First–Price Auctions when the Ranking of Valuations is Common Knowledge*

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Abstract

We consider an augmented version of the symmetric private value auction model with independent types. The augmentation, intended to illustrate reality, concerns information bidders have about their opponents. To the standard assumption that every bidder knows his type and the distribution of types is common knowledge we add the assumption that the ranking of bidders' valuations is common knowledge.

This set–up induces a particular asymmetric auction model that raises serious technical difficulties. We prove existence and uniqueness of equilibrium in pure strategies in the two bidder case. We also show that the model generally has no analytic solution. If the distribution of valuations is uniform, *both* bidders bid pointwise more aggressively relative to the standard symmetric case. However, this property does not apply to all distributions of valuations. Finally, we also provide a numerical solution of equilibrium bid functions for the uniform distribution case.

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1 Introduction

In the literature on private value auctions it is assumed that bidders know their type whereas information about other bidders' is restricted to the distribution from which types are drawn; the latter is assumed to be common knowledge. In reality, however, it is often the case that bidders know more about their opponents. In particular, bidders often have information about the ranking of their valuations.

For example, in a procurement setting, when engineering or architect firms bid for contracts, each of them may know who has the lower cost. This information may result from experience accumulated from other bidding occasions or industrial espionage. Firms often spend significant resources to acquire information about their rivals' cost and technology. Similarly, in art auctions bidders often revise their strategies when they learn that some wealthy collector participates ("Getty factor"). In privatization or takeover bidding, participants often have access to information about each others' financial resources or other idiosyncratic features that affect bidders' valuations. Finally, the ranking of valuations may also become known when several identical items are auctioned sequentially and, after each round of bidding, bidders find out who the winner was but do not observe the winning bid.

The present paper modifies the standard symmetric independent private value auction model by assuming that the ranking of bidders' valuations is common knowledge.[1] Surprisingly or not, this modification raises serious technical difficulties because it gives rise to a system of differential equations with non–standard singularities.

We prove existence and uniqueness of equilibrium in pure strategies in the case of two bidders. Generally, the differential equations do not have an analytic solution. The equilibrium bid function of the bidder with the lower valuation is always strictly above that of his rival. In the particular case of uniformly distributed valuations, the equilibrium bid functions of *both* bidders are above the equilibrium bid functions of the standard symmetric case. Consequently, in this environment, the auctioneer earns higher revenues in a first–price auction than in a second–price auction, not only on average but even pointwise, for each configuration of valutions. Hence, the auctioneer should be interested to inform bidders about the rank order of valuations if he has that information. This result is in the spirit of Theorem 8 in Milgrom & Weber [1982], that claims, in the context of a common value model, that if the auctioneer has private information, he can benefit by making it public. However, as we also show, the strong pointwise comparison obtained for the uniform distribution case does not apply to *all* distributions of valuations.

There are many ways to model the fact that bidders have typically more information about each other than assumed in standard auction models. In the present paper we focus on the one particular case where the ranking of bidders' valuations is common knowledge. This analysis is indicative of the complications and new results

[1] For basic results of the standard auction model see the surveys by McAfee and McMillan [1987], Milgrom [1989], Matthews [1995] and Wolfstetter [1996].

that may emerge if, to bridge the gap with reality, the standard model is augmented so that bidders are better informed.

There is an emerging literature on "asymmetric auctions" (see Maskin and Riley [1994], [1995a], [1995b] and Plum [1992])[2]. This literature assumes that bidders' valuations are drawn from *different* probability distributions that are common knowledge to all bidders. In our model, it is common knowledge that valuations are drawn from the *same* distribution, but once bidders learn about the ranking of their valuations, the subsequent *conditional* distribution functions differ but are not common knowledge since valuations which generate the conditional distributions are privately observed. In that technical sense our model differs from the asymmetric auctions considered in that literature.

2 The Model

Consider a symmetric first–price auction where an indivisible good is auctioned to two bidders. The seller's reserve price is equal to zero. Valuations v are realizations of a random variable V, independently drawn from a differentiable probability distribution function $G(v)$ with density $g := G'$ which is strictly positive on the support $[0,1]$. We also assume that G has a Taylor expansion around 0; $G(x) = \alpha x + \beta x^2 + \ldots$, with $\alpha > 0$.

Valuations are privately observed, but each bidder knows whether his valuation is the higher or lower of the two. Furthermore, the ranking of valuations is common knowledge.

Denote bidders by H and L, where H stands for the bidder with the higher valuation, and let b_H and b_L be the respective bid functions (pure strategies). We restrict the analysis to equilibria in pure and strict monotone increasing strategies.

Equilibrium conditions Suppose bidder H has valuation v and bids x. If the rival bidder plays the strict monotone increasing strategy b_L, the probability that H wins is

$$\begin{aligned}\Pr\{H\,\text{wins}\} &= \Pr\{b_L(V) < x \mid V < v\} \\ &= \frac{\Pr\{V < \min\{\phi(x), v\}\}}{G(v)},\end{aligned} \tag{1}$$

where $\phi := b_L^{-1}$. Therefore, the expected payoff of bidder H with valuation v when he bids x is

$$\Pi_H(x; v) = \frac{\Pr\{V < \min\{\phi(x), v\}\}}{G(v)}(v - x). \tag{2}$$

[2] Plum [1992] establishes closed–form solutions for some class of parametric probability distributions. Maskin and Riley [1994a], [1994b] address existence and uniqueness issues, and Maskin and Riley [1995] focus on a comparison of expected revenues when random valuations are ordered stochastically.

