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Effectivity functions, game forms, games, and rights

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Abstract. In this paper we offer an axiomatic approach for the investigation of rights by means of game forms. We give a new definition of constitution which consists of three components: the set of rights, the assignment of rights to groups of members of the society, and the distribution of power in the society (as a function of the distribution of rights). Using the foregoing definition we investigate game forms that faithfully represent the distribution of power in the society, and allow the members of the society to exercise their rights simultaneously. Several well-known examples are analyzed in the light of our framework. Finally, we find a connection between Sen's minimal liberalism and Maskin's result on implementation by Nash equilibria.

1. Introduction

This paper consists of an attempt to use the axiomatic approach in investigating rights by means of game forms (for a recent paper which explores the relationship between rights and game forms see Hammond 1994). The assignment of rights to the members of a society is, usually, part of the constitution of the society. Therefore, in order to investigate it we need a definition of constitution. Such a definition is given in Arrow (1967). However, if we adopt Arrow's definition of constitution (i.e., that a constitution is a "well-behaved" social welfare function – see Arrow 1967), then we have to accept the conclusion of Arrow's Impossibility Theorem that there is no satisfactory constitution. We quote from (Arrow 1967, p. 228): "This conclusion is quite embarrassing, and it forces us to examine the conditions which have been

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stated as reasonable. It's hard to imagine anyone quarreling either with the Pareto Principle or the condition of Non-Dictatorship. The principle of Collective Rationality may indeed be questioned. One might be prepared to allow that the choice from a given environment be dependent on the history of previous choices made in earlier environments, but I think many would find that situation unsatisfactory. There remains, therefore, only the Independence of Irrelevant Alternatives, which will be examined in greater detail in Section 4 below". In that Section we find (Arrow 1967, p. 231): "Unfortunately, it is clear, as I have already suggested, that social decision processes which are independent of irrelevant alternatives have strong practical advantages, and it remains to be seen whether a satisfactory social decision procedure can really be based on other information." The only way to resolve this impasse is to use a different, less known, definition of constitution. We follow this path in this paper. Our approach is based on Gardenfors (1981).

We now briefly review the contents of the paper. Section 2 introduces a definition of constitution which is a generalization of Gardenfors's definition of rights system. In our model rights are common knowledge and preferences, that may be private information, do not enter the definition of constitution. We also compute the constitutions for several examples in Section 2. We show, in Section 3, how a constitution leads in a natural way to an effectivity function which describes the "distribution of power" in a given society as a result of the the assignment of rights (see Deb 1994 for somewhat similar ideas). We then proceed to describe how game forms, that faithfully represent the foregoing effectivity function, are used by the members of the society to simultaneously exercise their rights. Section 4 is devoted to a study of games that are related to Gibbard's Paradox (Gibbard 1974). Also, we show that it is possible to choose constitutions whose game forms have a non-empty set of equilibria for each profile of preferences of the members of the society. In Section 5 we investigate the connection between Sen's Liberal Paradox and the implementability of social choice correspondences. A comparison of our paper with some closely related contributions to the theory of rights is presented in Section 6.

2. The model and examples

The legal rules of a democratic society are given in terms of rights of subgroups of members of the society (including, of course, individuals). We attempt to precisely define in such a situation the notion of constitution. Later (in Section 5) we shall compare the current definition of this notion in terms of preferences (see, e.g., Arrow 1967 and Gibbard 1974), with our definition.

A *society*, $\mathscr{S}$ is a list $\langle N, A, \rho, \alpha, \gamma \rangle$ where:

(i) $N = \{1, \ldots, n\}$ is the set of *members* of $\mathscr{S}$.

(ii) A is the set of *social states* (which may be finite or infinite).

(iii) $\rho = \{\rho_1, \ldots, \rho_l\}$ is the (finite) set of *rights*.

(iv) $\alpha : 2^N \to 2^\rho$ is the (current) *assignment* of rights to groups of individuals.

(v) γ, the *access correspondence*, determines the sets of attainable social states by groups of members of $\mathscr{S}$ as a function of their rights. Thus, γ: $2^N \times 2^\rho \to\to 2^A$. ($\to\to$ denotes a correspondence, i.e., a set-valued function.)

We always assume the following: (1) $\alpha(\emptyset) = \emptyset$, and (2) $\gamma(\emptyset, \theta) = \gamma(S, \emptyset) = \{A\}$ for all $\theta \subset \rho$ and $S \subset N$. (1) is a convenient agreement. (2) is true if A is the set of *all* possible social states.

The following remarks are in order. (We shall consistently use the foregoing notations.)

Remark 2.1. A social state is, intuitively, a complete description of all aspects relevant to the members of the society of a (possible) social situation. Formally, the set of ρ is an abstract set. However, intuitively, rights serve as vehicles for obtaining certain social states. Or, more concretely, they determine some major aspects of the "distribution of power" in $\mathcal{S}$.

Remark 2.2. The definition of the access correspondence deserves detailed explanation. If S is a coalition (of the members of the society), and $\theta \subset \rho$ is a set of rights, then $\gamma(S, \theta) = \{B_1, \ldots, B_m\}$ has the following interpretation. It is not (legally) excluded that the social outcome is in each of the sets $B_1, \ldots, B_m$ separately. For coherent models of rights a stronger interpretation is available (see Section 3).

Example 2.3. Let $\mathcal{S}$ be described in the following way: $N = \{1, 2\}$. Each member $i \in N$ has two shirts, white and blue, and he must wear one of the two. Denoting w for white and b for blue, the set A of social states is $A = \{(w, w), (w, b), (b, w), (b, b)\}$ (here, if $(x, y) \in A$ then x is the color of 1's shirt and y is the color of 2's shirt). $\rho = \{\rho_1\}$ where ρ_1 is the right to freely choose one's own shirt. (Henceforth, we shall denote a singleton $\{a\}$ by a.) α is given by $\alpha(\emptyset) = \emptyset$, $\alpha(1) = \alpha(2) = \alpha(N) = \rho_1$. Finally, γ is given by $\gamma(S, \emptyset) = A$ for all $S \subset N, \gamma(\emptyset, \rho_1) = A$, $\gamma(1, \rho_1) = \{\{(w, w), (w, b)\}, \{(b, w), (b, b)\}\}$, $\gamma(2, \rho_1) = \{\{(w, w), (b, w)\}, \{(w, b), (b, b)\}\}$, and $\gamma(N, \rho_1,) = 2^A \backslash \{\emptyset\}$.

