = Y'\left(\frac{x}{l}\right)\frac{-x\,l_y}{l^2},$$
or $$-\,l + y\,l_y = Y'\left(\frac{x}{l}\right)x\,l_y\,.$$

Finally we obtain:
$$\begin{cases} l_x = \dfrac{-\,lY'}{y - xY'} \\[2ex] l_y = \dfrac{l}{y - xY'} \end{cases}$$

Hence using (1) the GNS is given by
$$y'(x) = \frac{H_x(x,y)}{H_y(x,y)} = \frac{l_x(x,y)}{l_y(x,y)}$$
or $$y'(x) = -Y'\left(\frac{x}{l\,(x,y)}\right) \tag{7}$$

We now have to show that the solution of this equation (7) starting at the origin arrives at the same point as (x_N, $Y(x_N)$) defined above. We do this by showing that the solution in this case is a straight line.

Because of the homothetic property the tangential line to any curve Y_λ has the same slope along a straight line passing through the origin and the Nash Solution as is shown in the figure 5.

This means that at any point on the straight line $\left(\frac{x_N}{l}, \frac{y_N}{l}\right)$ the tangential line to the Y_λ curve has the same slope. In addition the vector that defines the direction of the bargaining process $-Y'(x/\lambda)$ has the same direction for all λ. Hence, the only property we have to verify is that the vector at the Nash point has the right direction, in other words that it points to the origin (because of the negative sign, the origin is on the line continuing this vector). This property means that for some α

$$\frac{Y(x_N)}{x_N} = aY'(x_N),$$

which is exactly the property we had in (5). Thus the straight line from the origin to the Nash Solution is the bargaining path. Finally, note that the same result holds for any D(t), and not only for t =1. To prove this simply repeat the argument with respect to a GBP H for which D(t) is the final pie. **Q.E.D.**

Below is the proof that establishes the relationship between our solution and the Asymmetric Nash Solution.

Proof of Proposition 2: The proof is similar to the one we have given in Lemma 2. However because of the unequal crumbs the equations are slightly different. Using the same notations

$$x^j(t_{i+1}) = x^j(t_i)+\Delta x,\ y^j(t_{i+1}) = y^j(t_i)$$

$$x^j(t_{i+2}) = x^j(t_{i+1}),\ y^j(t_{i+2}) = y^j(t_{i+1})+\Delta y$$

At the first step the first player takes the whole crumb of size $p\delta$ and at the second step the second player takes the whole crumb of equal size $q\delta$.

$$H(x^j(t_{i+1}), y^j(t_{i+1})) = H(x^j(t_i), y^j(t_i)) + p\delta$$

$$H(x^j(t_{i+2}), y^j(t_{i+2})) = H(x^j(t_{i+1}), y^j(t_{i+1})) + q\delta$$

Using the Taylor series and keeping only first order terms we obtain

$$H(x^j(t_i), y^j(t_i)) + H_x(x^j(t_i), y^j(t_i))\Delta x = H(x^j(t_i), y^j(t_i)) + p\delta$$

$$H(x^j(t_i), y^j(t_i)) + H_x(x^j(t_i), y^j(t_i))\Delta x + H_y(x^j(t_i), y^j(t_i))\Delta y = H(x^j(t_i), y^j(t_i)) + (p+q)\delta$$

Or after simplifying

$$H_x(x^j(t_i), y^j(t_i))\Delta x = p\delta$$

$$H_y(x^j(t_i), y^j(t_i))\Delta y = q\delta$$

Thus leading to the asymmetric gradual bargaining equation

$$\frac{dy}{dx} = \frac{q}{p}\frac{H_x(x,y)}{H_y(x,y)}.$$

Since this equation has a unique solution satisfying the initial condition x(0) = y(0) = 0, the sequence of discrete gradual solutions tends to the continuous solution described by this equation. **Q.E.D.**