To compute the best reply of H, note that in equilibrium $\min\{\phi(x), v\} = \phi(x)$ because otherwise H could lower his bid and still win with certainty. Consequently, the best reply is obtained by solving

$$\max_x G(\phi(x))(v - x). \tag{3}$$

It is readily seen that there is a unique local maximum to this problem thus, differentiating (3) with respect to x, and using the fact that in equilibrium x is equal to $b_H(v)$ (or equivalently, $v = \sigma(x)$), one obtains the differential equation

$$\phi'(x)g(\phi(x))\,(\sigma(x) - x)) = G(\phi(x)). \tag{4}$$

Next, consider bidder L with valuation v who bids x, and write σ for the inverse of the bidding strategy b_H. The probability of winning is

$$\begin{aligned} \Pr\{L\,\text{wins}\} &= \Pr\{b_H(V) < x \mid V > v\} \\ &= \frac{\Pr\{V > v \text{ and } \sigma(x) > V\}\}}{1 - G(v)}. \end{aligned} \tag{5}$$

Note that, in equilibrium, the bid x must satisfy $\sigma(x) > v$, because otherwise, in order to have a positive probability of winning, L would have to raise his bid. Therefore,

$$\Pr\{L\,\text{wins}\} = \frac{G(\sigma(x)) - G(v)}{1 - G(v)}. \tag{6}$$

Computing the best reply, as before, one obtains the differential equation

$$g(\sigma(x))\sigma'(x)(\phi(x) - x) = G(\sigma(x)) - G(\phi(x)). \tag{7}$$

Two boundary conditions apply:

$$\begin{aligned} &\sigma(0) = \phi(0) = 0 \quad \text{and} \\ &\sigma(b) = \phi(b) = 1 \quad \text{for some } b \leq 1. \end{aligned} \tag{8}$$

The first boundary condition in (8) follows from the fact that in equilibrium a bidder with $v = 0$ does not make a positive bid i.e., $b_H(0) = b_L(0) = 0$. The other boundary condition is due to the fact that in equilibrium the maximum bid, b (that of valuation $v = 1$), must be the same for both bidders, because if it differed, the bidder with the higher bid could lower it, still win the object with probability 1, and thus strictly increase his expected payoff. Consequently,

$$b_H(1) = b_L(1) = b. \tag{9}$$

3 Non-existence of an Analytic Solution

In this section we prove that the model does not have a closed–form solution even if the distribution of valuations is uniform.

Assume G is uniform on $[0,1]$. Rewriting (4), (7) and (8), we obtain

$$\begin{aligned}
\phi'(x) &= \frac{\phi(x)}{\sigma(x)-x} \\
\sigma'(x) &= \frac{\sigma(x)-\phi(x)}{\phi(x)-x} \\
\phi(0) &= \sigma(0) = 0 \\
\phi(b) &= \sigma(b) = 1 \quad \text{for some} \quad b \leq 1.
\end{aligned} \tag{10}$$

Lemma 1 *System (10) does not have an analytic solution.*

Proof Evidently, $\phi(x) = \frac{4}{3}x$ and $\sigma(x) = 2x$ is an analytic solution of the two differential equations in (10) which satisfy the first boundary condition but violates the second one. We now show that this is the only analytic solution of the system in (10) without the second boundary condition.

Since both differential equations in (10) are singular at $x = 0$, to obtain $\sigma'(0)$ and $\phi'(0)$ we apply L'Hôpital's rule which gives $\sigma'(0) = 2$ and $\phi'(0) = \frac{4}{3}$. Since every solution around 0 can be expressed as an asymptotic expansion in x, the power series expansions of $\phi(x)$ and $\sigma(x)$ around 0 are

$$\phi(x) = \frac{4}{3}x + \alpha x^k + \ldots \tag{11}$$

$$\sigma(x) = 2x + \beta x^r + \ldots. \tag{12}$$

where $k, r > 1$ are the first powers whose coefficients are non–zero.

Substitute (11) and (12) into the first differential equation in (10), rearrange terms, and one obtains

$$\frac{4}{3}x + \alpha k x^k + \frac{4}{3}\beta x^r + \beta\alpha k x^{k+r-1} + \ldots = \frac{4}{3}x + \alpha x^k + \ldots. \tag{13}$$

If $r > k > 1$ then $\alpha = 0$, which is a contradiction. If $k > r > 1$ then $\beta = 0$, again a contradiction. Therefore, $k = r$ which in turn implies

$$\alpha(k-1) + \frac{4}{3}\beta = 0. \tag{14}$$

Applying similar reasoning to (11), (12) and the second differential equation in (10), one obtains

$$(\frac{k}{3} - 1)\beta + 3\alpha = 0. \tag{15}$$

These two linear equations in α, β have a non–trivial solution if and only if the determinant of coefficients vanishes, which implies $4-(k-1)(\frac{k}{3}-1)=0$, i.e.

$$k = 2+\sqrt{13}. \tag{16}$$

This proves the Lemma since for a solution to be analytic, the exponents must be integers. ∎

Having proved that there is no analytic solution for the uniform distribution case does not prove that there is no such solution for other distributions, although we believe that this is indeed the case.

4 Existence and Uniqueness of Equilibrium

In this section we prove existence and uniqueness of equilibrium in pure strategies for arbitrary probability distributions. We thus consider the system:

$$\begin{aligned}
&\phi'(x) = \frac{G(\phi(x))}{g(\phi(x))(\sigma(x)-x)} \\
&\sigma'(x) = \frac{G(\sigma(x))-G(\phi(x))}{g(\sigma(x))(\phi(x)-x)} \\
&\exists b \in (0,1) \quad \text{such that} \quad \phi(b)=\sigma(b)=1 \\
&\sigma(0)=\phi(0)=0.
\end{aligned} \tag{17}$$

We prove that this system, to which we refer as the *constrained system*, has a unique solution. The proof follows from a sequence of Lemmas.