Example 2.3 plays an important role in Gibbard (1974), and Gaertner et al. (1992). Also, in choosing $\gamma(N, \rho_1)$ we have assumed that 1 and 2 may exercise ρ_1 *simultaneously*. Such a possibility may not always exist. The exact relationship between the foregoing example and Gibbard's (first) paradox (in Gibbard 1974), will be clarified in Section 4.

Definition 2.4. Let $\mathcal{S} = \langle N, A, \rho, \alpha, \gamma \rangle$ be a society. The triple $\langle \rho, \alpha, \gamma \rangle$ is called a *constitution*.

Thus, a constitution consists of a set of rights, an assignment of rights to groups of members of the society, and a function which specifies for each coalition (of members) its set of attainable (sets of) outcomes.

Remark 2.5. In our model rights are personal because α and γ depend on the names of the members. In real-life situations this is usually not the case. To render our model more realistic we may assume that there is set of parameters π such that each member i of the society is completely specified, for the sake of the analysis of rights and power, by a non-empty subset π_i of π. Under this assumption, two members $i, j \in N$ will be *symmetric* if $\pi_i = \pi_j$. Also, the constitution $\langle \rho, \alpha, \gamma \rangle$ satisfies *equal-treatment* (ET), if for every pair of symmetric players $i, j \in N$ the transposition (i, j) is a symmetry of the pair $\langle \alpha, \gamma \rangle$ (more precisely, if $i, j \in N$, $\pi_i = \pi_j$, and $S \subset N \backslash \{i, j\}$, then $\alpha(S \cup \{i\}) = \alpha(S \cup \{j\})$ and $\gamma(S \cup \{i\}, \theta) = \gamma(S \cup \{j\}, \theta)$ for every $\theta \subset \rho$).

In the sequel we shall use this approach of describing members of a society $\mathscr{S}$ by sets of parameters. Also, rights are very often associated with roles of certain members in $\mathscr{S}$ (e.g., mother, student, policeman, etc.). In the foregoing approach we may include the role of a member i of $\mathscr{S}$ in his set of personal parameters π_i.

Remark 2.6. In our model rights should be interpreted in a broad sense: All obligations to society (e.g., paying taxes) are rights. Thus, our notion of constitution is similar to the usual one. The observation that a constitution must contain both rights and obligations is not new (see, e.g., Kanger and Kanger 1972). (I am indebted to the referee of this paper for this remark.)

The following example illustrates Remarks 2.5 and 2.6.

Example 2.7. A set $N = \{1, \ldots, n\}$ of workers share the same office. Let $\pi = \{\sigma, v\}$ be the set of the following two habits: $\sigma \equiv$ smoker, and $v \equiv$ non-smoker. The set of smokers N_1 is determined by a function $L: N \to \pi$, that is, $N_1 = \{i \in N | L(i) = \sigma\}$. Thus, $N_2 = \{i \in N | L(i) = v\} = N \backslash N_1$. The set A describes the possible states of the air at the office, that is, $A = \{\text{smoky, clear}\}$. Assume that ρ is a singleton which is the following obligation: "Refrain from smoking, at the office, in the presence of at least one non-smoker." Furthermore, assume that α, the assignment of rights, is given by $\alpha(\emptyset) = \emptyset$, and $\alpha(S) = \rho$ for all $S \subset N$, $S \neq \emptyset$. If we follow the usual meaning of the foregoing assumptions, then the access function γ is given by

$$\gamma(S, \rho) = \begin{cases} 2^A \backslash \{\emptyset\}, S \cap N_2 = \emptyset, S \neq \emptyset, \\ \{\{\text{clear}\}, A\}, S \cap N_2 \neq \emptyset, \end{cases}$$

and $\gamma(S, \emptyset) = \gamma(\emptyset, \rho) = A$ for all $S \subset N$.

Notice that in Example 2.7 every two (non-)smokers are symmetric (in the sense of Remark 2.5), and this is, indeed, reflected by α and γ. Also, formally we could define γ in an arbitrary manner. However, that might render our example senseless. Moreover, the values $\gamma(S, \theta)$ where $\theta \neq \alpha(S)$ do not enter the analysis of a society at a given date. However, if $\alpha(\bullet)$ changes over time, then all the values of γ matter. Finally, the description of Example 2.7 may be shortened. We chose the foregoing way in order to illustrate the use of parameters in describing the members of the society.

Remark 2.8. A constitution $\langle \rho, \alpha, \gamma \rangle$ is, at a given point of time, the result of the past continuous political process. In a democracy, at a given time, $\langle \rho, \alpha, \gamma \rangle$ represents the status quo of the rules of the state. Thus, it may be changed by the legislative institutions by voting or by other procedures (e.g., a referendum). Therefore, in our model rights are politically determined (see Sen 1994). At each point in time t of change the members of $\mathscr{S}$ have a profile of preferences (on A), $R^N(t)$, that determines the direction of change. So, in our framework the problem of choosing the constitution does not arise because the constitution at a given period determines all possible (legal) constitutions at the next period. Illegal changes (e.g., coups d'etat) are not covered by our model. However, we do not investigate in this work the dynamics of constitutions.

