האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

NOISE SENSITIVITY AND CHAOS IN SOCIAL CHOICE THEORY

by

GIL KALAI

Discussion Paper # 399 August 2005

מרכז לחקר הרציונליות

CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

NOISE SENSITIVITY AND CHAOS IN SOCIAL CHOICE THEORY

Gil Kalai *†

Abstract

In this paper we study the social preferences obtained from monotone neutral social welfare functions for random individual preferences. It turns out that there are two extreme types of behavior. On one side, there are social welfare functions, such as the majority rule, that lead to stochastic stability of the outcome in terms of perturbations of individual preferences.

We identify and study a class of social welfare functions that demonstrate an extremely different type of behavior which is a completely chaotic: they lead to a uniform probability distribution on all possible social preference relations and, for every $\epsilon > 0$, if a small fraction ϵ of individuals change their preferences (randomly) the correlation between the resulting social preferences and the original ones tends to zero as the number of individuals in the society increases. This class includes natural multi-level majority rules.

Keywords: Social choice, social welfare functions, simple games, Banzhaf power index, noise sensitivity, chaos, indeterminacy, noise stability.

*Hebrew University of Jerusalem and Yale University

†I wish to thank Itai Benjamini, Guy Kindler, Nati Linial, Elchanan Mossel, Motty Perry, Oded Schramm, and Yuval Peres for fruitful discussions. This research was supported by an ISF Bikura grant and by an NSF grant.

1 Introduction

How likely is it that small random mistakes in counting the votes in an election between two candidates will reverse the election's outcome? And if there are three alternatives and the society prefers alternative a to alternative b and alternative b to alternative c how likely is it that a will be preferred to c? We will show that for general social welfare functions these two questions are closely related.

In this paper we consider the behavior of general social welfare functions with respect to random uniform voter profiles. Namely, the individual preferences on a set of alternatives are uniformly and independently distributed among all order relations. It turns out that there are two extreme types of behavior: social welfare functions, such as the majority rule, that lead to stochastic stability of the outcome in terms of perturbations of individual preferences and social welfare functions that lead to what we refer to as "social chaos." [1]

These two types of behavior in the case of two alternatives have been studied in the mathematical literature by Benjamini, Kalai and Schramm (1999).

A simple game (or voting game) G defined on a set N of players (voters) is described by a function v that assigns to every subset (coalition) S of players the value "1" or "0". We assume that $v(\emptyset) = 0$ and $v(N) = 1$. A candidate is elected if the set S of voters that voted for him is a winning coalition in G, i.e., if $v(S) = 1$. We will always assume that the game G

[1] There are many studies of chaos and chaotic dynamics in economics but they appear to be rather different from this one. In the context of social choice theory, the huge number of "voting paradoxes" and the fact that "everything can happen" are often referred to as chaotic phenomena and were studied extensively by Saari; see, e.g., Saari (2001). Saari's works are concerned mainly with specific voting methods like majority, plurality, Borda's method, and some weighted versions of them. We refer to the property that "anything can happen," as "indeterminacy". Indeterminacy is different from yet related to our notion of social chaos; see Section 7 and Kalai (2004).

is monotone, i.e., that if $v(R) = 1$ and $R \subset S$ then $v(S) = 1$. Recall that a simple game is *proper* if $v(S) + v(N \backslash S) \leq 1$ for every coalition S, i.e., if the complement of a winning coalition is a losing one. A simple game G is *strong* if $v(S) + v(N \backslash S) = 1$ for every coalition S, i.e., if it is proper and the complement of a losing coalition is a winning one.

It is convenient to let $N = \{1, 2, \ldots, n\}$ and regard v as a *Boolean function* $v(x_1, x_2, \ldots, x_n)$ where each variable x_i is a Boolean variable: $x_i \in \{0, 1\}$, and thus $v(x_1, x_2, \ldots, v_n)$ stands for $v(S)$, where $S = \{i \in N : x_i = 1\}$. This notation is used in much of the relevant mathematics and computer science literature. It is also consistent with the basic economic interpretation of the variables x_i as signals the voters receive regarding the superior alternative.

