Games and Economic Behavior **28,** 203–216 (1999)
Article ID game.1998.0695, available online at http://www.idealibrary.com on IDEAL®

Sowing Doubt Optimally in Two-Person Repeated Games

Eitan Israeli*

Department of Operation Research, Naval Postgraduate School, Monterey, California 93943

Received August 28, 1996

Consider a two-person repeated game, where one of the players, P1, can sow doubt, in the mind of his opponent, as to what P1's payoffs are. This results in a two-person repeated game with incomplete information. By sowing doubt, P1 can sometimes increase his minimal equilibrium payoff in the original game. We prove that this minimum is maximal when only one payoff matrix, the negative of the payoff matrix of the opponent, is added (the opponent thus believes that he might play a zero-sum game). We obtain two formulas for calculating this maximal minimum payoff. *Journal of Economic Literature* Classification Numbers: C7, D8.

1. INTRODUCTION

Repeated games with incomplete information have received considerable attention in the last 20–30 years. These games can be seen as theoretical idealizations of many social and economic situations. A standard two-player repeated game is a bimatrix game played an infinite number of times. A game with incomplete information is a game in which at least one of the players lacks part of the relevant data. A standard two-person game with incomplete information on one side consists of a finite set K of states of nature. Each state of nature is described by a bimatrix game, i.e., one payoff matrix for each player. A state of nature is chosen according to a probability vector p. Only player 1 is told which game was chosen.

Repeated games and games with incomplete information are two independent classes of games, but the combination of both is particularly

*This work is part of the author's M.Sc. dissertation, written in Tel Aviv University in 1989 under the helpful supervision of Prof. Sergiu Hart. The author wishes to thank Dr. Igal Milchtaich for his invaluable help in writing this paper, and the three anonymous referees for their useful comments.

0899-8256/99 $30.00

interesting. In a repeated game with incomplete information, the complicated structure of the repeated game allows the lack of information to come into play. Depending on the way player 1, the "informed" player, is playing, all the information he has, part of it, or no information, is revealed to the other player in the course of the game.

Incomplete information naturally raises the issue of cooperation between the two players, which is also related to repeated games. Recall the equivalence between equilibrium points in a repeated game with complete information and feasible and individually rational points in a one-shot game when cooperation is allowed, as expressed by the Folk Theorem.

Repeated games with incomplete information were first studied by Aumann and Maschler (1966); see Aumann and Maschler (1995, Chap. I). Their work was mainly concerned with the case of two-person zero-sum games. The two-person non-zero-sum case was first studied by Aumann, Maschler, and Stearns (1968); see Aumann and Maschler (1995, Chap. V). A complete characterization of the set of equilibria in these games, with standard one-sided information, was obtained by Hart (1985). The general case of this model assumes that the "uninformed" player monitors the actions of both players but does not know what his own realized payoffs are; otherwise he could deduce what state of nature was chosen. The current work looks at a special case where the uninformed player has the same payoff matrix in all states of nature. Thus, he knows his own payoffs but cannot deduce from them what the state of nature is. This kind of a game is called a repeated game with standard one-sided information and "known own payoffs."

Explicitly, the game consists of a set K of payoff matrices for player 1, A^1 to $A^{|K|}$, and a single payoff matrix, B, for player 2. A payoff matrix for player 1 is chosen according to a probability vector p (over K). Only player 1 is informed which matrix was chosen. Shalev (1994) characterized the set of equilibrium points for games in this class and proved that every such game has at least one equilibrium point.

One can build a game of this kind by extending a repeated game with complete information. Suppose that we have a repeated game with complete information, with payoff matrices $A \equiv A^1$ to player 1, and B to player 2. If we allow player 1 to create uncertainty by adding payoff matrices, A^2 to $A^{|K|}$, and a probability vector p (with all components strictly positive) for choosing among them, then we get a repeated game with one-sided information and known own payoffs. We will refer to this as the "extended game."

The set of equilibrium payoffs for player 1 in the original game (i.e., the game with payoff matrices A and B) is given by the Folk Theorem. It follows from this theorem that this set has a minimum, $\underline{a}$, and a maximum. One result of Shalev (1994, Lemma 5) is that, in the extended incomplete information game, the minimum equilibrium payoff for player 1 in case that the original state of nature is chosen (i.e., $k = 1$) is no less than $\underline{a}$, and can be strictly greater then $\underline{a}$. In other words, it is possible that, by adding new payoff matrices, the payoff guaranteed to player 1 in the case that the original state of nature is chosen, increases. Another result of Shalev (1994) is that the increase does not depend on the value of p.