Remark 2.9. Let $S \subset N, S \neq \emptyset$, be a coalition, and let $\theta \subset \rho, \theta \neq \emptyset$. Then $\gamma(S, \theta)$ is a collection of subsets of A (i.e., a subset of 2^A). The reader may ask why do we need such a "complex" definition. In order to convince him that we have the right concept, let us consider the following example. Let N be the population of Israelis who have finished high school, are not older than 22, and look for a job in Israel. Let A be the set of all possible states of the job market in Israel and ρ be "the right to work". All the members of N have the right ρ, which guarantees a certain (low) payment in case of unemployment. A member $i \in N$ exercises his right in the following way. First i should choose a profession. Each choice determines a set of possible outcomes for i. (Clearly, the expected income depends on the choice of a profession.) Also, the workers in the same profession compete with each other. Thus, i's future income will be determined by competition with his peers, and may depend on factors that are not controlled by him. Hence, the "right to work" does not determine i's profession and income directly; these are determined by i's decision and efforts that may be guided by signals of the labor market. Thus, in a capitalistic state the connection between the basic "right to work" and the actual state of the labor force is highly indeterminate. This indeterminacy reflects the "freedom of choice" that is embodied in a free market system (for a recent discussion of the problems of freedom of choice see, e.g., Puppe 1994). We should add that the Israeli governments also try to regulate the distribution of the labor force in Israel by other ways: massive support of high education and vocational training, direct subsidies to exporting industries, and other means (which affect the attractivity of various sectors). The estimation of an "optimal" degree of freedom may be very difficult. However, a system without freedom at all (people are assigned to jobs by the government) may be highly inefficient. We conclude from this example that the freedom aspect of a constitution is reflected, in our model, by the assumption that γ is a set-valued function (see Kanger and Kanger 1972 for a different approach to the concept of freedom).

Now we give an example where $|\rho| = \ell > 1$. (If D is a finite set, then $|D|$ denotes the number of elements in D.)

Example 2.10. This example of a society $\mathscr{S}$ also is taken from Gibbard (1974) (see also Hammond 1994). N consists of three individuals: A (Angelina), E (Edwin), and J (the male Judge). There are three social states: $0, e, j$, where 0 indicates that Angelina remains single, e that she marries Edwin, and j that she marries the judge. The set of rights is $\rho = \{\rho_1, \rho_2\}$, where ρ_1 is the right to remain single, and ρ_2 is the right to marry. In Gibbard's example the assignment of rights is given by: $\alpha(\emptyset) = \emptyset$, $\alpha(A) = \alpha(E) = \alpha(J) = \rho_1$, $\alpha(\{A, E\}) = \alpha(\{A, J\}) = \rho$, $\alpha(\{E, J\}) = \rho_1$, and $\alpha(N) = \rho$. Using the usual interpretation of the foregoing data we may compute the access function γ in the following way. For each $B \subset \{0, e, j\}$ let $B^+ = \{\hat{B} \subset \{0, e, j\} | \hat{B} \supset B\}$. Then $\gamma(\emptyset, \theta) = \{0, e, j\}$ for all $\theta \subset \rho$, and $\gamma(S, \emptyset) = \{0, e, j\}$ for each $S \subset N$. The other values of γ are given by:

$\gamma(A, \rho_1) = \gamma(A, \rho) = \{0\}^+$, and $\gamma(A, \rho_2) = \{0, e, j\}$; $\gamma(E, \rho_1) = \gamma(E, \rho) = \{0, j\}^+$, and $\gamma(E, \rho_2) = \{0, e, j\}$; $\gamma(J, \rho_1) = \gamma(J, \rho) = \{0, e\}^+$, and $\gamma(J, \rho_2) = \{0, e, j\}$; $\gamma(\{A, E\}, \rho_1) = \{0\}^+$, $\gamma(\{A, E\}, \rho_2) = \{e\}^+$, and $\gamma(\{A, E\}, \rho) = \{0\}^+ \cup \{e\}^+$; $\gamma(\{A, J\}, \rho_1) = \{0\}^+$, $\gamma(\{A, J\}, \rho_2) = \{j\}^+$, and $\gamma(\{A, J\}, \rho) = \{0\}^+ \cup \{j\}^+$; $\gamma(\{E, J\}, \rho_1) = \{0\}^+$, $\gamma(\{E, J\}, \rho_2) = \{0, e, j\}$, and $\gamma(\{E, J\}, \rho) = \{0\}^+$; $\gamma(N, \rho_1) = \{0\}^+$, $\gamma(N, \rho_2) = \{e\}^+ \cup \{j\}^+$, and $\gamma(N, \rho) = \{0\}^+ \cup \{e\}^+ \cup \{j\}^+$.

3. Representation by game forms

Let $\mathscr{S} = \langle N, A, \rho, \alpha, \gamma \rangle$ be a society. An *effectivity function* (EF) is a correspondence E: $2^N \to\to 2^A$ that satisfies the following conditions: (i) $E(\varnothing) = A$; (ii) $E(N) = 2^A \backslash \{\varnothing\}$; (iii) $\varnothing \notin E(S)$ for all $S \subset N$; and (iv) $A \in E(S)$ for all $S \subset N$. Under very mild conditions the constitution $\langle \rho, \alpha, \gamma \rangle$ defines an *EF* $E = E(\bullet; \alpha, \gamma)$ in the following way:

(3.1) $E(S; \alpha, \gamma) = E(S) = \gamma(S, \alpha(S))$

Condition (i) above is satisfied as $\gamma(\varnothing, \alpha(\varnothing)) = A$. The next condition $\gamma(N, \alpha(N)) = 2^A \backslash \{\varnothing\}$ is the familiar condition of citizen's sovereignty (or non-imposition) for EF's (see Peleg 1984, Remark 6.1.3). Condition (iii) is obvious: There is always some social state that prevails. The same (trivial) argument also justifies (iv). In summary, under the assumption of non-imposition (3.1) defines an EF. We shall now define some basic properties of α and γ. α satisfies *monotonicity* if

(3.2) $S \subset T \Rightarrow \alpha(S) \subset \alpha(T)$ for all $S, T \subset N$.
$\gamma: 2^N \times 2^\rho \to\to 2^A$ is *monotonic with respect to* (*w.r.t.*) *the alternatives if* for all $S \in 2^N$ and all $\theta \in 2^\rho$.