Proof of Proposition 3: The proof is similar to the one in Lemma 3. However because of the unequal probabilities the equations are slightly different. Using the same notation the expected next intermediate agreement point is

$$(x^j(t_i) + p\Delta^* x,\ y^j(t_i) + q\Delta^* y)$$

where again

$$\begin{cases} H(x^j(t_i) + \Delta^* x, y^j(t_i)) = c + d \\ H(x^j(t_i), y^j(t_i) + \Delta^* y) = c + d \end{cases}$$

using the Taylor series decomposition we find the relationship between increments Δx and Δy

$$\frac{\Delta y}{\Delta x} = \frac{q}{p}\frac{H_x(x,y)}{H_y(x,y)}.$$

The same uniqueness property implies that the limiting solution of the discrete problems is the solution of this differential equation. **Q.E.D.**

Proof of Proposition 4: The proof extends the homothetic case. Consider a symmetric continuous bargaining procedure over a quasi-homothetic pie with parameter p. This means that any level curve {(x,y) | H(x,y) = c} can be represented in the form {(λ^{1-p}x, λ^{p}y) | H(x,y) = 1}. Denote the final level curve {(x, y) | H(x,y) = 1} by Y(x) and then all level curves can be represented as a family $Y_\lambda(x)$ given by

$$Y_l(x) = l^p Y\left(\frac{x}{l^{1-p}}\right).$$

The solution to the standard asymmetric Nash problem is at the point (x,y) on Y(x) where $x^p y^{1-p}$ achieves its maximum. This can also be written as the first order condition

$$\frac{d\left(x^p Y^{1-p}(x)\right)}{dx} = 0,$$

which means that at the asymmetric Nash solution (x_{AN}, y_{AN}) the following equation must hold $$p x_{AN}^{p-1} Y^{1-p}(x_{AN}) + (1-p) x_{AN}^{p} Y^{-p}(x_{AN}) Y'(x_{AN}) = 0,$$

or alternatively $pY(x_{AN}) + (1-p) x_{AN} Y'(x_{AN}) = 0$,

which implies that $\frac{y_{AN}}{x_{AN}} = \frac{Y(x_{AN})}{x_{AN}} = \frac{p-1}{p} Y'(x_{AN})$. This defines the asymmetric Nash solution.

According to Remark 2 any function H with the same level curves must generate the same bargaining process. Let's choose a specific form for H. Similarly to what we did in the homothetic case, for any point (x, y) there is a number λ such that the point (x/λ^{1-p},y/λ^{p}) is located on the final frontier Y(x). This λ is a function of x and y. Set H(x, y) = λ(x, y). Then for all pairs (x, y) on the same curve $Y_\lambda(x)$ the coefficient λ is the same and hence automatically the value of H is the same, meaning that $Y_\lambda(x)$ are level curves of the function H as desired. We can now write

$$\frac{y}{l^p(x,y)} = Y\left(\frac{x}{l^{1-p}(x,y)}\right)$$

Differentiating this expression with respect to x and y we obtain

$$l_x = \frac{-Y' l^{p-1}}{yp l^{-p-1} + Y' x(p-1) l^{p-2}}$$

$$l_y = \frac{l^{-p}}{yp l^{-p-1} + Y' x(p-1) l^{p-2}}$$

Recall that we have chosen H(x, y)=λ(x, y), and hence equation (2) becomes

$$\frac{dy}{dx}=\frac{H_x(x,y)}{H_y(x,y)}=\frac{l_x(x,y)}{l_y(x,y)}=-Y'\left(\frac{x}{l^{1-p}(x,y)}\right)l^{2p-1}(x,y)$$

Verify that the curve x(t) = $t^{1-p}x_{AN}$, y(t) = $t^p y_{AN}$ satisfies this equation

$$\frac{dy(t)}{dx(t)}=\frac{pt^{p-1}y_{AN}}{(1-p)t^{-p}x_{AN}}=\frac{pt^{p-1}}{(1-p)t^{-p}}\frac{(p-1)}{p}Y'\left(x_{AN}\right)=-Y'\left(\frac{x(t)}{t^{1-p}}\right)t^{2p-1},$$

which shows that t = λ is the right parameterization. Thus we have proved that a quasi-homothetic pie with a symmetric bargaining procedure leads to the Asymmetric Nash Solution on D(1). **Q.E.D.**

Appendix A.4

Lemma 4 asserts that a player's payoff according to the Nash solution in a standard problem is an increasing function of his disagreement point and a decreasing function of the other player's disagreement point.