The main question that the present work addresses is the following: Given A and B, which matrix or matrices should player 1 add if he wants to maximize the payoff guaranteed to him in the case that the chosen state of nature is the original one (i.e., in the case that his payoff matrix is A)? Our main result is that it is optimal for player 1 to add a single matrix, namely, $-B$, the negative of player 2's payoff matrix.

The minimum equilibrium payoff for player 1 in this case is equal to the payoff guaranteed to him in the one-shot game with payoff matrices A and B, when player 2's responses to player 1's strategy are restricted to those that yield player 2 a payoff of at least his value (i.e., the minmax value of B). This is similar, but not identical, to a non-zero-sum static Stackelberg games (Simaan and Cruz, 1973).

We can view the addition of matrices as a way of sowing doubt in player 2's mind. Thus, the main result is interpreted as follows: It is best for player 1 to make player 2 believe that, with some positive probability, he is involved in a zero-sum game, namely, that player 1 wants to keep player 2's payoff at its lowest. Surprisingly, this is what maximizes the guaranteed payoff of player 1 in the original game.

Notation. $\mathbf{R}$ is the real line. For a finite set L, the number of elements of L is denoted $|L|$, and $\mathbf{R}^L$ is the $|L|$-dimensional Euclidean space with coordinates indexed by the elements of L. Thus, we write $x = (x_l)_{l \in L}$ for $x \in \mathbf{R}^L$. For vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, $y \le x$ means $y_i \le x_i$ for $i = 1, \dots, n$. $x \cdot y$ is the scalar product $\sum_{i=1}^n x_i y_i$. $\Delta^L = \{x \in \mathbf{R}^L \mid x_l \ge 0 \text{ for all } l \in L \text{ and } \sum_{l \in L} x_l = 1\}$ is the unit simplex in $\mathbf{R}^L$. For finite sets I and J, $\delta \in \Delta^{I \times J}$, and matrix M of order $|I| \times |J|$, we define $M(\delta) = \sum_{i \in I} \sum_{j \in J} \delta_{ij} M(i, j)$.

2. BACKGROUND

Our work and results are based on a model that was introduced by Shalev (1994). In this section we present this model and some of Shalev's results that we are going to use in later sections. Shalev's model is based on the model of Aumann, Maschler, and Stearns (1968) and Hart (1985) and it consists of the following:

(i) Two players, player 1 and player 2.

(ii) A finite set I of actions for player 1, and a finite set J of actions for player 2. Each set contains at least two elements.

(iii) A finite set $K = \{A^1, A^2, \dots, A^{|K|}\}$ of payoff matrices of dimension $|I| \times |J|$ for player 1, and one payoff matrix B of dimension $|I| \times |J|$ for player 2.

(iv) A probability vector $p = (p_k)_{k \in K} \in \Delta^{|K|}$. We assume that $p_k > 0$ for every k.

A *game with incomplete information on one side and known own payoffs* $\Gamma_\infty(A^1, A^2, \dots, A^{|K|}; B; p)$ [or shortly $\Gamma_\infty(p)$] is defined as follows:

(v) An element $\mathbf{k}$ of K (interpreted as the state of nature) is chosen according to the probability vector p; $\mathbf{k}$ is told to player 1 but not to player 2.

(vi) At each stage t of the game, player 1 chooses an element i_t in I and player 2 chooses an element j_t in J. The choices are made without either player knowing the choice of the other player. Then, both players are told the pair (i_t, j_t).

(vii) Both players have perfect recall.

Next we describe the set of strategies for the players in $\Gamma_\infty(p)$. For $t = 1, 2, \dots$ let $H_t = (I \times J)^{t-1}$ be the set of histories up to (but not including) stage t. A *pure strategy* for player 1 is a sequence $\sigma = (\sigma_t)_{t=1}^\infty$, where

$$\sigma_t : H_t \times K \to I \qquad \text{for } t = 1, 2, \dots . \tag{2.1}$$

Thus, for every $k \in K$ (the "true" game being played), $\sigma_t(h_t, k)$ is the action taken by player 1 at stage t. Similarly, a pure strategy for player 2 is a sequence $\tau = (\tau_t)_{t=1}^\infty$, where

$$\tau_t : H_t \to J \qquad \text{for } t = 1, 2, \dots . \tag{2.2}$$

As usual, a mixed strategy is a probability distribution over the set of pure strategies. Since $\Gamma_\infty(p)$ is a game with perfect recall, one can restrict the study to *behavior strategies* (Kuhn, 1953; Aumann, 1964), where players make independent randomizations at each stage. A behavior strategy is defined in the same way as a pure strategy but with (2.1) replaced by

$$\sigma_t : H_t \times K \to \Delta^I \tag{2.3}$$

and (2.2) replaced by

$$\tau_t : H_t \to \Delta^J . \tag{2.4}$$

In the following, "strategy," with no qualifier, will always mean "behavior strategy."