The main idea of the proof is to start from an arbitrary boundary point b, as defined by the first boundary condition in (17), and move along the trajectories governed by the differential equations in (17). In what follows, we refer to this system as the *partially constrained system* (that is the system (17) without the boundary condition at 0). Note, the partially constrained system can be written in the form

$$\frac{dG(\phi(x))}{dx} = \frac{G(\phi(x))}{\sigma(x)-x} \tag{18}$$

$$\frac{dG(\sigma(x))}{dx} = \frac{G(\sigma(x))-G(\phi(x))}{\phi(x)-x} \tag{19}$$

$$\phi(b)=\sigma(b) = 1. \tag{20}$$

By a standard property of ordinary differential equations[3], for every $b\in(0,1)$ the partially constrained system (18)–(20) has a unique solution which we denote by ϕ_b and σ_b. We show that there is exactly one b at which the second boundary condition in (17) is also satisfied i.e., $\phi_b(0)=\sigma_b(0)=0$.

We distinguish between two kinds of solutions to which we refer as Ω_1, Ω_2:

[3] See e.g. Corduneanu, [1971].

Definition 1 *A solution of the partially constrained system belongs to* Ω_1 *if* $\phi_b(x) > x$ *for all* $x \in (0, b]$, *; it belongs to* Ω_2 *if* $\phi_b(x) = x$ *for some* $x \in [0, b)$.

Lemma 2 *Both sets* Ω_1 *and* Ω_2 *are not empty.*

Proof We shall show that for b sufficiently small $(\phi_b, \sigma_b) \in \Omega_1$, while for b sufficiently close to 1, $(\phi_b, \sigma_b) \in \Omega_2$.

(i) By our assumption $\mu := \min_x g(x) > 0$. We claim that for $b = \mu/4$ we have $(\phi_b, \sigma_b) \in \Omega_1$. In fact we show that, for this value of b, both functions ϕ_b and σ_b are above the line $\ell: \; y = 1/2 + (2/\mu)x$ (see Figure 1). By (17), the derivatives of the functions are bounded by

$$\phi_b'(x) \leq 1/(\sigma_b(x) - x)\mu \quad \text{and} \quad \sigma_b'(x) \leq 1/(\phi_b(x) - x)\mu$$

In particular $\phi_b'(b) \leq 1/\mu$ and $\sigma_b'(b) \leq 1/\mu$ which is smaller than $2/\mu$, the slope of ℓ , and hence both functions are above ℓ in the neighborhood of $b = \mu/4$ (see Figure 1).

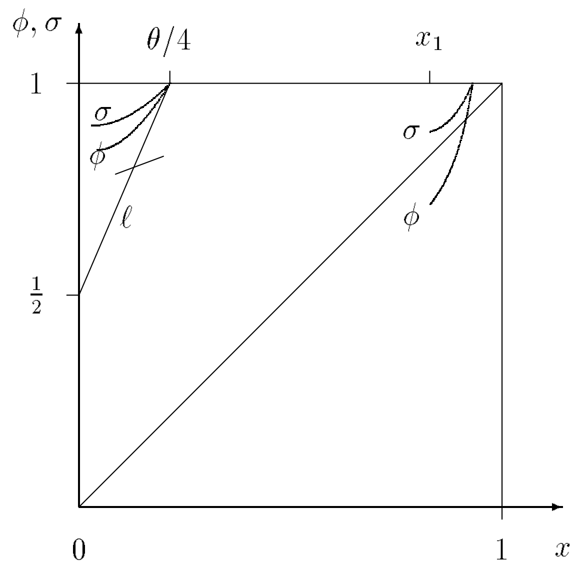

Figure 1: The solutions of the partially constrained system for small and large values of b.

Assume, contrary to our claim, that the functions do not lie entirely above ℓ. Let x_0 be the largest crossing points of one of the two functions, say ϕ_b with ℓ. Note that since $\mu \leq 1$, the slope of ℓ is at least 2, so $\sigma_b(x_0) - x_0 > 1/2$ and hence $\phi_b'(x_0) < 2/\mu$ which means that $\phi_b(x)$ is bellow ℓ at the right of x_0 which is in contradiction to the fact that x_0 is the largest crossing point of ϕ_b.

(ii) Let $c := \min_{\frac{1}{2} \leq x \leq 1} \frac{G(x)}{g(x)}$, then, by (17), $\phi_b'(x) > c/(1-x)$ for all $b > 1/2$ and $x > 1/2$. Take $x_1 > 1/2$ such that $c/(1-x_1) > 2$ and $b = (1+x_1)/2$ (see figure 1), then

$$\phi_b(x_1) < \phi_b(b) - 2(b - x_1) = x_1,$$

and hence $(\phi_b, \sigma_b) \in \Omega_2$. ∎

For any $b \in (0,1)$ define $x_b \in [0,b)$ by

$$x_b = \begin{cases} 0 & \text{if } \phi_b(x) > x \ \forall x \in [0,b) \\ \min\{x | \phi_b(x) = x\} & \text{otherwise.} \end{cases}$$

Lemma 3 *$\sigma_b(x) > \phi_b(x)$, for all $b \in (0,1)$ and for all $x \in (x_b, b)$.*

Proof By the two differential equations and the first boundary condition in (17), $\phi_b'(b) > 0$ and $\sigma_b'(b) = 0$. Therefore, $\sigma_b(x) > \phi_b(x)$ at least in a neighborhood of b. If our claim is false then let $\bar{x}$ be the largest x in (x_b, b) such that $\sigma_b(x) = \phi_b(x)$. Then again $\phi_b'(\bar{x}) > 0$ (since $\phi_b(x) > x$ in (x_b, b) by definition of x_b), and $\sigma_b'(\bar{x}) = 0$ and hence, $\sigma_b(\bar{x} + \epsilon) < \phi_b(\bar{x} + \epsilon)$ for sufficiently small ϵ, in contradiction to the definition of $\bar{x}$. ∎

For $b \in (0,1)$ let $I_b = \{x \in (0,b] | \phi_b(x) \geq x\}$.