Lemma 4: Let B_1 be a standard Nash bargaining problem with the set of payoffs S and a disagreement point (x_0,y_0). Let B_2 be a bargaining problem with respect to the same set of payoffs S but with the disagreement point being ($x_0+\Delta x$, $y_0-\Delta y$), where $\Delta x,\Delta y > 0$. If N(·) denotes the Nash bargaining solution, then $N(B_1)\leq N(B_2)$.

Proof: Recall first that the Nash solution of the problem B_1 is $\arg\max_S(y-y_0)(x-x_0)$. Let (x,g(x)) be the point on the Pareto frontier of S. Then the Nash payoff for player 1 is the solution of $\max(g(x)-y_0)(x-x_0)$. Differentiate with respect to x we get: $g'(x)(x-x_0)=y_0-g(x)$. But as we change the disagreement point (x_0, y_0), the agreement point (x, y) changes. Hence the agreement point can be considered as a function of the point (x_0, y_0). We thus have:

$$g'(x(x_0))(x(x_0)-x_0)=y_0(x_0)-g(x(x_0))$$

Differentiating with respect to x_0 we get

$$g''(x)x'(x-x_0)+g'(x)(x'-1)=y_0'-g'(x)x'$$

and $$x'(x_0) = \frac{y_0' - g'(x)}{g''(x)(x - x_0) + 2g'(x)}.$$

When x_0 increases y_0 decreases and so $y_0' < 0$, or $g' < 0$. Moreover $g'' \leq 0$, and $x > x_0$, and so the derivative $x'(x_0)>0$, which means that the payoff of the first player increases with x_0. **Q.E.D.**

Proof of Proposition 5: Along the unique equilibrium of the model above each player takes the whole crumb at each stage in which he is making the proposal, and accepts any offer made by the other player at all other stages. This follows from the fact that the proposer has an ultimatum power and from the fact that the higher a player's share is from stage t_j's crumb, the higher his ultimate (total) payoff is. Therefore we can in fact treat each crumb as a separate ultimatum game. Hence the subgame perfect equilibrium of the above game gives rise to the same bargaining path as that of arbitration scheme 2, which by Lemma 2, we know converges to the gradual Nash solution. **Q.E.D.**

Proof of Proposition 6: Using the same argument as in proposition 5 we establish that ex-post after the proposer has been randomly chosen at stage j, he acquires the whole crumb for himself at the unique subgame perfect equilibrium of the game. This means that the discrete path of expected payoffs resulting from a subgame perfect equilibrium is identical to the path generated by the arbitration scheme discussed in Lemma 3, and the result follows directly from that Lemma. **Q.E.D.**

Proof of Proposition 7:Take a discrete GBP and consider the game induced by a single stage t_j of model 2 after an agreement $s(t_{j-1})$ has been reached on stage t_{j-1}. This game is identical to the one used by Binmore Rubinstein and Wolinsky (1986) to support the Nash solution. It follows from their result that as p goes to zero the subgame perfect equilibrium of this game converges to the Nash solution when the disagreement point is taken to be $s(t_{j-1})$. By Lemma 4 the Nash payoff for a player is increasing with respect to his payoff in the disagreement point, which means that the subgame perfect equilibrium of the whole game can be derived by looking at the game of each stage separately.

Consequently after sending p to zero the agreement path derived from the subgame perfect equilibrium of the game coincides with the path generated by arbitration scheme 1 (in section 4), which by Lemma 1, we know converges to the GNS of B when B^i converge to B. **Q.E.D.**

Appendix A.5

Proof of Proposition 8: Denote by ω(t) the total share of the assets that player 1 possesses at time t with respect to the GNS. Then the two players' utility level at time t is:

$$\begin{cases} u = U(t, w(t)) \\ v = V(t, t - w(t)) \end{cases}$$

(remember that the second player always gets t – ω(t)).

For an arbitrary solution we have:

$$\begin{cases} u = U(t, w) \\ v = V(t, t - w) \end{cases} \qquad \text{(A)}$$

Let:

$\tilde{V}(t, w) = V(t, t - w)$ as defined earlier.