We have not yet defined payoffs in $\Gamma_\infty(p)$, only payoff streams. Given a history $h_{t+1} \in H_{t+1}$ and a state of nature $k \in K$, the average payoff for player 1 in stages 1 through t is

$$a_{k,t} = (1/t) \sum_{s=1}^{t} A^k(i_s, j_s) \tag{2.5}$$

and the average payoff for player 2 in stages 1 through t is

$$\beta_t = (1/t) \sum_{s=1}^{t} B(i_s, j_s). \tag{2.6}$$

Both averages depend on σ and on τ, and $a_{k,t}$ depends also on k. Let $E^k_{\sigma,\tau}(a_{k,t})$ and $E_{\sigma,\tau,p}(\beta_t)$, respectively, be their expectations.[1] A pair of strategies (σ, τ) is an *equilibrium point* of $\Gamma_\infty(p)$ if, for every strategy σ' of player 1 and every $k \in K$,

$$\liminf_t E^k_{\sigma,\tau}(a_{k,t}) \geq \limsup_t E^k_{\sigma',\tau}(a_{k,t}) \tag{2.7}$$

and for every strategy τ' of player 2

$$\liminf_t E_{\sigma,\tau,p}(\beta_t) \geq \limsup_t E_{\sigma,\tau',p}(\beta_t). \tag{2.8}$$

In particular [take $\sigma' = \sigma$ in (2.7) and $\tau' = \tau$ in (2.8)], if (σ, τ) is an equilibrium point, then there exist a vector $a = (a_k)_{k \in K}$ and a scalar β such that, for every $k \in K$,

$$\lim_t E^k_{\sigma,\tau}(a_{k,t}) = a_k \text{ and } \lim_t E_{\sigma,\tau,p}(\beta_t) = \beta. \tag{2.9}$$

We write $a = \{a_k\}_{k \in K}$ for the vector payoff of player 1, and we will call (a, β) the *equilibrium payoffs* at the equilibrium point (σ, τ).

For a probability vector $p = (p_1, \dots, p_{|K|}) \in \Delta^{|K|}$, let $p \cdot A$ be the matrix whose (i, j)th element is $\sum_{k \in K} p_k A^k(i, j)$. The value of the (one-shot) two-person zero-sum game with payoff matrix $p \cdot A$ is denoted $(\mathrm{val}_1\, A)(p)$.

[1] $E_{\sigma,\tau,p}$ is the expectation with respect to the probability measure induced by the strategies (σ and τ) and by the choice of nature of $\mathbf{k}$ (according to $\mathbf{p}$). $E^k_{\sigma,\tau}$ denotes the expectation conditional on $\mathbf{k} = k$.

Explicitly,

$$(\mathrm{val}_1\ A)(p) = \max_{x\in\Delta^I}\ \min_{y\in\Delta^J}(p\cdot A)(x,y) = \min_{y\in\Delta^J}\ \max_{x\in\Delta^I}(p\cdot A)(x,y), \tag{2.10}$$

where $(p\cdot A)(x,y) = \sum_{i\in I}\sum_{j\in J} x_i y_j \sum_{k\in K} p_k A^k(i,j)$. Similarly, $\mathrm{val}_2\ B$ is the value to player 2 of the two-person zero-sum game with payoff matrix B, i.e.,

$$\mathrm{val}_2\ B = \max_{y\in\Delta^J}\ \min_{x\in\Delta^I} B(x,y) = \min_{x\in\Delta^I}\ \max_{y\in\Delta^J} B(x,y), \tag{2.11}$$

where $B(x,y) = \sum_{i\in I}\sum_{j\in J} x_i y_j B(i,j)$. A payoff vector $a = (a_k)_{k\in K} \in \mathbf{R}^{|K|}$ is *individually rational* (IR) *for player* 1 in $\Gamma_\infty(p)$ if

$$q\cdot a \geq (\mathrm{val}_1\ A)(q) \text{ for every } q\in\Delta^K. \tag{2.12}$$