Lemma 4 *If $b' > b$, then $\phi_b(x) > \phi_{b'}(x)$ and $\sigma_b(x) > \sigma_{b'}(x)$, for all x in $I_b \cap I_{b'}$.*

Proof By the differential equations, $\sigma_b(x)$, $\phi_b(x)$ are strictly increasing in I_b so $\phi_b(b) = 1 = \phi_{b'}(b') > \phi_{b'}(b)$, and similarly $\sigma_b(b) = 1 = \sigma_{b'}(b') > \sigma_{b'}(b)$. Hence, the assertion is valid in some (left) neighborhood of b. Assume, it does not hold everywhere in $I_b \cap I_{b'}$. Then, there exists a largest (closest to b) crossing point z of either the ϕ functions or of the σ functions. If this last crossing is of the ϕ functions only, we have $\phi_b(z) = \phi_{b'}(z)$ and $\sigma_b(z) > \sigma_{b'}(z)$ and therefore,

$$\phi_b'(z) = \frac{G(\phi_b(z))}{g(\phi_b(z))(\sigma_b(z) - z)} < \frac{G(\phi_{b'}(z))}{g(\phi_{b'}(z))(\sigma_{b'}(z) - z)} = \phi_{b'}'(z), \tag{21}$$

implying that in the right neighborhood of z we have $\phi_b(x) < \phi_{b'}(x)$, in contradiction to the fact that z is the largest crossing point. Similarly, we can rule out the possibility that $\sigma_b(z) = \sigma_{b'}(z)$, and $\phi_b(z) > \phi_{b'}(z)$. Finally, we rule out the possibility that z is the crossing point both of the ϕ functions and of the σ functions i.e., $\sigma_b(z) = \sigma_{b'}(z)$ and $\phi_b(z) = \phi_{b'}(z)$. In fact this means that the two differential equations in (17), with the boundary condition at z, have two distinct solutions in $[z, b]$ namely (ϕ_b, σ_b) and $(\phi_{b'}, \sigma_{b'})$, violating the uniqueness of the solution to such a system (guaranteed by standard results). ■

Lemma 5 *There exists a unique b^* that induces a solution of the partially constrained system which belongs to both sets Ω_1 and Ω_2. This solution satisfies $\phi_{b^*}(0) = 0$ and $\phi_{b^*}(x) > x \ \forall x \in (0, b^*]$.*

Proof For convenience we shall write $b \in \Omega_i$ for $(\phi_b, \sigma_b) \in \Omega_i$. The monotonicity properties established in Lemmas 3 and 4 imply that if $b \in \Omega_1$ and $b' < b$, then also $b' \in \Omega_1$. Similarly, if $b \in \Omega_2$, and $b' > b$, then also $b' \in \Omega_2$. Let $b^* := \sup\{b \mid b \in \Omega_1\}$. We claim that this is the desired b^*. In fact if $b^* \in \Omega_1 \cap \Omega_2$, then $\phi_{b^*}(0) = 0$ and $\phi_{b^*}(x) > x$ for all $x > 0$. We have thus to rule out the possibility $b^* \notin \Omega_1 \cap \Omega_2$ which (since $\Omega_1 \cup \Omega_2 = [0, 1]$) consists of two cases:

Case 1: $b^* \in \Omega_1$ and $b^* \notin \Omega_2$. Then, $\phi_{b^*}(x) > x$ for all $x \in (0, b^*]$, and hence (by continuity) $\phi_{b^*}(0) > 0$. Since the solution $(\phi_{b^*}, \sigma_{b^*})$ is C^∞, a known result about stability of smooth solutions to ordinary differential equations with respect to changes in initial conditions, implies the existence of nearby solutions (see Corduneanu [1971], Theorem 3.4). In the present case this implies that there exist a $b > b^*$ and yet $b \in \Omega_1$, in contradiction to the definition b^*.

Case 2: $b^* \in \Omega_2$, $b^* \notin \Omega_1$. Then, by definition, there exist $0 < x_{b^*} < b^*$ where $\phi_{b^*}(x_{b^*}) = x_{b^*}$ (recall that, by its definition, x_{b^*} is the largest value in $(0, b^*)$ satisfying this equality). Hence by Lemma 3, $\sigma_{b^*}(x_{b^*}) \geq x_{b^*}$ since the opposite inequality would imply (by continuity) $\sigma_{b^*}(x_{b^*} + \epsilon) < \phi(x_{b^*} + \epsilon)$, in contradiction to Lemma 3. We now show that this inequality leads to a contradiction:

(i) If $\sigma_{b^*}(x_{b^*}) > x_{b^*}$ then (by the differential equation) $\phi'_{b^*}(x_{b^*})$ is finite, and therefore $\phi_{b^*}(x)$ is C^1 on $[x_{b^*}, b^*]$, and for x in the neighborhood $[x_{b^*}, x_{b^*} + \delta]$, it can be expressed in the form $\phi_{b^*}(x) = x_{b^*} + \gamma(\tilde{x} - x_{b^*})$, where $\tilde{x} \in [x_{b^*}, x]$. But then it follows from the second differential equation in (17) that the singularity of $\sigma_{b^*}(x)$ at x_{b^*} is not integrable, and therefore σ_{b^*} cannot attain a finite value at x_{b^*}, a contradiction.