A neutral social welfare function $F(A)$ on a set A of m alternatives based on a proper strong simple game G is defined as follows: the function $F(A)$ is a map that assigns an asymmetric relation R on the alternatives to every profile of individual preferences. Given a monotone proper simple game G and a set A of alternatives, the society prefers alternative a to alternative b if the set of voters that prefers a to b forms a winning coalition.[2]

We study probabilistic properties of social welfare functions for random uniform voter profiles. Thus, to every individual, we assign the same probability $(1/m!)$ for each one of the $m!$ order preference relations on the alternatives. Furthermore, the individual preference relations are independent. In other words, if there are n individuals, we assign the same probability for each one of the $(m!)^n$ possible profiles of individual preferences. This is a standard probabilistic model which has been extensively studied in the literature especially in the case of the majority rule (see Gehrlein (1997)). It is worth noting at this point that we will be studying the asymptotic properties of strong simple games and the associated social welfare functions, and it will be convenient to describe our results in terms of sequences $(G_k)_{k=1,2,\ldots}$ of simple games.[3]

[2] The neutral social choice we consider satisfies the basic Pareto and IIA conditions. Of course, we do not assume that the social preference is necessarily an order relation.

[3] For our results we will not require any relation between the games G_k for different

We can describe our notion of social chaos using several equivalent definitions and we will start with the following simple definition based on the probability of cyclic social preferences when there are three alternatives. Let G be a monotone strong simple game with n players and let F be a neutral social welfare function on three alternatives based on G. Consider a random uniform profile for the individual order preferences between the three alternatives. Let $p_{\mathbf{cyc}}(G)$ be the probability that the social preferences are cyclic. Arrow's theorem asserts that unless G is a dictatorship, $p_{\mathbf{cyc}}(G) > 0$.

Definition 1.1. The sequence (G_k) of monotone strong simple games leads to *social chaos* if

$$\lim_{n\to\infty} p_{\mathbf{cyc}}(G_k) = 1/4. \tag{1.1}$$

There are altogether eight possible social asymmetric relations for three alternatives, six of which are order relations and two of which are cyclic. Consider a social welfare function based on a strong monotone simple game G on three alternatives and the uniform distribution on individual order preferences. Symmetry considerations imply that the probabilities of obtaining each of the six order relations as the social preference relation are equal, as are the two probabilities for the two cyclic relations. Therefore, another way of stating relation (1.1) is that for a social welfare function based on (G_k) on three alternatives under the uniform distribution on individual order preferences, for every asymmetric relation R, the probability, for a random voter profile, that the social preferences are described by R tends to 1/8.

A result of Gulibaud (see Gehrlein (1997)) asserts that for the social welfare functions on three alternatives based on simple majority, the probability for a cyclic order relation as the number of voters tends to infinity is

values of k. It is useful to think of the results applied to the case where the various G_k's represent the "same" type of voting rule, such as "simple majority" for different population sizes. But giving a formal definition of a "voting rule" that applies to populations of different sizes is not an easy matter and, in any case, is not needed in this paper.

$1/4 - (3/(2 \cdot \pi)) \cdot \arcsin(1/3) \approx .08744$. In other words, if we assume that the voter profile is uniform and observe that the society prefers alternative a to b and b to c, our a posteriori probability of the society preferring a to c tends to 0.876. In contrast, in a similar scenario where the sequence G_k satisfies relation (1.1) our a posteriori probability of the society preferring a to c will tend to 0.5 (which is the a priori probability).[4]

We will now move from three alternatives to two and consider the effect of random noise. Suppose that there are two candidates and that the voters' preferences are random and uniform or equivalently that $x_i = 1$ with probability 1/2 independently for all voters. The *noise sensitivity* of f is defined as the effect of random independent mistakes in counting the votes. Formally, for a strong simple game G and $t > 0$, consider the following scenario: first choose the voter signals $x_1, x_2, \ldots, x_n$ randomly such that $x_i = 1$ with probability $p = 1/2$ independently for $i = 1, 2, \ldots, n$. Let $S = v(x_1, x_2, \ldots, x_n)$. Next let $y_i = x_i$ with probability $1 - t$ and $y_i = 1 - x_i$ with probability t, independently for $i = 1, 2, \ldots, n$. Let $T = v(y_1, y_2, \ldots, y_n)$. Define $N_t(G)$ to be the probability that $S \neq T$.

Definition 1.2. A sequence $(G_k)_{k=1,2,\ldots}$ of strong simple games is *asymptotically noise-sensitive* if, for every $t > 0$,

$$\lim_{k \to \infty} N_t(G_k) = 1/2. \tag{1.2}$$

Our first theorem shows a surprising relation between noise sensitivity for two alternatives and the probability of Condorcet's paradox for three alternatives.