Then the system (A) can be written as

$$\begin{cases} u = U(t, w) \\ v = \tilde{V}(t, w) \end{cases}$$

This system of equations can be inverted by considering t and w as functions of u, v. Specifically, given u, v we can express the time and the division w using the two functions T(u,v) and W(u,v). We therefore have:

$$\begin{cases} u = U(T(u, v), W(u, v)) \\ v = \tilde{V}(T(u, v), W(u, v)) \end{cases}$$

Differentiating these two equations with respect to u and v we get:

$$\begin{cases} 1 = U_t T_u + U_w W_u \\ 0 = U_t T_v + U_w W_v \\ 0 = \tilde{V}_t T_u + \tilde{V}_w W_u \\ 1 = \tilde{V}_t T_v + \tilde{V}_w W_v \end{cases}$$

Expressing the partial derivatives of W and T as a function of the partial derivatives of U and $\tilde{V}$ we obtain:

$$\begin{cases} W_u = \dfrac{-\tilde{V}_t}{U_t \tilde{V}_w - U_w \tilde{V}_t} \\ W_v = \dfrac{U_t}{U_t \tilde{V}_w - U_w \tilde{V}_t} \\ T_u = \dfrac{\tilde{V}_w}{U_t \tilde{V}_w - U_w \tilde{V}_t} \\ T_v = \dfrac{-U_w}{U_t \tilde{V}_w - U_w \tilde{V}_t} \end{cases}$$

Denote by X(t), Y(t) the GNS in its parametric form.. According to our choice of parameterization we have H(X(t),Y(t)) = t. But we have already constructed a function with this property – the T function. Thus we can set H(u,v) = T(u,v) and we obtain the equation:

$$T(X(t),Y(t)) \equiv t$$

(the sign ≡ refers to the fact that the identity holds for all t).

Differentiating this equation with respect to t we get:

$$T_u X'(t) + T_v Y'(t) \equiv 1$$

According to the equation defining the GNS:

$$\frac{dY(t)}{dX(t)} = \frac{Y'(t)dt}{X'(t)dt} = \frac{T_u(X(t),Y(t))}{T_v(X(t),Y(t))}$$

And the two equations above jointly yield:

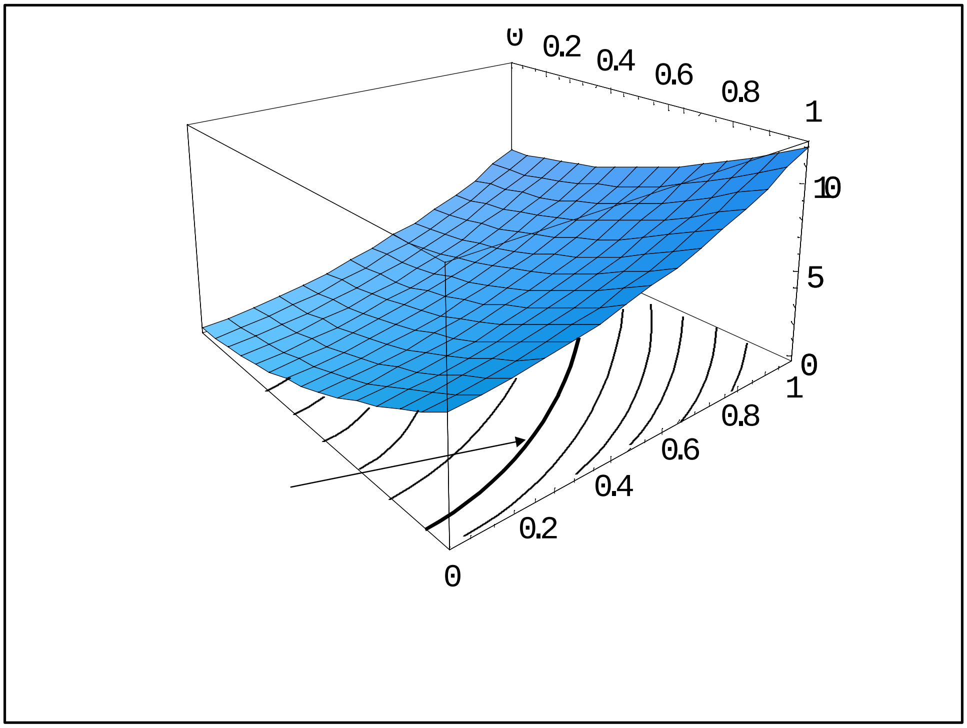
$$T_u Y'(t)\frac{T_v}{T_u} + T_v Y'(t) \equiv 1$$