(ii) The only case that remains to be considered is $\sigma_{b^*}(x_{b^*}) = \phi_{b^*}(x_{b^*}) = x_{b^*}$ for some fixed $x_{b^*} > 0$. We prove that $(\phi_{b^*}, \sigma_{b^*})$ are C^0 in $[x_{b^*}, b^*]$: In fact these functions are smooth in $(x_{b^*}, b^*]$, monotone increasing and bounded from below by x_{b^*} in $[x_{b^*}, b^*]$. Therefore, for any monotone sequence $x_n \downarrow x_{b^*}$ we have also the convergence: $\phi_{b^*}(x_n) \to \phi_*$ and $\sigma_{b^*}(x_n) \to \sigma_*$, with $x_{b^*} \leq \phi_* \leq \sigma_*$. If $\phi_* > x_{b^*}$, then we could pass to the limit $x \downarrow x_{b^*}$ in the differential equations (17) and thus, obtain a solution starting at b^* which is different from $(\phi_{b^*}, \sigma_{b^*})$, in contradiction to the uniqueness theorem. The case $\phi_* = b_{b^*} < \sigma_*$ is ruled out by the same argument used in (Case 1), and we conclude that $x_{b^*} = \phi_* = \sigma_*$, establishing the continuity.

The continuity implies that for a fixed small $\epsilon > 0$ (to be chosen later), there is a small $\eta(\epsilon) > 0$ such that $\phi_{b^*}(x_{b^*}+\epsilon) - x_{b^*} < \eta(\epsilon)$, and $\sigma_{b^*}(x_{b^*}+\epsilon) - x_{b^*} < \eta(\epsilon)$ (and hence also $\phi_{b^*}(x_{b^*}+\epsilon) - (x_{b^*}+\epsilon) < \eta(\epsilon)$, and $\sigma_{b^*}(x_{b^*}+\epsilon) - (x_{b^*}+\epsilon) < \eta(\epsilon)$), with $\eta(\epsilon) \to 0$ as $\epsilon \to 0$.

Consider now the partially constrained system on $[x_{b^*}+\epsilon, b^*]$. Since the solution is smooth there, we can find $b < b^*$ (and very close to it), so that $\phi_b(x_{b^*}+\epsilon) - x_{b^*} < \delta(\epsilon)$ and $\sigma_b(x_{b^*}+\epsilon) - x_{b^*} < \delta(\epsilon)$, with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. But then $\phi_b'(x_{b^*}+\epsilon) > c/\delta(\epsilon)$ (for some $c > 0$), and since σ_b decreases as we move to the left of $x_{b^*}+\epsilon$, the inequality $\phi_b'(x) > c/\delta(\epsilon)$ still holds in some small (but ϵ–independent) interval to the left of x_{b^*}. Since $\phi_b(x_{b^*})$ is $\eta(\epsilon)$–close to the diagonal $y = x$, and its derivative is $\delta^{-1}(\epsilon)$ large, it must cross the diagonal at some point $0 < \bar{x} < x_{b^*}$ and thus $b \notin \Omega_1$, contradicting the definition $b^* := \sup\{b \mid b \in \Omega_1\}$.

■

Combining the above Lemmas we prove that:

Theorem 1 *The auction game has a unique pure strategy equilibrium.*

Proof We need to show that the full system in (17) has a unique solution. By Lemma 5 we know already that $(\phi_{b^*}, \sigma_{b^*})$ solve the partially constrained system and satisfy $\phi_{b^*}(0) = 0$. It remains to be shown that $\sigma_{b^*}(0) = 0$. Recall that $\sigma_{b^*}(x) > x$ for $x > 0$. Assume $\sigma_{b^*}(0) > 0$. Then, by the first differential equation in (17), $\phi_{b^*}'(0) = 0$ (because $G(0) = 0$ and the denominator is positive), and therefore $\phi_{b^*}(x) < x$ in some small interval $[0, \epsilon)$, in contradiction with Lemma 5. Therefore, $(\phi_{b^*}, \sigma_{b^*})$ satisfy also the second boundary condition in (17) as well. Uniqueness follows from the uniqueness of b^*. ■

5 Strategy Comparison

How does the introduction of common knowledge concerning the ranking of valuations affect equilibrium bid functions? Does the bidder with the lower valuation always bid more agressively? Does it affect the efficiency of the first–price auction? And how do equilibrium bid functions differ from those obtained in the standard symmetric independent private value case?