Theorem 1.3. *A sequence (G_k) of monotone strong simple games leads to social chaos if and only if it is noise-sensitive.*

In proving Theorem 1.3 we obtain a simple formula which, for every strong simple game G, directly relates the probability for Condorcet's paradox and noise sensitivity when the probability for an error is 1/3.

[4] (Kalai (2002) proved that for every monotone strong simple game this a posteriori probability is always larger than 0.5.)

Proposition 1.4. *For every strong simple game G,*

$$2P_{\mathbf{cyc}}(G) + 1 = 3N_{1/3}(G). \tag{1.3}$$

We will now define the complementary notion of noise (or stochastic) stability.

Definition 1.5. A class $\mathcal{G}$ of strong simple games is uniformly noise-stable if for every $s > 0$ there is $u(s) > 0$ such that $N_{u(s)}(G) \leq s$.

Theorem 1.6. *The family of simple majority functions M_n is uniformly noise-stable. Moreover, $N_{u(s)}(M_n) \leq s$ for $u(s) = K \cdot s^2$ for some constant K.*

Our next theorems provide several additional equivalent properties of social chaos which we believe justify this notion. We first note that we can consider slightly larger classes of games. A sequence G_k of monotone proper simple games has a *strongly diminishing bias* if for a random uniform voter profile, the probability of a "tie" tends to zero as n tends to infinity. An example is the class of simple majority games. All our results extend from strong simple games to the case of monotone proper simple games with strongly diminishing bias.

An asymmetric relation R on a finite set X is a binary relation such that every pair of elements $x, y \in X$ is ascribed one and only one of the relations xRy or yRx. Clearly, there are $2^{\binom{m}{2}}$ asymmetric relations on a set of m alternatives. For a monotone strong simple game G and an asymmetric relation R on a set of m alternatives, let $p_R(G)$ be the probability under the uniform distribution on voter profiles that the social welfare function based on G will lead to R as the social preference relation.

Theorem 1.7. *Let (G_k) be a sequence of proper strong simple games. The following properties P_m are equivalent for $m \geq 3$:*

(P_m): *For a social welfare function based on (G_k) on a set A of m alternatives and every asymmetric relation R on A,*

$$\lim_{k\to\infty} p_R(G_n) = 1/2^{\binom{m}{2}}. \tag{1.4}$$

Note that (P_3) is simply our definition of social chaos. It is easy to see that (P_m) implies ($P_{m'}$) when $m' < m$. Surprisingly, the reverse implication also holds when $m' \geq 3$.

Consider a neutral monotone social welfare function and a random voter profile U, and let R be the resulting preference relation for the society. Consider the following scenario:

Suppose that every voter with a small probability t reconsiders and reverses the order of the two alternatives he ranked in the two *last* places. Let R' be the new social preference relation.

Theorem 1.8. *Let $m \geq 2$ be a fixed integer and t, $0 < t < 1/2$, be a fixed real number. A sequence G_k leads to social chaos if and only if any one of the following equivalent properties is satisfied:*

(A_m) The correlation between R and R' tends to zero with n.

(B_m) For every two asymmetric relations R_1 and R_2 on A, the probability that $R' = R_2$, conditioned on $R = R_1$, tends to $1/2^{\binom{m}{2}}$ as n tends to infinity.

(Properties A_2 and B_2 both coincide with the notion of noise sensitivity.)

In Section 2 we provide the mathematical tools and proofs for our results concerning the equivalence between the different properties of social chaos. We also describe there relations with the Banzhaf power index and additional characterization in terms of the correlation with weighted majority functions. Benjamini, Kalai and Schramm (1999) showed that the class of weighted majority (strong) games is uniformly noise-stable. A stronger quantitative version was proved by Peres (2004). Benjamini, Kalai and Schramm (1999) also proved that if (G_k) is sequence of strong simple games that is asymptotically uncorrelated with weighted majority games, then it is noise-sensitive.

In Section 3 we will analyze several examples. Consider the class of multi-level majority-based voting rules described inductively as follows: (i) Simple majority on an odd number of at least three voters has one layer. (ii) Suppose that the players are divided into an odd number of three or more groups and on each group we consider a game with r layers. If G is defined by the simple majority of the outcomes, then G is said to be a multi-layer majority-based simple game with $r+1$ layers.