(3.3) $[B \in \gamma(S, \theta)$ and $B^* \supset B] \Rightarrow [B^* \in \gamma(S, \theta)]$.
γ is *monotonic w.r.t. rights* if for all $S \subset N$ and $\theta, \theta^* \subset \rho$

(3.4) $[\theta^* \supset \theta] \Rightarrow [\gamma(S, \theta^*) \supset \gamma(S, \theta)]$.
Finally, γ is *monotonic w.r.t. coalitions* if for all $\theta \in 2^\rho$ and $S, S^* \in 2^N$

(3.5) $[S^* \supset S] \Rightarrow [\gamma(S^*, \theta) \supset \gamma(S, \theta)]$.
(3.2) is intuitively acceptable: Usually larger groups have more rights. Essentially, it follows from the usual interpretation of rights.

(3.3) is always satisfied if the constitution is coherent, that is E (see (3.1)) can be represented by a game form (see Definition 3.4). In this case $B \in \gamma(S, \theta)$ may be interpreted as follows: If S has the set of rights θ, then, by exercising its rights "properly", S may force the social outcome to be an element of B. This argument will be precisely formulated when we shall discuss in the sequel representations of EF's.

(3.4) generally does not hold. As rights in our model include also obligations having more rights may diminish the set of possible outcomes. Also, (3.5) may not be true. The members of $S^* \backslash S$ may have rights conflicting those of the members of S and thereby excluding some of the outcomes in $\gamma(S, \theta)$. For example, a taxi driver has the right to smoke when he is alone in his car. However, passengers may prevent him from smoking by objecting (or by law) (see also Example 2.7). Nevertheless, we consider such conflicts as being "marginal", and we shall usually assume (3.5).

Definition 3.1. An EF $E: 2^N \to\to 2^A$ is *superadditive* if for all $S_1, S_2 \in 2^N$, $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$,

(3.6) $[S_1 \cap S_2 = \varnothing] \Rightarrow [B_1 \cap B_2 \in E(S_1 \cup S_2)]$.
A superadditive EF is monotonic w.r.t. coalitions, that is, if $S \subset T \subset N$ then $E(S) \subset E(T)$ (the proof is straightforward). Hence, an EF that is derived

from a constitution by (3.1) might not be superadditive. Nevertheless, we shall deal with EF's that correspond to constitutions and are superadditive, because superadditivity is satisfied quite often.

In order to exercise their rights the members of $\mathcal{S}$ use (legal) strategies. These strategies must be, in addition, compatible with the constitution. The formulation of this compatibility condition is achieved in the following way.

Definition 3.2. A *game form* (GF) is a list $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A\rangle$ where N is the set of members of the society; Σ^i is the non-empty set of *strategies* of $i \in N$; $g: \Sigma^1 \times \cdots \times \Sigma^n \to A$, is the *outcome function*; and A is the set of social states.

A GF Γ is *legal* if for each $i \in N$ every strategy $\sigma^i \in \Sigma^i$ does not contradict the assignment of rights $\alpha(\bullet)$. For example, if Adam has the obligation to support his family, and stealing is forbidden by rule (i.e. by the assignment α(Adam)), then Adam cannot support his family by stealing. Henceforth, we shall only consider legal GFs. Moreover, we shall assume that also coalitions cannot break the law (by coordination of strategies).

Definition 3.3. Let $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A\rangle$ be a GF, let $S \subset N$, $S \neq \emptyset$, and let $B \subset A$. S is α-*effective* for B if there exists $\sigma_0^S \in \Sigma^S = \times_{i \in S}\Sigma^i$ such that for all $\sigma^{N\backslash S} \in \Sigma^{N\backslash S}$ $g(\sigma_0^S, \sigma^{N\backslash S}) \in B$. Now assume that g is surjective (onto A), and denote for each $S \subset N$, $S \neq \emptyset$,

$$E_\alpha(S;\Gamma) = \{B \subset S \mid S \textit{ is } \alpha\text{-effective for } B\}$$

and $E_\alpha(\emptyset;\Gamma) = A$. Then $E_\alpha(\bullet;\Gamma)$ is the α-*EF* of Γ (in particular, it is an EF).

A GF Γ is compatible with the constitution $\langle \rho, \alpha, \gamma\rangle$ if it has the following property.

Definition 3.4. A (legal) GF Γ is a *representation* of the constitution $\langle \rho, \alpha, \gamma\rangle$ if $E_\alpha(\bullet;\Gamma) = E(\bullet)$, where E is defined by (3.1).

A representation of the constitution may be considered as a permissible mechanism that enables all the members of the society to exercise their rights simultaneously. The basic EF E, which is defined by (3.1), may be represented by many GFs. Each representation may be considered as a (legal) translation of the constitution into strategic behavior. If the society $\mathcal{S}$ is, for example, geographically divided into several communities, then each community may choose its own representation of the constitution.

The main result on existence of representations is the following theorem.

Theorem 3.5. *Let $E: 2^N \to\to 2^A$ be the EF which is derived by* (3.1). *Then there exists a GF $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A\rangle$ such that $E_\alpha(S;\Gamma) = E(S)$ for all $S \in 2^N$ iff the following two conditions hold*:

(i) γ *is monotonic w.r.t. the alternatives*;
(ii) E *is superadditive.*

The proof is given in the appendix. For a proof when A is finite see Moulin (1983).

Example 3.6. Consider the society of Example 2.3. (3.1) yields the following EF $E: E(\emptyset) = A$ and $E(S) = \gamma(S, \rho_1)$ for $S \neq \emptyset$. Let $\hat{w}(\hat{b})$ be the strategy "choose a white (blue) shirt". Then the GF $\Gamma = \langle N; \{\hat{w}, \hat{b}\}, \{\hat{w}, \hat{b}\}; g; A\rangle$, where g is given by $g(\hat{x}^1, \hat{x}^2) = (x^1, x^2)$ for all $x^1, x^2 \in \{w, b\}$, is a representation of E. According to Γ, the two members choose their shirt simultaneously.