5.1 Inefficiency of the first–price auction

Although Lemma 3 was established as part of the proof of Theorem 1, it is of independent interest. It shows that the low valuation bidder bids higher than his rival that is, in equilibrium, the bid functions b_L and b_H satisfy:

$$b_L(v) > b_H(v) \quad \forall v \in (0, 1).$$

The straightforward intuitive explanation is that otherwise the low valuation bidder would stand no chance of winning, which cannot be part of an equilibrium. An important consequence of this relationship between the bid functions is:

Corollary 1 *When the ranking of valuations is common knowledge, the object is awarded to the bidder with the lower valuation with positive probability; therefore, the first price auction is inefficient.*

Proof For any $x \in (0,1)$ we have $\sigma(x) > \phi(x)$. Using the monotonicity of these functions, this implies that any Low bidder with valuation higher than $\phi(x)$ bids more than x while any High bidder with valuation smaller than $\sigma(x)$ bids less than x. It follow that whenever both valuations are in the interval $(\phi(x), \sigma(x))$, which occurs with positive probability, the low valuation bidder wins the auction and gets the object. ■

The inefficiency is due to the fact that the two bid functions are apart, and hence the distance between σ and ϕ is related to the "degree" of inefficiency. For example, if we measure the inefficiency by the probability that bidder L, with the lower valuation, gets the object then it can be easily verified that:

$$\Pr\{\text{The object is awarded to } L\} = 2\int_0^b \left(G(\sigma(x)) - G(\phi(x))\right) \sigma'(x) g(\phi(x))\, dx.$$

For the uniform distribution case, this measure simplifies to $2\int_0^b \left(\sigma(x) - \phi(x)\right) \sigma'(x)\, dx$.

5.2 Comparison with the standard symmetric model

Does this modification result in more aggressive bidding by one or both bidders, relative to the standard symmetric independent private value model?

On the intuitive level, there is no indication for an unequivocal prediction. It seems pretty safe to conjecture that the bidder with the lower valuation will bid more aggressively. However, the impact of knowing that one has the higher valuation is ambiguous.

To see why the bidder with the lower valuation should bid more aggressively, start from the equilibrium of the game where the rank order is not known, and suppose that one bidder is "secretly" informed that he has the lower valuation. Should he revise his bid if he assumes that the rival continues to play the symmetric equilibrium strategy? The answer is that he should bid more aggressively. This is so, because if he adheres to the symmetric equilibrium strategy he stands no chance of winning. By slightly raising his bid he can earn a profit with positive probability. Of course, the argument is not yet complete since this information is common knowledge, and the bidder with the higher valuation should not be expected to stick to the symmetric information case behavior.

What about the high valuation bidder? On the one hand, knowing that his rival has a lower valuation may suggest that he can afford to make lower bids and still have a good chance to win. On the other hand, expecting his rival to bid more aggressively (by the previous heuristic argument) may lead him to bid higher. It is not clear, intuitively, which of the two effects is stronger. Taking into account higher levels of knowledge (knowing that the rival knows that he has higher valuation etc.) makes the intuitive case for more agressive bidding even more problematic and less convincing.

Propositions 1 and 2 show that both bidders *may* bid more aggressively but *need not* to.

Proposition 1 *If valuations are drawn from a uniform distribution, then, both bid functions of high and the low valuation bidders are above the equilibrium bid function in the symmetric case* $b(v)$

$$b(v) < b_H(v) < b_L(v). \tag{22}$$

Proof See Appendix. ∎

Note, even though the item is not always awarded to the bidder with the higher valuation, the winning bid is always higher than in the standard symmetric case, for each configuration of valuations. Therefore, the auctioneer's revenue is also higher for each configuration of valuations. Since bidding is unaffected by the assumed common knowledge in a second–price auction, it follows immediately that a first–price auction generates higher revenue to the auctioneer. This is in line with the observed predominance of the first–price auction.

To further characterize the equilibrium bid functions we generated a numerical solution of the system (10). The numerical integration of this system requires some care because of the singularity at the origin. We used an asymptotic expansions given by (11), (12), (15) and (16) to compute the value of ϕ and σ at $dx > 0$ for an arbitrary choice of the free parameter α. We then divided $[dx, 1]$ into small subintervals of size dx, and integrated the partially constrained system obtained from (10) by ignoring the second boundary condition, by a finite difference scheme. We started with a small positive α and repeated the procedure outlined above with increasing values of this parameter, until we hit a numerical solution that also satisfies the second boundary condition in (10). To ensure that the solution obtained is not a numerical artifact, we performed similar calculations with different choices of dx. The solution is plotted in Figure 2.

Returning to the main question of this section, the following proposition shows that this pattern of both players bidding more aggressively is not valid for all distributions of valuations.

Proposition 2 *There exists a distribution for which in equilibrium* $b_H(v) < b(v)$*, for some values of* v *(i.e. the high valuation bidder bids* less *than what he would bid in the symmetric case, somewhere).*

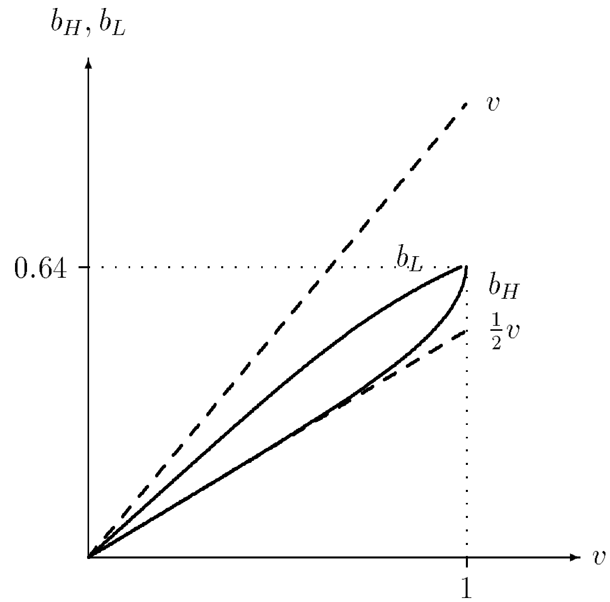

Figure 2: Equilibrium bid functions for the uniform distribution case

Proof See Appendix.