Theorem 1.9. *A sequence (G_k) of multi-layer voting rules based on simple majority leads to social chaos if and only if the number of layers tends to infinity as k does.*

The reason for the chaotic behavior is that any new level of majority will amplify the probability of a cyclic outcome in three alternatives and by a similar argument every new level will amplify the noise introduced by mistakes in counting the votes.

When we base our rule on supermajority, social chaos may already appear for two layers. Consider a set of n individuals that is divided into many committees of size k. The decision between two alternatives is based on the majority of committees that prefer one of the alternatives by a 2/3-supermajority. This rule defines a proper simple game and in order to have a strongly diminishing bias we require that $k = o(\log n)$. We will show in Section 3 that this example leads to complete social chaos when the size of the committees tends to infinity with n. The reason for the chaotic behavior in this case is that the outcomes of the elections are determined by a small number of committees with "decisive views;" i. e., they are sufficiently biased towards one of the alternatives. Such a strong bias in favor of one of the alternatives is unlikely to survive following a small perturbation of the individual preferences. We note that hierarchical or recursive aggregation of preferences similar to those in the first example, and cases where the social preferences heavily depend on a small number of small communities with decisive views, can both represent realistic situations of preference aggregation.

The assumption of uniform and independent voter behavior is a standard one and is the basis of a large body of literature. Most of this literature deals with the majority voting rule (see, e.g., Bell (1981) and Gehrlein (1997)). Assuming uniform and independent voter behavior is unrealistic, though it can be argued that for issues concerning noise sensitivity and Condorcet's paradox this assumption represents a realistic way of comparing different voting methods. In Sections 4 and 5 we examine our probabilistic assumptions and discuss the situation under more general probability distributions for voter signals/behavior. Section 4 is devoted to the case that either the probability distribution or the voting rule are a priori biased. In Section 5 we present an example of chaotic behavior under the majority voting rule, when voter behavior is not independent.

Perhaps the most important form of "noise" in real-life elections and other forms of aggregated choice is abstention. In Section 6 we extend our model to allow for individual indifference between alternatives. Analyzing this model, we find out that noise sensitivity implies that a small change in the fraction of voters who decide to abstain has a dramatic effect on the outcomes, similar to the effect of random mistakes in counting their votes. Section 7 is devoted to a discussion of issues related to noise stability, information aggregation, indeterminacy, the asymptotic nature of our results, and possible extensions to other economic models. Section 8 concludes.

2 Proofs of the equivalence theorems

In this section we prove the various equivalent properties of sequences of simple games that lead to social chaos. We also state additional equivalent formulations. The proofs rely on two methods. A rather simple coupling argument allows us to move from noise sensitivity to the conclusions concerning more than two alternatives given in Theorems 1.3, 1.7, and 1.8. The only remaining step is to show that our definition of social chaos in terms of the probability of Condorcet's paradox for three alternatives implies noise sensitivity. I am not aware of a direct probabilistic argument and the proof of this implication relies on a rather elementary harmonic analysis of Boolean functions. This part of the proof requires mathematical techniques and concepts developed in Benjamini, Kalai and Schramm (1999) and in Kalai (2002).

Let (G_k) be a sequence of strong simple games. (All our arguments apply unchanged to the case of proper simple games with strongly diminishing bias.) We consider the following properties:

- (NS) Noise sensitivity
- (P_m) Asymptotically uniform distribution of social preference relations when there are m alternatives, $m \geq 3$
- Properties $(A_m), (B_m)$, $m \geq 2$.

We wish to prove that all these properties are equivalent. The basic arguments are presented in this section with some technical proofs presented in the Appendix.

2.1 A coupling argument

Proof of the implication$(NS) \Rightarrow (P_k)$:

We rely here on a simple (coupling) idea which nonetheless requires careful application. Consider a random voter profile. For two alternatives a and b consider the set V of voters for whom alternatives a and b are consecutive. We will try to determine the effect of reversing the order of the two alternatives a and b with probability $1/2$ for all voters in V.