Example 3.7. Let $\Sigma^1 = \{\hat{w}, \hat{b}\}$ (see Example 3.6), and let $\Sigma^2 = \{f \mid f: \Sigma^1 \to \{w, b\}\}$. Further define $g: \Sigma^1 \times \Sigma^2 \to A$ by $g(\hat{x}, f) = (x, f(\hat{x}))$. The GF $\Gamma_0 = \langle N; \Sigma^1, \Sigma^2, g, A\rangle$ describes the following sequential procedure:

Step 1. 1 chooses her shirt.

Step 2. 2 chooses her shirt after observing 1's choice.

Γ_0 is *not* a representation of E (see, again, Example 3.6). Indeed, by choosing the strategy $f(\hat{x}) = x$ 2 can force the outcome to be in the set $B = \{(w, w), (b, b)\}$. However, $B \notin E(2)$.

Example 3.8. Now we compute the EF of the society of Example 2.10 by means of (3.1): $E(\varnothing) = \{0, e, j\}$; $E(A) = \gamma(A, \rho_1) = \{0\}^+$ (see Example 2.10); $E(E) = \gamma(E, \rho_1) = \{0, j\}^+$; $E(J) = \gamma(J, \rho_1) = \{0, e\}^+$; $E(\{A, E\}) = \gamma(\{A, E\}, \rho) = \{0\}^+ \cup \{e\}^+$; $E(\{A, J\}) = \gamma(\{A, J\}, \rho) = \{0\}^+ \cup \{j\}^+$; $E(\{E, J\}) = \gamma(\{E, J\}, \rho_1) = \{0\}^+$, and $E(N) = \gamma(N, \rho) = \{0\}^+ \cup \{e\}^+ \cup \{j\}^+$. As the reader may easily verify, $E(\bullet)$ is superadditive and monotonic w.r.t. the alternatives. Hence, by Theorem 3.5, E is representable. Also, if A is finite then the proof of Theorem 3.5 is constructive and, therefore, may be used to obtain representations of E.

4. Games and rights

Let $\mathscr{S} = \langle N, A, \rho, \alpha, \gamma\rangle$ be a society, let the EF E be derived by (3.1), and let $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A\rangle$ be a representation of E. If $i \in N$ then a preference ordering of i is a complete and transitive binary relation on A. Let Q be the set of all preference orderings on A. Then, Q^N is the set of all preference profiles. If $R^N \in Q^N$ then the pair $\langle \Gamma, R^N\rangle$ determines an (ordinary) game in strategic form $G(\Gamma, R^N) = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A; R^N\rangle$ (in the usual way). Every situation of simultaneous exercising of rights by the members of $\mathscr{S}$ is a play of a game of the foregoing type (i.e., a play of a game $G(\Gamma, R^N)$ where Γ is a representation of E and $R^N \in Q^N$).

Example 4.1. Consider the GF Γ of Example 3.6. The set $\{G(\Gamma, R^N) \mid R^N \in Q^N\}$ is isomorphic to the set of all ordinal types of 2×2 (two-person) games. (Two 2×2 (two-person) games with numerical payoffs are ordinally equivalent if one can be obtained from the other by strictly increasing (individual) transformations of the payoffs.) In particular, we can obtain the game of "matching pennies", which is Gibbard's first paradox, and the "prisoner's dilemma", which is Gibbard's second paradox. This observation is not new (see, e.g., Gaertner (1993) for the same observation and a list of references for earlier discussions of this example).

Remark 4.2. Let us consider "matching pennies" in the framework of Example 4.1. The effectivity function E is given by $E(\varnothing) = A$, $E(1) = \{\{(w, w), (w, b)\}, \{(b, w), (b, b)\}\}$, $E(2) = \{\{(w, w), (b, w)\}, \{(w, b), (b, b)\}\}$, and $E(\{1, 2\}) = 2^A \setminus \{\varnothing\}$

(see Examples 2.3 and 3.6). A profile of "matching pennies" is given by

R^1	R^2
b, w	w, w
w, b	b, b
w, w	b, w
b, b	w, b

The following claim is true:
(*) If $\Gamma = \langle N; \Sigma^1, \Sigma^2; g; A\rangle$ is a representation of E, then the game $G(\Gamma, R^N)$ has no Nash equilibrium (NE).

Proof of (*). Let $\Gamma = \langle N; \Sigma^1, \Sigma^2; g; A\rangle$ be a representation of E. Assume, on the contrary, that $\sigma_0 = (\sigma_0^1, \sigma_0^2)$ is a NE of $G(\Gamma, R^N)$. We have four possible values for $g(\sigma_0)$. We only will consider the case $g(\sigma_0) = (w, w)$. By assumption, 2 is not α-effective for $\{(w, w), (b, b)\}$. Hence, 1 has a strategy $\mu^1 \in \Sigma^1$ such that $g(\mu^1, \sigma_0^2) \in \{(w, b), (b, w)\}$, contradicting our assumption that σ_0 is an NE. Q.E.D.

Remark 4.3. Is the game which is considered in Remark 4.2 a paradox? Not according to game theory. In order to solve it we have to introduce mixed strategies. In our model we have to consider lotteries over A. If this is not possible, then the society may modify the constitution in order to avoid such inconsistent behavior (see Remark 2.8).