This terminology is based on the fact that (2.12) is a necessary and sufficient condition for the set $Q_a = \{x\in\mathbf{R}^K \mid x\leq a\}$ to be approachable by player 2 in the sense of Blackwell (1956). A scalar β is an *individually rational payoff to player* 2 in $\Gamma_\infty(p)$ if

$$\beta \geq \mathrm{val}_2\ B. \tag{2.13}$$

Shalev (1994) obtained a characterization of the set of equilibrium points in $\Gamma_\infty(p)$ and proved its nonemptiness:

PROPOSITION 1 (Shalev, 1994). *A repeated game with incomplete information on one side and known own payoffs has at least one equilibrium point.*

PROPOSITION 2 (Shalev, 1994). *Let $\Gamma_\infty(A^1, A^2, \ldots, A^{|K|}; B; p)$ be a repeated game with incomplete information on one side and known own payoffs. Then, $(a,\beta)\in\mathbf{R}^K\times\mathbf{R}$ are equilibrium payoffs in $\Gamma_\infty(A^1, A^2, \ldots, A^{|K|}; B; p)$ if and only if for every $k\in K$ there exists a probability distribution $\delta^k\in\Delta^{I\times J}$ such that*

(i) $A^k(\delta^k) = a_k$ *for every* $k\in K$,

(ii) $\sum_{k\in K} p_k B(\delta^k) = \beta$,

(iii) $q\cdot a \geq (\mathrm{val}_1\ A)(q)$ *for every* $q\in\Delta^K$ (*i.e., a is IR for player* 1),

(iv) $B(\delta^k) \geq \mathrm{val}_2\ B$ *for every* $k\in K$, *and*

(v) $A^k(\delta^k) \geq A^k(\delta^{k'})$ *for every* $k, k'\in K$.

We now outline one direction of the proof of Proposition 2 and show how, given $\delta^1, \ldots, \delta^{|K|}$ that satisfy the above conditions, one can construct a completely revealing equilibrium point with equilibrium payoffs (a,β).

For each $k \in K$, fix a joint plan for achieving the frequencies δ^k. The plan will consist of two sequences of actions, one sequence for each player. The strategy σ of player 1 will be the following: Signal k in the first $[\log_2 |K|] + 1$ stages (by converting k to a binary number and signaling digit after digit), and then play according to the joint plan, corresponding to δ^k. If player 2 deviates from the plan, then play a mixed strategy that holds player 2 to $\text{val}_2\, B$. The strategy τ of player 2 will be the following: Play arbitrarily in the first $[\log_2 |K|] + 1$ stages, and then play according to δ^k. If player 1 deviates from the plan, then play a Blackwell strategy ensuring that player 1 will get no more than a; the existence of such a strategy follows from condition (iii) (Blackwell, 1956; Aumann and Maschler, 1966).

It is clear that neither player can benefit from a detectable deviation. Player 1 can make an undetectable deviation by signaling $k' \neq k$, but condition (v) ensures that he cannot benefit from such a deviation.

Conclusion. Every equilibrium payoff is achievable by a completely revealing strategy of player 1.

3. DESCRIPTION OF THE PROBLEM

Given payoff matrices A and B, let a game $\Gamma_\infty^*(A, B)$ be defined as follows: First, in stage 0 of the game, player 1 chooses a finite set of payoff matrices that includes A, say, $\{A^1 \equiv A, A^2, \dots, A^{|K|}\}$, and a probability vector $p \in \Delta^K$ with all components strictly positive. Then the game $\Gamma_\infty(A, A^2, \dots, A^{|K|}; B; p)$ is played. If player 1 does not add any payoff matrix, i.e., if $|K| = 1$, then by the Folk Theorem the set of equilibrium payoffs is the intersection of the feasible set $\{(a, \beta) \mid \exists \delta \in \Delta^{I \times J} \text{ s.t. } A(\delta) = a \text{ and } B(\delta) = \beta\}$ and the individual rational set $\{(a, \beta) \mid a \geq \text{val}_1\, A \text{ and } \beta \geq \text{val}_2\, B\}$. The guaranteed payoff for player 1 in this case is therefore $\underline{a} := \min\{A(\delta) \mid \delta \in \Delta^{I \times J}, A(\delta) \geq \text{val}_1\, A \text{ and } B(\delta) \geq \text{val}_2\, B\}$.