Let R and R' be two asymmetric relations that differ only for the two alternatives a and b, namely, for a set $\{c, d\} \neq \{a, b\}$ of two alternatives cRd if and only if $cR'd$ and for the two alternatives a and b, aRb and bRa. We will derive from noise sensitivity that

$$\lim_{k \to \infty} p_R(G_k) = \lim_{k \to \infty} p_{R'}(G_k). \tag{2.1}$$

Consider the following process of determining two profiles P and P' of individual preference relations:

Step (1): Determine the preference order relations between alternatives a and b for every voter randomly and uniformly.

Step (2): Choose the order preference relations for all voters subject to the voters' preference between a and b that was determined in step (1).

Let P be the resulting profile of voters' order relations obtained by this process. Of course, P is a uniformly distributed random profile.

Let V be the set of voters for whom alternatives a and b are consecutive in the preference relation. The set V is a random subset of the set of voters. The probability that a voter i belongs to V is $1/(m-1)$ and these events are independent. Moreover, the set V and the random voter preference relations are independent.

Let P' be the voter profile obtained from P by reversing the order relations between alternatives a and b for every voter in V. Note that P' is also a uniformly distributed random profile.

Let R be the asymmetric relation obtained by applying the social welfare function F to the profile P and let R' be the asymmetric relation obtained by applying F to P'. Note that the two relations R and R' coincide for every pair of alternatives except possibly a and b.

If we focus on these two variables we note that the distribution of voter preference relations between a and b in P is random. We claim that the correlation between the events aRb and $aR'b$ tends to zero as n tends to infinity. Indeed, this follows from the fact that $\lim_{n\to\infty} N_t(f_n) = 0$ for $t = 1/(m-1)$. *Q.E.D.*

The proof of the implication $(NS) \Rightarrow (B_m)$ is similar to the proof presented above and is presented in Section 9.1 in the Appendix.

Proof of the implication $(P_m) \Rightarrow (P'_m)$ **for** $m' \leq 3$, $m' < m$ **:**

Let A be a set of m alternatives and let A' be a subset of A containing m' alternatives. A random uniform voter profile on A induces a random uniform voter profile on A'. Every asymmetric relation R' on A' can be extended to precisely $2^{\binom{m}{2}-\binom{m-1}{2}}$ asymmetric relations R on A. Given such an asymmetric relation R, the probability that a social welfare function based on G_k will lead to R tends to $1/2^{\binom{m}{2}}$. (By our assumption (P_m).) Therefore the probability that the restriction on A' will lead to R' tends to $1/2^{\binom{m'}{2}}$ as required. *Q.E.D.*

The same argument shows the implications: $(A_m) \Rightarrow (A_{m'})$, for $2 \leq m' < m$.

2.2 An elementary harmonic analysis argument

Proof of Theorem 1.3:

As mentioned in the Introduction, a simple game can be described by a *Boolean function*, namely, $f(x_1, x_2, \ldots, x_n)$, where the variables x_k take the values 0 or 1 and the value of f itself is also either 0 or 1. Every 0-1 vector $x = (x_1, x_2, \ldots, x_n)$ corresponds to a subset of players $S = \{k : x_k = 1\}$ and we let $f(x_1, x_2, \ldots, x_2) = v(S)$.

Let Ω_n denote the set of 0-1 vectors $(x_1, \ldots, x_n)$ of length n. Let $L_2(\Omega_n)$ denote the space of real functions on Ω_n, endowed with the inner product

$$< f, g >= \sum_{(x_1,x_2,\ldots,x_n)\in\Omega_n} 2^{-n} f(x_1, \ldots, x_n) g(x_1, \ldots, x_n).$$

The inner product space $L_2(\Omega_n)$ is 2^n-dimensional. The L_2-norm of f is defined by

$$\|f\|_2^2 =< f, f >= \sum_{(x_1,x_2,\ldots,x_n)\in\Omega_n} 2^{-n} f^2(x_1, x_2, \ldots, x_n).$$

Note that if f is a Boolean function, then $f^2(x)$ is either 0 or 1 and therefore $\|f\|_2^2 = \sum_{(x_1,\ldots,x_n)\in\Omega_n} 2^{-n} f^2(x)$ is simply the probability that $f = 1$ (with respect to the uniform probability distribution on Ω_n). If the Boolean function f represents a strong simple game then $\|f\|_2^2 = 1/2$.

For a subset S of N consider the function

$$u_S(x_1, x_2, \ldots, x_n) = (-1)^{\sum\{x_i:\ i\in S\}} .$$

It is not difficult to verify that the 2^n functions u_S for all subsets S form an orthonormal basis for the space of real functions on Ω_n.