Remark 4.4. Let $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A\rangle$ be a GF and let $R^N \in Q^N$. $\sigma^N \in \Sigma^N$ is a *strong* Nash equilibrium (SNE) of $G(\Gamma, R^N)$ if for every $S \subset N, S \neq \emptyset$, and for every $\mu^S \in \Sigma^S$ there exists $i \in S$ such that $g(\sigma^N) R^i g(\mu^S, \sigma^{N \setminus S})$. Γ is *strongly consistent* if for every $R^N \in Q^N G(\Gamma, R^N)$ has an SNE. The following class of EFs have strongly consistent representations. An EF $E: 2^N \to\to 2^A$ is *maximal* if for every $S \subset N$, and for every $B \subset A$

$$B \notin E(S) \Leftrightarrow A \setminus B \in E(N \setminus S)$$

E is *convex* if E is superadditive and for all $S_1, S_2 \subset N$ and all $B_1 \in E(S_1)$ and $B_2 \in E(S_2)$,

$$B_1 \cup B_2 \in E(S_1 \cap S_2) \text{ or } B_1 \cap B_2 \in E(S_1 \cup S_2).$$

If E is maximal and convex and A is finite, then E has a strongly consistent representation (see Peleg (1984, Theorems 6.A.7 and 6.4.2)). Thus, a strongly consistent behavior is possible for the class of societies which yield, according to (3.1), a maximal and convex EF. Finally, we remark that the result of Peleg (1984) has been generalized to infinite sets of social states by Abdou and Keiding in several papers (see, e.g., Abdou 1987 and Keiding 1986).

Remark 4.5. A GF $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^N; g; A\rangle$ is *Nash-consistent* if for every $R^N \in Q^N$ $G(\Gamma, R^N)$ has an NE. Clearly, we are interested in the EFs of Nash-consistent GFs. If $|N| = 2$, then a complete characterization of Nash-consistent GFs in terms of their EF is given by Gurvich (1989) and Abdou (1993a, b). Also, a GF Γ is *acceptable* if: (i) it is Nash consistent; and (ii) for every $R^N \in Q^N$ and for every NE σ of $G(\Gamma, R^N)$, the outcome $g(\sigma)$ is Pareto-optimal (w.r.t. the profile R^N). Acceptable GFs are constructed and characterized in Hurwicz and Schmeidler (1978) and Dutta (1984). They may also be useful in our framework.

5. Implementation and Sen's liberal paradox

Let $\mathscr{S} = \langle N, A, \rho, \alpha, \gamma \rangle$ be a society. A *social choice rule* (*SCR*) *is a* function $C: 2^A \times Q^N \to 2^A$ that satisfies $C(B, R^N) \subset B$ and $C(B, R^N) \neq \varnothing$ for all $B \neq \varnothing$, $B \subset A$, and $R^N \in Q^N$, and $C(\varnothing, R^N) = \varnothing$ for all $R^N \in Q^N$. Let C be an SCR, let $i \in N$, and $x, y \in A$, $x \neq y$. i is *decisive for* x *over* y if for all $R^N \in Q^N$, and all $B \subset A$, the following condition is satisfied: If xP^iy, $x \in B$, and $x \notin C(B, R^N)$, then $y \notin C(B, R^N)$ (here xP^iy if xR^iy and not yR^ix). Sen's (weakest) condition of *liberalism* is: (ML) There are at least two persons i and j and two ordered pairs of alternatives (x, y) and (z, w), with $x \neq z$ and $y \neq w$, and such that i is decisive for x over y and j is decisive for z over w (see Sen 1970, p. 88).

Now we shall recall Maskin's definition of implementability (see Maskin 1977). Again let C be an SCR and let $B \subset A$, $B \neq \varnothing$. A GF $\Gamma_B = \langle N; \Sigma_B^1, \ldots, \Sigma_B^n; g_B; A \rangle$ *implements* the social choice correspondence $C_B(\bullet): Q^N \to 2^A$ (defined by $C_B(R^N) = C(B, R^N)$ for all $R^N \in Q^N$), if for every $R^N \in Q^N$ the following condition is satisifed: $x \in C_B(R^N)$ iff there is an NE $\sigma \in \Sigma_B^N$ of the game $G(\Gamma_B, R^N) = \langle N; \Sigma_B^1, \ldots, \Sigma_B^n; g_B; A; R^N \rangle$ such that $g_B(\sigma) = x$. C is *implementable* if for every $B \subset A$, $B \neq \varnothing$, there exists a GF Γ_B that implements $C_B(\bullet)$.

Also, we recall that an SCR C satisfies *unanimity* if for every $B \subset A$, $B \neq \varnothing$, $x \in B$, and $R^N \in Q^N$ such that xP^iy for all $y \in B \backslash \{x\}$ and $i \in N$, $C(B, R^N) = \{x\}$. Now we may formulate the following result.

Theorem 5.1. *If an SCR C satisfies ML and unanimity , then it is not implementable.*

Proof. Assume, on the contrary, that C is implementable. Let $B_0 = \{x, y\} \cup \{z, w\}$ where (x, y) and (z, w) satisfy (ML). Then the social choice correspondence $C_{B_0}(\bullet)$ satisfies unanimity and it is implementable. Hence, by Maskin (1977), $C_{B_0}(\bullet)$ is strongly monotonic (see also Definition 2.3.15, Lemma 2.3.25, and Lemma 6.5.1 of Peleg (1984)). Now, it follows from Lemma 3.2.12 of Peleg (1984) that $C_{B_0}(\bullet)$ satisfies the Pareto criterion. Therefore, by Sen (1970, Theorem 6*3), we have obtained the desired contradiction. Q.E.D.

Remark 5.2. The reader may easily construct SCRs that satisfy both unanimity and ML.

Remark 5.3. In Arrow (1967) constitutions are defined as social welfare functions. (A social welfare function is a function $F: Q^N \to Q$.) However, it seems to us that this approach might face some difficulties. The reader is referred to Gardenfors (1981) for criticism of Sen's definition of liberalism. Sen's model is based on the foregoing definition of Arrow.