or
$$\frac{dY(t)}{dt} = \frac{1}{2T_v(X(t),Y(t))}$$

We can now express the physical divisions of the assets (according to the GNS) as a function of time:

$$\frac{dw(t)}{dt} = W_u(X(t),Y(t))\frac{dX(t)}{dt} + W_v(X(t),Y(t))\frac{dY(t)}{dt} =$$
$$W_u(X(t),Y(t))\frac{dX(t)}{dY(t)}\frac{dY(t)}{dt} + W_v(X(t),Y(t))\frac{dY(t)}{dt} =$$
$$\frac{-\tilde{V}_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}\frac{T_v}{T_u}Y'(t) + \frac{U_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}Y'(t) =$$
$$\frac{-\tilde{V}_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}\frac{T_v}{T_u}\frac{1}{2T_v} + \frac{U_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}\frac{1}{2T_v} =$$
$$\frac{-\tilde{V}_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}\frac{U_t\tilde{V}_w - U_w\tilde{V}_t}{\tilde{V}_w}\frac{1}{2} + \frac{U_t}{U_t\tilde{V}_w - U_w\tilde{V}_t}\frac{U_t\tilde{V}_w - U_w\tilde{V}_t}{-U_w}\frac{1}{2} =$$
$$-\frac{1}{2}\left(\frac{\tilde{V}_t}{\tilde{V}_w} + \frac{U_t}{U_w}\right)$$

Setting $\omega(0) = 0$ as an initial condition we obtain the differential equation asserted in Proposition 8. **Q.E.D.**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1
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0
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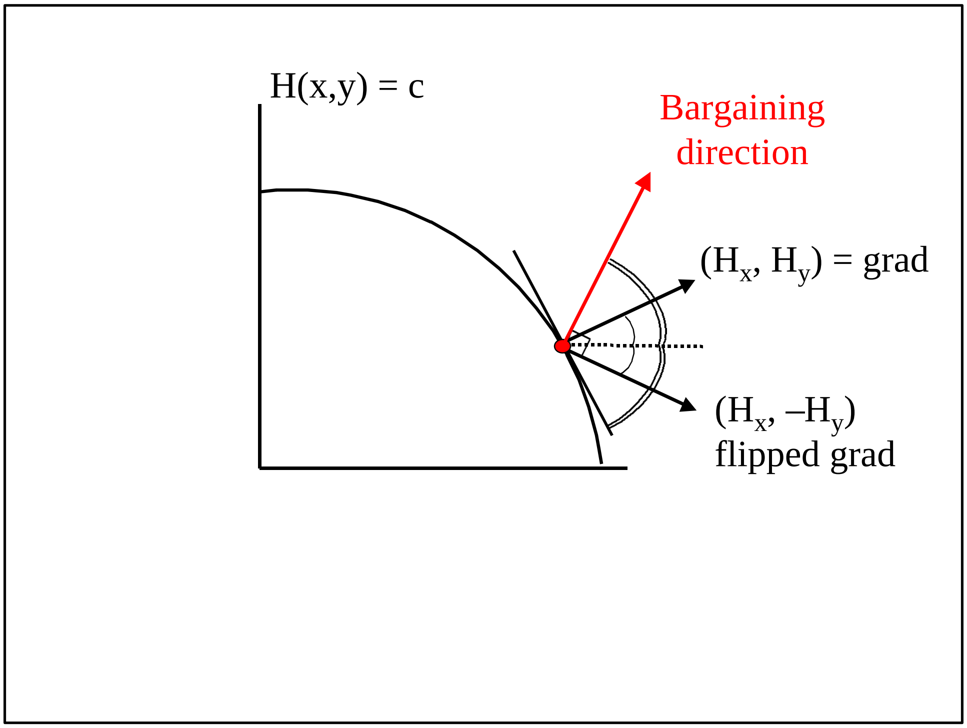
0