6 Conclusions

We have modified the standard symmetric independent private value model by assuming that bidders know the rank order of their valuations. We proved existence and uniqueness of equilibrium in pure strategies for the first–price sealed bid auction with two bidders. In this equilibrium the low valuation bidder always bids higher. Consequently, the first–price auction is not efficient since it occurs with positive probability that the object is sold to the bidder with the lower valuation. In the case of uniformly distributed valuations, we showed that *both* bidders bid higher than in the standard symmetric case. Therefore, the auctioneer's revenue is higher for all configurations of valuations. Since bidding is unaffected by our modification in a second–price auction, this indicates that the auctioneer should prefer the first–price to the second–price auction. However, as we also showed, the strong pointwise comparison obtained for the uniform distribution case does not apply to *all* distributions of valuations.

A Appendix

Proof of Proposition 1: The proof of the proposition is obtained as part of a proof of existence of a unique solution of constrained system (10). As a by product we thus have two quite different proofs of existence and uniqueness for the uniform distribution case.

In general, a system like (10) has a solution if the corresponding partially constrained system (i.e., when the second boundary condition is ignored) has infinitely many solutions . We will show that this is indeed the case.

Restricting attention to the partially constrained system in (10) we observe the following:

(i) $\sigma_2 = \frac{4}{3}x, \ \phi_2 = 2x$ is a solution.

(ii) every solution must satisfy $\sigma'(0) = 2, \ \phi'(0) = \frac{4}{3}$ (using L'Hôpital's rule and the differential equations).

(iii) the asymptotic behavior of every solution, near $x = 0$, is described by the following power series expansions which are obtained by inserting k from (16) into (11), (12)

$$\phi(x) = \frac{4}{3}x + \alpha x^{2+\sqrt{13}} + \ldots \tag{23}$$

$$\sigma(x) = 2x + \beta x^{2+\sqrt{13}} + \ldots . \tag{24}$$

Inserting $k = 2 + \sqrt{13}$ into (14) and (15) yields two linearly dependent equation and therefor they have one parameter family of solutions:

$$\beta(\alpha) = -\frac{3\alpha(\sqrt{13}+1)}{4}, \quad \text{for all } \alpha \neq 0. \tag{25}$$

Define the functions $g(x) := \phi(x)/x$ and $h(x) := \sigma(x)/x$. Rewriting the two first differential equation in (10) in terms of $g(x)$ and $h(x)$ gives

$$xg'(x) = \frac{g(x)}{h(x)-1} - g(x) \tag{26}$$

$$xh'(x) = \frac{h(x)-g(x)}{g(x)-1} - h(x) \tag{27}$$

and dividing (26) by (27) yields

$$\frac{dg}{dh} = \frac{g-1}{h-1} \ \frac{g(2-h)}{h(2-g)-g}. \tag{28}$$

Note that (26) and (27) are invariant to changes in the scale of x, i.e. if (h, g) is a solution, so is $(h_c, g_c) = (h(cx), g(cx))$, for all $c > 0$. If we interpret the variable x as 'time' then c is the 'speed' of motion along the trajectory of the solution in the (h, g) plane. We shall use this phase–plane to show that there exist infinitely many solutions to (10) when the second boundary condition is ignored, and then show that one and only one of them satisfies the second boundary condition in (10).

Although the following argument is self contained, it is based on methods explained in Boyce and DiPrima [1992, Ch. 9] where more detailed discussion and examples can be found.

First we make the dynamic system (26) and (27) an autonomous system by changing variable from x to t; letting $x = e^t$, $\tilde{g}(t) := g(e^t)$ and $\tilde{h}(t) := h(e^t)$ we obtain

$$\tilde{g}'(t) = \frac{\tilde{g}(t)}{\tilde{h}(t) - 1} - \tilde{g}(t) \tag{29}$$

$$\tilde{h}'(t) = \frac{\tilde{h}(t) - \tilde{g}(t)}{\tilde{g}(t) - 1} - \tilde{h}(t) \tag{30}$$

Since dividing these two equations yield the same differential equation (28), the two dynamic systems $\{(26), (27)\}$ and $\{(29), (30)\}$ have *the same trajectories* in the phase plane (g, h) or $(\tilde{g}, \tilde{h})$; they are governed by the differential equation. This phase diagram is given in Figure 3.

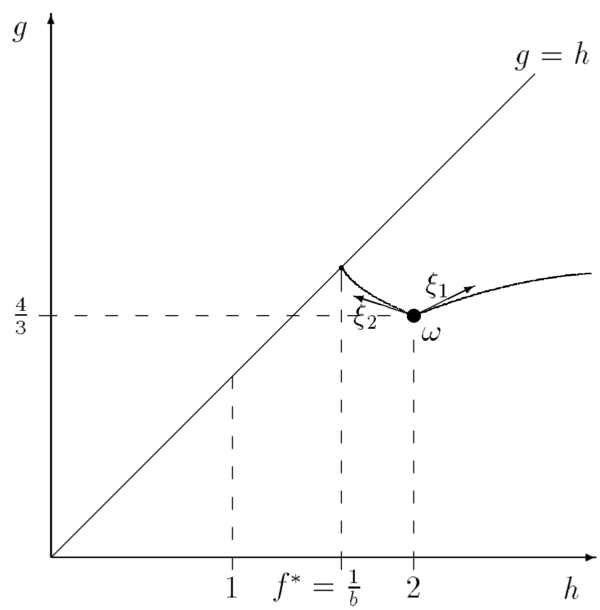

Figure 3: Trajectories in the phase plane.