6. Discussion

6.1 The connection with Gardenfors' model

Let N be a society and A a set of social states. A right, according to Gardenfors (1981), is the possibility of a group S, $S \subset N$, to restrict the choice of a social state to a subset B of A. His main concept is *rights-system*, which is a subset of $2^N \times 2^A$; that is, a rights-system is a set of pairs (S, B), $S \in 2^N$ and $B \in 2^A$. Thus, a rights-system is, essentially, an EF (see Section 3). Therefore, Gardenfors's starting point is our Definition (3.1). Because our analysis is Sections 3 and 4 is based mainly on (3.1), we see that our model is similar to that of Gardenfors. However, there are some important differences.

(a) Unlike Gardenfors, we do not distinguish between rights and obligations. Now, in a constitution rights are usually supported by obligations (e.g., to maintain property rights we should forbid stealing). Hence, our model is more general.

(b) We explicitly model the set of rights and the assignment of rights to groups of members of society. This may enable (future) analysis of the dynamics of systems of rights (see Remark 2.8).

(c) Our notion of representation of an EF (see Definition 3.4) allows us to immediately apply quite a few results on strategic games to our model. In Gardenfors (1981) the connection between a rights-system and the associated strategic games is much more complicated; in particular, the preferences of the players must be extended to 2^A.

6.2 Relationships with other works on game forms

In a series of recent papers rights are (formally) modelled via use of game forms (see, e.g., Deb 1994; Gaertner et al. 1992, and Hammond 1994). As far as we could check only Hammond (1994) contains a formal definition of systems of rights in terms of sets of alternatives. Therefore, all the works on game forms (except Hammond's) are not directly comparable with our work. Briefly, the main difference is as follows. We, essentially, endorse Gardenfors's definition of rights (because most of our work is based on the EF given by (3.1)). Other authors consider the mere availability of strategies in a GF to be equivalent to the existence of rights (see, e.g., Deb et al. 1993, p. 7). We do not wish at this point to enter a debate on which approach is more suitable. Hence we only point out the main difference.

Hammond's model is different from our model in two respects: (a) His definition of *rights profile*, which describes the distribution of rights among coalitions (i.e., groups of members of the society), is *not* an effectivity function; and (b) as a result of (a) Hammond lacks the notion of "representation" of a rights system by a game form. Therefore, his analysis of coherent rights does not seem to be directly linked to known existence results for strategic equilibrium (see our Section 4 for such results).

Our main contribution to the literature on modelling rights via game forms is as follows. First, we generalize Gardenfors' work and reformulate his definition of rights system by means of effectivity functions. Secondly, by introducing representations of effectivity functions we are able to obtain a rich (formal) theory of rights exercising in a society. The use of representations also considerably limit the set of game forms that are consistent with a given constitution.

6.3 Minimal liberalism

Let $\mathscr{S} = \langle N, A, \rho, \alpha, \gamma \rangle$ be a society and let the EF E be given by (3.1). The constitution $\langle \rho, \alpha, \gamma \rangle$ satisfies *minimal liberalism* if there exist, $i, j \in N, i \neq j$, and $B_i, B_j \in 2^A \backslash \{A\}$ such that $B_i \in E(i)$ and $B_j \in E(j)$. This definition is related to Definition 2.3 of Deb et al. (1993) in the following (straightforward) way. Let $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A \rangle$ be a GF, let $i \in N$, and let $a \in A$. *i vetoes a if* there exists $\sigma_0^i \in \Sigma^i$ such that for all $\sigma^{N \backslash \{i\}} \in \Sigma^{N \backslash \{i\}} g(\sigma_0^i, \sigma^{N \backslash \{i\}}) \neq a$. Denote by $V(i)$ the set of all the alternatives that are vetoed by i. Γ satisfies *minimal liberalism* if there exist $i, j \in N, i \neq j$, such that $V(k) \neq \emptyset$, $k = i, j$ (see Definition 2.3 of Deb et al. 1993).

The following claim is true.

Claim 6.1. E satisfies minimal liberalism iff every representation of E (see Definition 3.4) satisfies minimal liberalism.

The proof of Claim 6.1 is left ot the reader. For an analysis of the relationship between strategic equilibrium, liberalism, and Pareto-optimality the reader is referred to the beautiful paper by Deb et al. (1993). The following (weak) Liberal Paradox for game forms is a corollary of their Proposition 4.2.

Corollary 6.2. *Let* $\Gamma = \langle N; \Sigma^1, \ldots, \Sigma^n; g; A \rangle$ *be a GF, let* $i, j \in N, i \neq j$, *and let* $V(i) \cap V(j) \neq \emptyset$. *Then there exists* $R^N \in Q^N$ *such that the game* $G(\Gamma, R^N)$ *has a Nash equilibrium whose outcome is Pareto dominated.*

7. Conclusion

Our work generalizes the model of Gardenfors in three respects: (i) By explicitly introducing the set of rights and the assignment of rights we allow time dependent constitutions that may change as a result of the legislative process. (ii) We do not distinguish between rights and obligations to society. Hence our definition of constitution is similar to the ordinary concept. (iii) By introducing the notion of representation of a rights system we can easily apply results on strategic games to our model. In particular, we do not have to extend the preferences of the players.

Our contribution to the literature on the analysis of rights by means of game forms consists of two parts: (i) the concise description of each constitution by means of an effectivity function; and (ii) the use of representations of effectivity functions by game forms in order to model simultaneous exercising of rights by all members of a society. The use of representations (by GF's) allows to apply the theory of strategic equilibrium to constitutions.