The main issue of this paper is to find an optimal play for player 1 in stage 0 of the game $\Gamma_\infty^*(A, B)$, i.e., a play that maximizes his minimum equilibrium payoff in case that the state of nature is the original one [according to proposition 1 in Shalev (1994) the *maximum* equilibrium payoff does not change].

Thus, for a given choice of $\{A^2, \dots, A^{|K|}\}$ let

$$\tilde{a}(A, A^2, \dots, A^{|K|}; B) := \min\Big\{A(\delta^1) \mid a, \delta^1, \dots, \delta^{|K|} \text{ satisfy (i)}, \text{(iii)}, \text{(iv)}, \text{and (v) in Proposition 2}\Big\}.^{2} \quad (2.14)$$

[2]As shown in Shalev (1994), $\tilde{a}$ does not depend on p, as long as all its coordinates are positive. Also, note that condition (ii) is irrelevant here.

Then we are looking for the maximum of these $\tilde{a}$, that is,

$$\hat{a} := \sup\left\{\tilde{a}(A, A^2, \dots, A^{|K|}; B) \,\middle|\, |K| \geq 2,\ A^2, \dots, A^{|K|} \text{ are matrices of order } |I| \times |J|\right\}. \quad (2.15)$$

Our main goal is to characterize this $\hat{a}$.

4. THE MAIN RESULT

THEOREM 1. *Let A and B be payoff matrices of order $|I| \times |J|$. Then $\hat{a}$ satisfies*

$$\hat{a} = \tilde{a}(A, -B; B), \quad (2.16)$$

$$\hat{a} = \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y), \text{ where } Y(x) = \left\{y \in \Delta^J \mid B(x, y) \geq \operatorname{val}_2 B\right\}, \text{ and} \quad (2.17)$$

$$\hat{a} = a^* = \sup_{\gamma \geq 0} \operatorname{val}_1(A - \gamma(B - \beta E)),$$

where $\beta = \operatorname{val}_2 B$ and E is the matrix of order $|I| \times |J|$ with all entries 1. (2.18)

The content of Theorem 1 is that an optimal play for player 1 is to add the single matrix $-B$. The theorem also suggests two ways for computing $\hat{a}$. We now give an example in which $\hat{a}$ is strictly greater than $\underline{a}$.

EXAMPLE. Consider the two-person repeated game with payoff matrices

$A =$

	L	R
T	0	0
B	1	0

and $B =$

	L	R
T	0	0
B	0	-10

By the Folk theorem, the set of equilibrium payoffs in this game is $\{(a, 0) \mid 0 \leq a \leq 1\}$. However, consider for example the equilibrium payoffs $(0, 0)$. The players get these payoffs when player 1 plays T and player 2 plays L. If player 1 deviates and plays B, then player 2 "punishes" him by playing R. But, in this case player 2 gets -10 and player 1 still gets 0, as before.

The addition of the payoff matrix $-B$ for player 1 eliminates this "problematic" equilibrium point. According to (2.17) in Theorem 1, $\hat{a}$ can be calculated by looking at the original game (the game with payoff matrices A and B) and assuming that player 2 is "individually rational," in the sense that he always plays in a way that ensures him at least $\mathrm{val}_2\, B$.

If player 1 plays B, then player 2 must play L. Therefore the only equilibrium payoffs after the addition of $-B$ are $(1, 0)$; so player 1 gets a payoff of 1. Note that (2.18) implies the same result:

$A - \gamma B =$

	L	R
T	0	0
B	1	10γ

$\mathrm{val}_1(A - \gamma B) = \min\{1, 10\gamma\}$, and $\sup_{\gamma \geq 0} \mathrm{val}_1(A - \gamma B) = 1$.

Proof of Theorem 1. Let $\beta = \mathrm{val}_2\, B$. By definition, $\tilde{a}(A, -B; B)$ is the solution of the following minimization problem (where $\delta^1, \delta^2 \in \Delta^{I \times J}$):

Minimize $A(\delta^1)$, subject to the constraints

$$\begin{aligned}
&\text{(i)} \quad B(\delta^1) \geq \beta,\\
&\text{(ii)} \quad B(\delta^2) \geq \beta,\\
&\text{(iii)} \quad A(\delta^1) \geq A(\delta^2),\\
&\text{(iv)} \quad (-B)(\delta^2) \geq (-B)(\delta^1), \text{ and}\\
&\text{(v)} \quad pA(\delta^1) + (1-p)(-B)(\delta^2) \geq \mathrm{val}_1(pA - (1-p)B) \text{ for all } 0 \leq p \leq 1.
\end{aligned} \tag{2.19}$$