H(x,y) = c
Bargaining direction
(H_x, H_y) = grad
$(H_x, -H_y)$
flipped grad

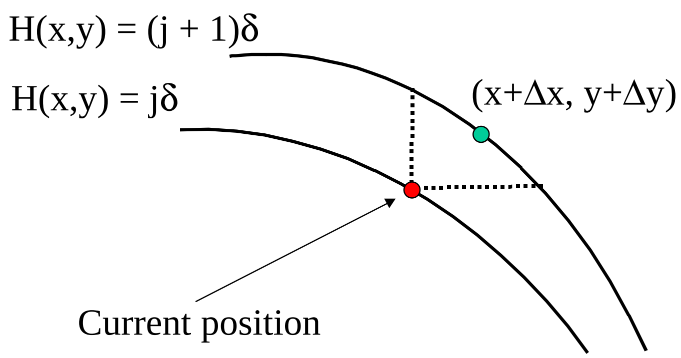


Figure 3.
Nash on each crumb

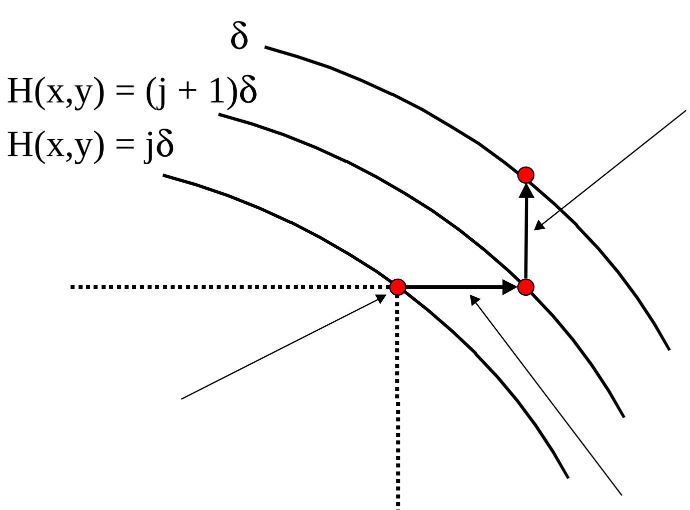


Figure 4.
Alternating turns

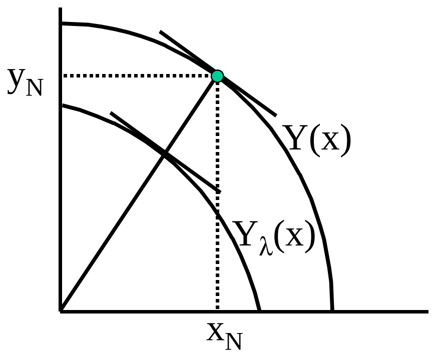


Figure 5.
Homothetic level curves

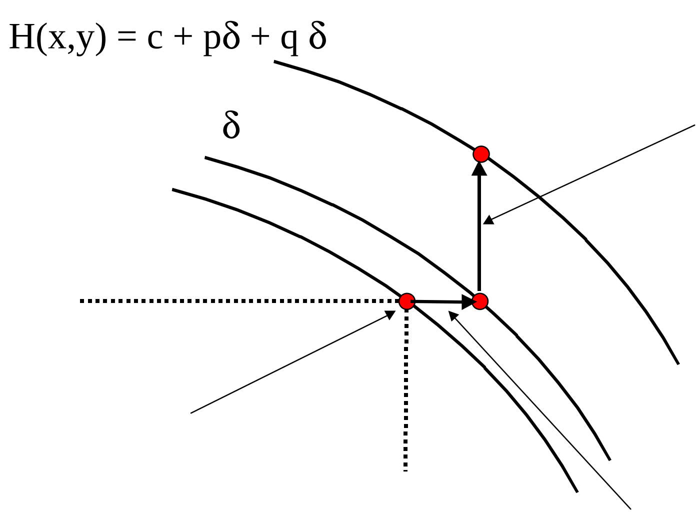


Figure 6
Arbitration Scheme I