For a function $f \in L_2(\Omega_n)$, let

$$\widehat{f}(S) =< f, u_S >$$

($\widehat{f}(S)$ is called a Fourier-Walsh coefficient of f). Since the functions u_S form an orthogonal basis it follows that

$$< f, g >= \sum_{S \subset N} \widehat{f}(S)\widehat{g}(S). \tag{2.2}$$

In particular,

$$\|f\|_2^2 = \sum_{S \subset N} \widehat{f}^2(S). \tag{2.3}$$

This last relation is called Parseval's formula.

Remark: We will demonstrate now the notions introduced here with a simple example. Let f_3 denote the payoff function of the simple majority game with three players. Thus, $f_3(x_1, x_2, x_3) = 1$ if $x_1 + x_2 + x_3 \geq 2$ and $f(x_1, x_2, x_3) = 0$, otherwise. The Fourier coefficients of f_3 are easy to compute: $\widehat{f_3}(\emptyset) = \sum(1/8)f_3(x) = 1/2$. In general if f is a Boolean function then $\widehat{f}(\emptyset)$ is the probability that $f(x) = 1$ and when f represents the payoff function of a strong simple game $\widehat{f}(\emptyset) = 1/2$. Next, $\widehat{f_3}(\{1\} = 1/8(f_3(0,1,1) - f_3(1,0,1) - f_3(1,1,0) - f_3(1,1,1)) = (1-3)/8$ and thus $\widehat{f_3}(\{j\}) = -1/4$, for $j = 1, 2, 3$. Next, $\widehat{f_3}(S) = 0$ when $|S| = 2$ and finally $\widehat{f_3}(\{1, 2, 3\}) = 1/8(f_3(1, 1, 0) + f_3(1, 0, 1) + f_3(0, 1, 1) - f(1, 1, 1)) = 1/4$.

We point out the following simple result:

Proposition 2.1. *If f is a Boolean function representing a strong simple game then $\widehat{f}(S) = 0$ whenever $|S|$ is an even positive integer.*

Proof: Let $g = f - 1/2$ $(= f(x) - 1/2u_\emptyset)$. Since f represents a strong simple game $g(1 - x_1, 1 - x_2, \ldots, 1 - x_n) = -g(x_1, x_2, \ldots, x_n)$. When $|S|$ is even, consider the contributions of $(x_1, x_2, \ldots, x_n)$ and $(1-x_1, 1-x_2, \ldots, 1-x_n)$ to the expression $\widehat{g}(S) =< g, u_S >$. Note that these two contributions cancel out and therefore $\widehat{g}(S) = 0$ for every set S of even size. It follows that when S is a nonempty set of even size, $\widehat{f}(S) = \widehat{g}(S) = 0$.

Let f be a Boolean function. For $k \geq 0$, let

$$W_k(f) = \sum \{\widehat{f}^2(S) : S \subset N, |S| = k\}. \tag{2.4}$$

Theorem 2.2. *The probability $P_{\mathbf{cyc}}(f)$ of a cyclic outcome of a social welfare function on three alternatives based on a strong simple game G with a payoff function f is*

$$P_{\mathbf{cyc}}(G) = 1/4 - \sum_{k=1}^{n} (1/3)^{k-1} W_k(f). \tag{2.5}$$

Theorem 2.2 is from Kalai (2002).

Theorem 2.3. *Let G be a strong simple game and let f be its payoff function. The probability that an ϵ-noise will change the outcome of an election is given by the formula:*

$$N_\epsilon(G) = 1/2 - 2 \cdot \sum_{k=1}^{n} (1-2\epsilon)^k W_k(f). \tag{2.6}$$

Theorem 2.3 is from Benjamini, Kalai and Schramm (1999). The proofs of Theorems 2.2 and 2.3 are quite elementary and rely essentially on relations (2.2) and (2.3). Note that we obtain the simple relation (Proposition 1.4) between the probability for Condorcet's paradox and noise sensitivity when the probability for a noisy bit is 1/3.

$$2P_{\mathbf{cyc}}(G) + 1 = 3N_{1/3}(G). \tag{2.7}$$

From Theorem 2.3 we can easily derive (see Benjamini, Kalai and Schramm (1999)):

Corollary 2.4. *A sequence (f_n) of Boolean functions that represent strong simple games is noise-sensitive if and only if for every $k > 0$*

$$\lim_{n \to \infty} \sum_{i=1}^{k} W_k(f_n) = 0. \tag{2.8}$$

As a matter of fact Corollary 2.4 holds as stated when the Boolean functions (f_n) do not necessarily represent strong simple games but rather satisfy the probabilities that $f_n = 1$ is bounded away from 0 and 1 as n tends to infinity.