Taking $p = 0$ in (v), we get $(-B)(\delta^2) \geq \mathrm{val}_1(-B) = -\beta$. In conjunction with (ii), this gives $B(\delta^2) = \beta$. Condition (v) therefore takes the form $pA(\delta^1) \geq \mathrm{val}_1[pA - (1-p)(B - \beta E)]$, for all $0 \leq p \leq 1$, which is equivalent to $A(\delta^1) \geq \sup_{\gamma \geq 0} \mathrm{val}_1(A - \gamma(B - \beta E)) = a^*$. We therefore conclude that

$$\tilde{a}(A, -B; B) \geq a^*. \tag{2.20}$$

Next we show that $a^* \geq \tilde{a}(A^1, A^2, \ldots, A^{|K|}; B)$ for every set of payoff matrices $K = \{A^1 \equiv A, A^2, \ldots, A^{|K|}\}$; This together with (2.20) completes the proof of (2.16) and (2.18). We prove this by constructing an equilibrium point for $\Gamma_\infty(A^1, A^2, \ldots, A^{|K|}; B; p)$ (for arbitrary p) in which the equilibrium payoff to player 1 in state of nature $k = 1$ is less than or equal to a^*. For $m \in K$, define

$$a_m = \max\left\{A^m(\delta) \mid \delta \in \mathbf{R}_+^{I \times J}, A(\delta) \leq a^*, B(\delta) \geq \beta, \text{ and } E(\delta) = 1\right\} \tag{2.21}$$

(the set on the right-hand side of (2.21) is nonempty because taking $\gamma = 0$ in (2.18) assures that $a^* \geq \text{val}_1 A$, and recall the Folk Theorem), and let δ^m be such that the maximum in (2.21) is attained at δ^m. It is clear from the definition that $\delta^1, \ldots, \delta^{|K|}$ satisfy conditions (iv) and (v) in Proposition 2. It remains to show that condition (iii) in Proposition 2 is satisfied too, i.e., that

$$\text{val}_1\left(\Sigma_{m \in K} p_m A^m\right) \leq \Sigma_{m \in K} p_m A^m(\delta^m) \text{ for every } p \in \Delta^K. \quad (2.22)$$

It follows from (2.21) that, for every $m \in K$ and every $\delta \in \mathbf{R}_+^{|I| \times |J|}$,

$$[(B - \beta E)(\delta) \geq 0 \text{ and } (a^*E - A)(\delta) \geq 0] \Rightarrow (a_m E - A^m)(\delta) \geq 0. \quad (2.23)$$

Therefore, by Farkas Lemma, for every $m \in K$ there exist non-negative constants, r_m and s_m, and a non-negative matrix C_m such that $a_m E - A^m = s_m(a^*E - A) + r_m(B - \beta E) + C_m$. Rearranging terms and using the monotonicity of the "val_1" operator, we conclude that, for every $p \in \Delta^K$,

$$\begin{aligned} &\text{val}_1\left(\Sigma_{m \in K} p_m A^m\right) \\ &\quad \leq \Sigma_{m \in K} p_m a_m + \text{val}_1\left[\Sigma_{m \in K} p_m\left(s_m A - r_m(B - \beta E)\right)\right] \\ &\quad\quad - \Sigma_{m \in K} p_m s_m a^*. \end{aligned} \quad (2.24)$$

It follows that (2.22) is satisfied if

$$\Sigma_{m \in K} p_m s_m a^* \geq \text{val}_1\left[\Sigma_{m \in K} p_m s_m A - \Sigma_{m \in K} p_m r_m(B - \beta E)\right]. \quad (2.25)$$

If $\Sigma_{m \in K} p_m s_m = 0$, then (2.25) holds. Otherwise, setting $\gamma = (\Sigma_{m \in K} p_m r_m)/(\Sigma_{m \in K} p_m s_m)$, (2.25) can be written as $a^* \geq \text{val}[A - \gamma(B - \beta E)]$, which is true by definiton of a^*.

It remains to prove (2.17).