The point $\omega := (2, \frac{4}{3})$ corresponds to the first boundary condition in (10); at $x = 0$ in the system $\{(26), (27)\}$ or $t = -\infty$ in the system $\{(29), (30)\}$. It is a critical point of both systems which means that at this point $(\tilde{h}', \tilde{g}')(-\infty) = (0, 0)$ and $(h', g')(0) = (0, 0)$. The local behavior of (28) can be studied by first finding the possible directions of the trajectories emanating from the critical point ω in the phase space. To do this we consider the linear approximation to the dynamic system around ω: Let

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} := \begin{pmatrix} g - \frac{4}{3} \\ h - 2 \end{pmatrix} = \begin{pmatrix} \tilde{g} - \frac{4}{3} \\ \tilde{h} - 2 \end{pmatrix},$$

be the vector of distance from ω, then the linear approximation of the dynamic system ($\{(26), (27)\}$) or $\{(29), (30)\}$) yields:

$$\begin{pmatrix} \delta_1' \\ \delta_2' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{3} \\ -9 & 2 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + O(\delta^2).$$

The matrix of coefficients in this equation has two eigenvalues:

- $r_1 = (1 - \sqrt{13}) \approx -2.6$ which corresponds to the eigenvector $\xi_1 = \begin{pmatrix} 1 + \sqrt{13} \\ 9 \end{pmatrix}$.
- $r_2 = (1 + \sqrt{13}) \approx 4.6$ which corresponds to the eigenvector $\xi_2 = \begin{pmatrix} 1 - \sqrt{13} \\ 9 \end{pmatrix}$.

The two directions ξ_1 and ξ_2 are indicated in Figure 3. The trajectory at the direction of ξ_1 is irrelevant to our dynamic system: Since $r_1 < 0$, a solution of the form $\delta = \xi_1 \exp(r_1 t)$ does not satisfy $\delta \to 0$ as $t \to -\infty$ (which is the first boundary condition at $x = 0$). On the other hand, the trajectory starting at ω in the direction ξ_2 is the "unstable manifold" of the system, that is, this is the solution that "leaves the critical point ω as t increases from $-\infty$ (or as x increases from 0). This corresponds to a negative value of β in (12) (see also (25)). The slope of this trajectory cannot become positive before it hits the line $g = h$, because a reversal requires that it becomes zero somewhere, which cannot occur as long as $h < 2$ (see Figure 3). Therefore, the trajectory must cross the line $g = h$ at some point that we denote by f^*. While there is a unique trajectory of this sort, it corresponds to infinitely many solutions of the system $\{(26), (27)\}$ (or to the two differential equations in (10)) that differ in their "speed" c. To choose the "right" speed we choose c for which

$$g_c(\frac{1}{f^*}) = h_c(\frac{1}{f^*}) = f^*. \tag{31}$$

Since other solutions move either faster or slower along the same trajectory, there is a unique solution that satisfies (31). Taking now $b^* = 1/f^*$ we have:

$$\phi_{b^*}(b^*) = b^* h_c(\frac{1}{f^*}) = \frac{1}{f^*} \cdot f^* = 1$$

and

$$\sigma_{b^*}(b^*) = b^* g_c(\frac{1}{f^*}) = \frac{1}{f^*} \cdot f^* = 1$$

Thus, we establish a unique solution to the full system (10) with $b^* \in (\frac{1}{2}, 1)$ (since $f^* \in (1, 2)$).

As we proved, along the equilibrium trajectory the function h is always smaller than 2 which implies that $h(x) = \frac{\sigma(x)}{x} < 2$, and therefore $\sigma(x) < 2x$. and by Lemma 3, $\phi(x) < \sigma(x) < 2x$. Recalling that $2x$ is the inverse of the symmetric case bidding function $b(v) = v/2$, we conclude that $b(v) < b_H(v) < b_L(v)$ ■

Proof of Proposition 2: Consider the constrained system (17) with the class of distributions for which we proved the existence and uniqueness of solution namely, distributions G which have a Taylor expansion around 0;

$$G(x) = \alpha x + \beta x^2 + \ldots,$$

with $\alpha > 0$. This implies that the density g has the expansion (around 0):

$$g(x) = \alpha + 2\beta x + \ldots.$$

Recall that in the symmetric case, the inverse bidding function $\rho := b^{-1}$ is determined by the following differential equation (which can be easily derived directly),

$$\rho'(x) = \frac{G(\rho(x))}{g(\rho(x))(\rho(x) - x)} \quad \text{with the boundary condition} \quad \rho(0) = 0. \tag{32}$$

Using the Taylor expansion for G in equation (32), we obtain the following asymptotic expansion of $\rho(x)$ near $x = 0$,

$$\rho(x) = 2x + Cx^2 + O(x^3) \quad \text{with} \quad C = -\frac{4\beta}{3\alpha}.$$

On the other hand carrying out an asymptotic expansion for $\phi(x)$ and $\sigma(x)$ (near $x = 0$), solving the differential equations of the system (17), we obtain:

$$\begin{aligned} \phi(x) &= \frac{4}{3}x + Ax^2 + O(x^3) \\ \sigma(x) &= 2x + Bx^2 + O(x^3) \end{aligned}$$

with

$$\begin{aligned} A + \frac{4}{3}B &= -\frac{16\beta}{9\alpha} \\ 3A - \frac{1}{3}B &= -\frac{4\beta}{9\alpha}, \end{aligned}$$

implying

$$B = -\frac{44\beta}{39\alpha},$$

and therefore for $\beta > 0$ we have $B > C$, and hence $\sigma(x) > \rho(x)$ at least for some interval near $x = 0$ that is, $b_H(v) < b(v)$ at least in some interval of valuations near $v = 0$. ■

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