Proof of Theorem 1.3: For this proof we need to compare the information given by Theorem 2.2 and Corollary 2.4. Suppose that $P_{\mathbf{cyc}}(f_n) \rightarrow 1/4$. Recall that since f_k represent a strong simple game $\widehat{f_k}(S) = 0$ if f is odd. It follows from relation (2.5) that for every $r > 0$, $\lim_{n\to\infty} W_r(f_n) = 0$, which by Corollary 2.4 is equivalent to noise sensitivity. On the other hand, if (f_n) satisfies relation (2.8) then it follows from relation (2.5) that $\lim_{n\to\infty} p_{\mathbf{cyc}}(f_n) = 1/4$. (This implication was also proved earlier.) $Q.E.D.$

We now complete the proof of Theorems 1.7 and 1.8. We have already proved the following implications: $(P_3) \Rightarrow (NS) \Rightarrow (P_m) \Rightarrow (P_3)$ and $(NS) \Rightarrow (B_m)$, for every $m \geq 3$. Of course, if property (P_m) holds then property (A_m) is equivalent to property (B_m) and therefore $(NS) \Rightarrow (B_m)$, $m \geq 3$. On the other hand, $(B_m) \Rightarrow (B_2) = (NS)$ and $(A_m) \Rightarrow (A_2) = (NS)$.

Remark: We now continue to demonstrate the notions and results introduced here with our simple example. As before, f_3 denote the payoff function of the simple majority game with three players. As we saw, the Fourier coefficients of f_3 are described by $\widehat{f_3}(\emptyset) = 1/2$, $\widehat{f_3}(\{j\}) = -1/4$, for $j = 1, 2, 3$, $\widehat{f_3}(S) = 0$ when $|S| = 2$ and $\widehat{f_3}(\{1, 2, 3\}) = 1/4$. When there are three alternatives the probability for cyclic social preferences given by formula (2.5) is $1/4 - 3 \cdot (1/4)^2 - (1/9) \cdot (1/4)^2 = 8/(9 \cdot 16) = 1/18$. This value agrees with the well-known outcome given by a direct computation. The value of $N_t(f_3)$ is $1/2 - 2 \cdot (1 - 2t)(3/16) - 2 \cdot (1 - 2t)^3 \cdot (1/16) = 3t/2 - 3/2t^2 + t^3$. (Again, this can be derived by a direct computation.)

2.3 Individual power

We conclude this section with a short discussion of the relation between noise sensitivity and the Banzhaf power index. Let $b_j(G)$ denote the Banzhaf power index of player j in a strong simple game G. Recall that $b_j(G)$ is defined as the probability that player j is pivotal, i.e., the probability that for a random coalition S not containing i it is the case that S is a losing coalition but $S \cup \{i\}$ is a winning one. Let $b_{\max}(G)$ denote the maximum value of the Banzhaf power indices and let $I(G)$ denote the sum of the Banzhaf power indices for all players in a simple game, known also as the total influence of the game. It is known that in the case of n players, $I(G)$ is maximized by a simple majority game. (See, e.g., Friedgut and Kalai (1996).) For simple majority games $I(G)$ is proportional to $\sqrt{n}$.

Definition 2.5. The sequence (G_k) of monotone simple games has a *diminishing individual Banzaf power* if

$$\lim_{k\to\infty} b_{\max}(G_k) = 0.$$

Proposition 2.6. *If the sequence (G_k) of monotone simple games leads to social chaos then it has a diminishing individual Banzhaf power.*

The proof is immediate: when there is a player with a substantial Banzhaf power the outcome of the game has a substantial correlation with this player vote. A small noise in counting the votes will miss the vote of this player with a substantial probability and therefore the correlation of the outcomes before and after the noise will also be substantial. (Here "substantial" means bounded away from zero.)

Definition 2.7. The sequence (G_k) of monotone simple games has a *bounded power ratio* if the ratio $b_{\max}(G_k)/b_{\min}(G_k)$ is uniformly bounded.