Appendix: Representations of effectivity functions

We now shall prove a generalization of Theorem 3.5. Let $N = \{1, \dots, n\}$, $n \geq 2$, be a set of players and let A be a set of alternatives. A may be finite or infinite. However, $|A| \geq 2$ if A is finite. Let $\mathbf{B} \subset 2^A$ such that $A \in \mathbf{B}$. (Such a set $\mathbf{B}$ is called a *structure* on A.) An EF $E: 2^N \to\to 2^A$ is *compatible with* $\mathbf{B}$ if $E(S) \subset \mathbf{B}\backslash\{\emptyset\}$ for all $S \in 2^N$; $E(\emptyset) = A$; $A \in E(S)$ for all $S \in 2^N$; and $E(N) = \mathbf{B}\backslash\{\emptyset\}$. $\mathbf{B}$ is *closed under finite intersections* (CFI) if $B_1, \dots, B_k \in \mathbf{B}$ imply that $\bigcap_{i=1}^k B_i \in \mathbf{B}$. If $(A, \mathbf{B})$ is a measurable space or a topological space, then $\mathbf{B}$ is closed under finite intersections. Now let $\mathbf{B} \subset 2^A$ satisfy $A \in \mathbf{B}$ and CFI.

Theorem 3.5*. *Let $E: 2^N \to\to 2^A$ be an EF that is compatible with* $\mathbf{B}$. *The following two conditions are equivalent*:
(3.8) *E is superadditive and monotonic w.r.t. the alternatives* (i.e. $S \in 2^N$, $B \in E(S)$, $C \in \mathbf{B}$, *and* $C \supset B$ *imply that* $C \in E(S)$).

(3.9) *There exists a GF* $\Gamma = \langle N; \Sigma^1, \dots, \Sigma^n; g; A\rangle$ *such that* $E_\alpha(S; \Gamma) \cap \mathbf{B} = E(S)$ *for all* $S \in 2^N$.

Proof. (3.9) ⇒ (3.8). To prove monotonicity w.r.t. the alternatives let $S \in 2^N$, $B \in E(S)$, $C \supset B$ and $C \in \mathbf{B}$. Then $B \in E_\alpha(\Gamma; S)$. Hence $C \in E_\alpha(\Gamma; S)$. Thus $C \in E_\alpha(\Gamma; S) \cap \mathbf{B} = E(S)$. Now let $S_i \in 2^N$, $i = 1, 2$, $S_1 \cap S_2 = \emptyset$, and $B_i \in E(S_i)$, $i = 1, 2$. Then $B_1 \cap B_2 \in \mathbf{B}$ by CFI and, also, $B_1 \cap B_2 \in E_\alpha(\Gamma; S_1 \cup S_2)$. Therefore $B_1 \cap B_2 \in E(S_1 \cup S_2)$. Thus E is superadditive.

(3.8) ⇒ (3.9). For $i \in N$ let

$$F^i = \{(S, B) | i \in S \subset N \text{ and } B \in E(S)\}$$

Further let $\mathbf{B}^* = \mathbf{B}\backslash\{\emptyset\}$ and

$$\Phi = \{\varphi: \mathbf{B}^* \to A | \varphi(B) \in B \text{ for all } B \in \mathbf{B}^*\}$$

Now we define Γ in the following way. $\Sigma^i = F^i \times N \times \Phi$ for all $i \in N$. Let $\sigma^N = (f^i, t^i, \varphi^i)_{i \in N}$. A coalition $S \subset N$, $S \neq \emptyset$, is σ^N-*consistent* if there is $B \in E(S)$ such that $f^i = (S, B)$ for all $i \in S$. Denote by $S_1(\sigma^N) = S_1, \dots, S_r(\sigma^N) = S_r$ the coalitions which are σ^N-consistent. Further, denote $S_0 = N\backslash\bigcup_{j=1}^r S_j$. Then $P(\sigma^N) = (S_0, S_1, \dots, S_r)$ is a partition of N. Now let $i_0 \equiv \sum_{i=1}^n t^i(n)$, and $f^i = (S_j, B_j)$ for $i \in S_j, j = 1, \dots, r$. So, we may define

$$g(\sigma^N) = \varphi^{i_0}\left(\bigcap_{j=1}^r B_j\right).$$

Because E is superadditive $\bigcap_{j=1}^r B_j \in E(\bigcup_{j=1}^r S_j)$, and therefore $\bigcap_{j=1}^r B_j \neq \emptyset$ and g is well defined. (If $j = 0$, i.e., there are no σ^N-consistent coalitions, then, by definition, $g(\sigma^N) = \varphi^{i_0}(A)$). Denote $\Gamma = \langle N; \Sigma^1, \dots, \Sigma^n; g; A\rangle$.

It is obvious that for each $S \subset N$, $E_\alpha(\Gamma, S) \cap \mathbf{B} \supset E(S)$. To prove the converse inclusion let $S \subset N$, $S \neq \emptyset$, and let $C \in \mathbf{B}$, $C \neq \emptyset$ and $C \notin E(S)$. Then $N \neq S$ and for every $B \in E(S)$, $B\backslash C \neq \emptyset$ (because E is monotonic w.r.t. the alternatives). Now let $\sigma^s = (f^i, t^i, \varphi^i)_{i \in S}$ be a member of Σ^S. Consider the following strategy $\sigma_0^{N\backslash S}$ of $N\backslash S$. $f_0^i = (N\backslash S, A)$ for all $i \in N\backslash S$. Let $j_0 \in N\backslash S$. Then we choose $t^i = 1$ for $i \in N\backslash S, j_0 \neq i$, and we choose t^{j_0} such that $t^{j_0} + \Sigma_{i \neq j_0} t^i \equiv j_0(n)$. Let $P(\sigma^S, \sigma_0^{N\backslash S}) = (S_0, N\backslash S, S_1, \dots, S_r)$. Then $S_j \subset S$,

$j = 1, \ldots, r$. For $i \in S_j$ let $f^i = (S_j, B_j), j = 1, \ldots, r$. Then, by the superadditivity of E, $B = \bigcap_{j=1}^{r} B_j \in E(S)$. Hence $B \backslash C \neq \emptyset$ and j_0 can choose $\varphi^{j_0} \in \Phi$ such that $\varphi^{j_0}(B) \notin C$. Thus $g(\sigma^S, \sigma_0^{N \backslash S}) \notin C$. The case $r = 0$ is straightforward. Q.E.D.

Clearly, Theorem 3.5* implies Theorem 3.5.

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