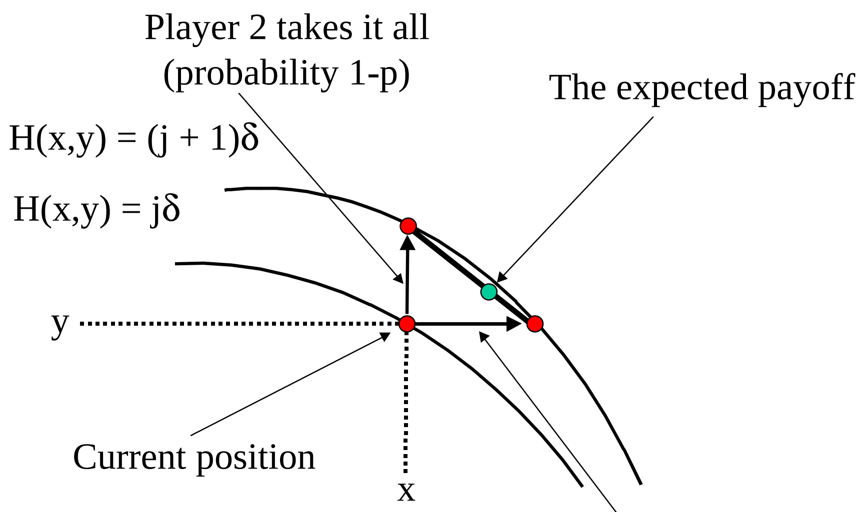
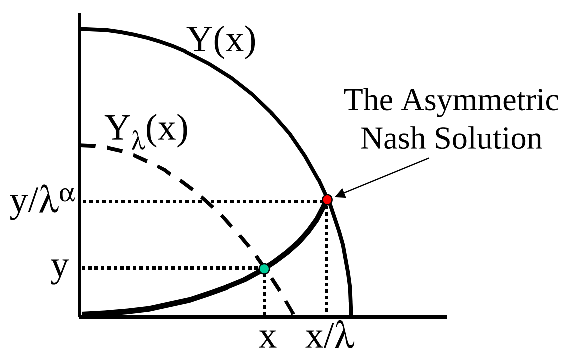


Figure 7
Arbitration Scheme II



The NGS on quasi-homogeneous
leads to the Asymmetric Nash solution.

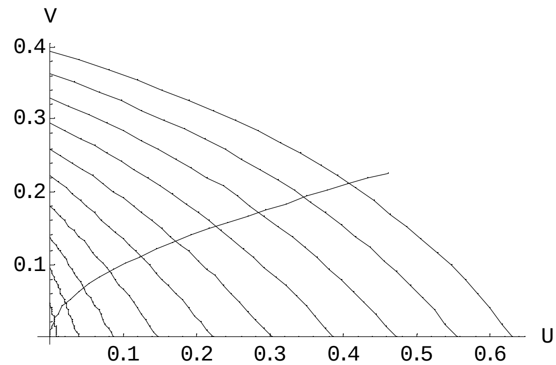


Figure 9.
Example A

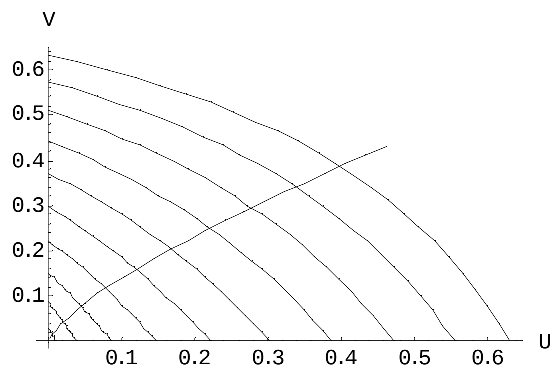


Figure 10.
Example B

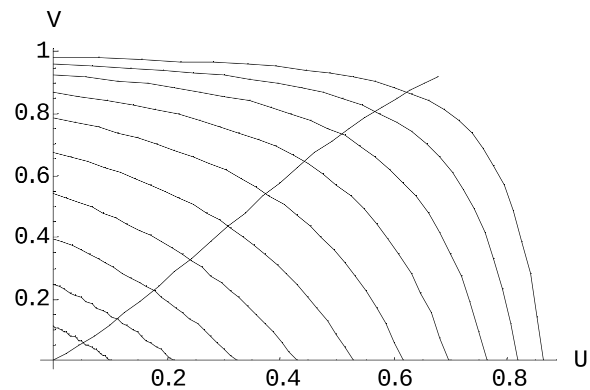


Figure 11.
Example C