First direction: $\hat{a} \leq \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y)$. For every $\varepsilon > 0$, there exist $\gamma(\varepsilon) \in \mathbf{R}$ and $x(\varepsilon) \in \Delta^I$ such that

$$\begin{aligned} &\sup_{\gamma \geq 0} \text{val}_1[A - \gamma(B - \beta E)] - \varepsilon \\ &\quad \leq \text{val}_1[A - \gamma(\varepsilon)(B - \beta E)] \\ &\quad = \min_{y \in \Delta^J} (A - \gamma(\varepsilon)(B - \beta E))(x(\varepsilon), y) \\ &\quad \leq \min_{y \in \Delta^J} \{(A - \gamma(\varepsilon)(B - \beta E))(x(\varepsilon), y) \mid B(x(\varepsilon), y) \geq \beta\} \\ &\quad = \min_{y \in Y(x(\varepsilon))} A(x(\varepsilon), y) \leq \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y). \end{aligned} \quad (2.26)$$

Second direction: $\hat{a} \geq \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y)$. For every $\varepsilon > 0$, there exists $x'(\varepsilon) \in \Delta^I$ such that $\min_{y \in Y(x')} A(x', y) \geq \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y) - \varepsilon$, and therefore, for every $y' \in \mathbf{R}^J_+$,

$$B(x'(\varepsilon), y') - \beta \Sigma_{j \in J} y'_j \geq 0$$
$$\Rightarrow A(x'(\varepsilon), y') - \left(\sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y) - \varepsilon \right) \Sigma_{j \in J} y'_j \geq 0.$$

It hence follows from Farkas Lemma that there exist a scalar $\gamma \geq 0$ and a vector $c \in \mathbf{R}^J_+$ such that, for every $y' \in \Delta^J$,

$$\begin{aligned} A(x'(\varepsilon), y') - \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y) + \varepsilon \\ = \gamma(B(x'(\varepsilon), y') - \beta) + c \cdot y' \\ \geq \gamma(B - \beta E)(x'(\varepsilon), y'). \end{aligned} \tag{2.27}$$

Therefore,

$$\hat{a} + \varepsilon \geq \mathrm{val}_1[A - \gamma(B - \beta E)] + \varepsilon \geq \sup_{x \in \Delta^I} \min_{y \in Y(x)} A(x, y). \tag{2.28}$$

Since ε is arbitrary, Eqs. (2.26) and (2.28) together imply (2.17). ■

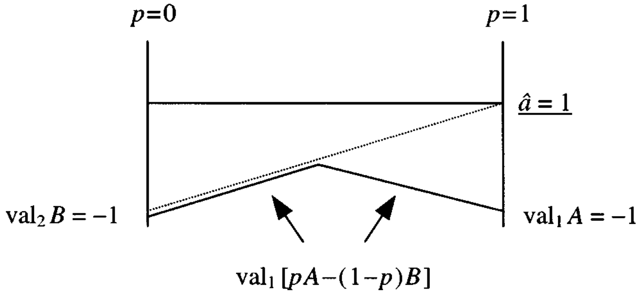
5. GEOMETRICAL INTERPRETATION

Equation (2.18) in Theorem 1 can be interpreted geometrically. This equation implies

$$p\hat{a} + (1 - p)(-\beta) \geq \mathrm{val}_1(pA - (1 - p)B) \text{ for all } 0 \leq p \leq 1, \text{ and} \tag{2.29}$$

$$p_0\hat{a} + (1 - p_0)(-\beta) = \mathrm{val}_1(p_0 A - (1 - p_0)B) \text{ for some } 0 < p_0 \leq 1. \tag{2.30}$$

Thus, the graph of the affine function (of the variable p) on the left-hand side of (2.29) lies above the graph of the function on the right-hand side of (2.29) and intersects it at $p = 0$ and at $p = p_0$. The former graph intersects the $p = 1$ line at the point $\hat{a}$. The latter graph intersects it at the point $\mathrm{val}_1 A$. This suggests the following geometrical way for computing $\hat{a}$: Plot the graph of $\mathrm{val}_1(pA - (1 - p)B)$, and then draw the line with minimum slope that lies above this graph and passes through the point $(0, -\beta)$. This line passes through $(1, \hat{a})$. Here is an example, where $\mathrm{val}_1 A = -1$, $\mathrm{val}_2 B = 1$, $\mathrm{val}_1[pA - (1 - p)B] = \min(p - 1, -p)$.



A	L	R
T	0	-1
B	-1	-1

B	L	R
T	1	0
B	1	0

$pA-(1-p)B$	L	R
T	$p-1$	$-p$
B	-1	$-p$

6. LEADER AND FOLLOWER INTERPRETATION

Equation (2.17) in Theorem 1 is similar but not identical to a non-zero-sum static Stackelberg game. According to this equation, the minimum equilibrium payoff for player 1 is equal to the payoff guaranteed to him in the one-shot game with payoff matrices A and B, when player 2 knows player 1's strategy and replies in such a way that his own payoff is at least the minimax value of B. Player 1 is the leader, because he is first to play. Player 2 is the follower.

In a non-zero-sum static Stackelberg game, the leader's strategy (called also the Stackelberg strategy) is optimal with respect to the assumption that the follower maximizes his objective function, after he knows what the leader did (Simaan and Cruz, 1973). Equation (2.17) represents a more extreme situation: The follower assumes to be indifferent between strategies when his own payoff is above the value of B, and the leader must be prepared to the worst case among all those follower's strategies.

On the other hand, in Stackelberg game, player 1, the leader, actually chooses his strategy. Hence he is limited to pure strategies. Equation (2.17) represents a situation where the leader can choose mixed strategies as well.

7. RELATED WORK

The work presented in this paper was done in 1989. Since then many related articles have been published. The interested reader should consult Cripps and Thomas (1995), Cripps, Schmidt, and Thomas (1996), and the references therein.

One possible extension of the work allows incomplete information on both sides. In this model, each player has a possible set of matrices. A couple of payoff matrices are chosen, one for each player, but each player is told only what his payoff matrix is. Koren (1989) characterized the set of equilibrium points for this general model. As in the one-sided case, all equilibrium points are achievable through a completely revealing strategies. Israeli (1989) explored the special case of "two-sided sowing doubt," where the possible matrices of the first player are A and $-B$ while the possible matrices of the second player are B and $-A$. An example without equilibrium point is shown.

REFERENCES

Aumann, R. J. (1964). "Mixed and Behaviour Strategies in Infinite Extensive Games," *Advances in Game Theory* (M. Dresher *et al*., Eds.), Annals of Mathematical Studies No. 52, pp. 627–650, Princeton: Princeton Univ. Press.

Aumann, R. J., and Maschler, M. (1966). "Game Theoretic Aspects of Gradual Disarmament," *Mathematica*, Reports to the U.S. Arms Control and Disarmament Agency, ST-80, Chap. V, pp. 1–55. Reprinted in Aumann and Maschler (1995, Chap. I).

Aumann, R. J., and Maschler, M. (1995). *Repeated Games with Incomplete Information*. Cambridge, MA: MIT Press.

Aumann, R. J., Maschler, M., and Stearns, R. E. (1968). "Repeated Games of Incomplete Information: An approach to the Non-Zero-Sum Case," *Mathematica*, Reports to the U.S. Arms Control and Disarmament Agency, ST-143, Chap. IV, pp. 117–216. Reprinted in Aumann and Maschler (1995, Chap. V).

Blackwell, D. (1956). "An Analogue for the Minimax Theorem for Vector Payoffs," *Pacific J. Math*. **6**, 1–8.

Cripps, M., and Thomas, J. (1995). "Reputation and commitment in two-person repeated games without discounting," *Econometrica* **63**, 1401–1419.

Cripps, M., Schmidt, K., and Thomas, J. (1996). "Reputation in perturbed repeated games," *J. Econ. Theory* **69**, 387–410.

Hart, S. (1985). "Nonzero-Sum Two-Person Repeated Games with Incomplete Information," *Math. Oper. Res*. **10**, 117–153.

Israeli, E. (1989). "Sowing Doubt Optimally in Two-Person Repeated Games" (in Hebrew), M.Sc. Thesis, School of Mathematics, Tel-Aviv University.

Koren, G. (1989). "Two-Person Repeated Games with Incomplete Information and Observable Payoffs," M.Sc. Thesis, School of Mathematics, Tel-Aviv University.

Kuhn, H. W. (1953). "Extensive Games and the Problem of Information," in *Contributions to the Theory of Games* (H. W. Kuhn and W. Tucker, Eds.), Vol. II, Annals of Mathematics No. 28, pp. 193–216. Princeton: Princeton Univ. Press.

Shalev, J. (1994). "Nonzero-Sum Two-Person Repeated Games with Incomplete Information and Known-Own Payoffs," *Games Econ. Behav.* **7**, 246–259.

Simaan, M., and Cruz, J. B. (1973). "On the Stackelberg strategy in nonzero-sum games," *J. Opt. Theory Appl.* **11**, 533–555.