Given a monotone simple game G, let $p(G)$ be the probability that a random uniform subset of players is a winning coalition. If G is strong, then $p(G) = 1/2$. Given two monotone simple games G and H on the same set N of players let $p(G, H)$ be the probability that a random uniform subset S of players is a winning coalition for *both* G and H. The correlation between G and H, denoted by $cor(G, H)$, is defined by

$$cor(G, H) = p(G, H) - p(G)p(H).$$

The well-known FKG inequality asserts that for every two monotone simple games $cor(G, H) \geq 0$.

Definition 2.8. Two sequences (G_k) and (H_k) of monotone strong simple games are *asymptotically uncorrelated* if

$$\lim_{n\to\infty} cor(G_k, H_k) = 0.$$

Theorem 2.9. *A sequence (G_k) of monotone strong simple games leads to social chaos if and only if it is asymptotically uncorrelated with every sequence of weighted majority games.*

An important special case is:

Theorem 2.10. *A sequence (G_k) with a bounded Banzhaf power ratio leads to social chaos if and only if it is asymptotically uncorrelated with the sequence of simple majority games.*

Theorem 2.10 together with our earlier results demonstrates the "robustness of majority":[5] Voting methods (based on proper simple games where the voters have comparable power and the probability of a tie tends to zero) either have a substantial correlation with simple majority or else lead to social chaos.

Theorem 2.11. *Let (G_n) be a sequence of monotone strong simple games with a bounded power ratio and suppose that G_n has n players. The sequence G_n leads to social chaos if and only if* $\lim_{n\to\infty} I(G_n)/\sqrt{n} = 0$.

This result can be derived directly from the following result from Benjamini, Kalai and Schramm (1999):

Theorem 2.12. *The sequence (G_k) is noise-sensitive if and only if*

$$\lim_{k\to\infty} \sum (b_k(G_k))^2 = 0.$$

Remark: Holzman, Lehrer and Linial (1988) proved that for every monotone simple game $\sum (b_k(G_k))^2 \leq 1$.

[5] A term coined by Dasgupta and Maskin in another context.

3 Multi-level majority

3.1 Simple majority

Our principal examples of social welfare functions that exhibit social chaos are based on multi-level majority. For their analysis we will have to start with the case of simple majority. The outcomes of simple majority under random individual voter profiles have been the subject of intense study. We will first require a few results concerning the behavior of simple majority. Given a monotone strong simple game G recall that $p_{\mathbf{cyc}}(G)$ denotes the probability that for a social welfare function on three alternatives A, B and C and a random uniform voter profile the social preferences will be cyclic.

We start with a simple result concerning the majority voting rule with an odd number of voters.

Lemma 3.1. *Let M_n be a simple majority game on an odd number n of players, $n \geq 3$. Then,*

(i) $N_\epsilon(M_n) \geq N_\epsilon(M_3) = 3/2 \cdot \epsilon - 3/2\epsilon^2 + \epsilon^3$;

(ii) $p_{\mathbf{cyc}}(M_n) \geq p_{\mathbf{cyc}}(M_3) = 1/18$.

The proof is given in Section 9.2 in the Appendix. We point out that Fishburn, Gehrlein and Maskin proved that $p_{\mathbf{cyc}}(M_{2k+1}) \geq p_{\mathbf{cyc}}(M_{2k-1})$ for every $k \geq 3$. This implies, in particular, part (ii) of Lemma 3.1. The next result describes the limiting behavior of majority when the number of voters tends to infinity.

Proposition 3.2. *(1) (Sheppard (1899)) For every $\epsilon > 0$*

$$\lim_{n\to\infty} N_\epsilon(M_n) = \frac{\arccos(1-2\epsilon)}{\pi}. \tag{3.1}$$

(2) Gulibaud (see Gehrlein (1997))

$$\lim_{n\to\infty} p_{\mathbf{cyc}}(M_n) = 1/4 - (3/(2\cdot\pi)) \cdot \arcsin(1/3) \approx .08744.$$

Proposition 1.4 gives a direct link between Gulibaud's formula and Sheppard's formula. Relation (2.7) asserts that for every n, $p_{\mathbf{cyc}}(M_n) = 3/2N_{1/3}(M_n) - 1/2$. Together with the case $\epsilon = 1/3$ of Sheppard's formula $\lim_{n\to\infty} N_{1/3}(M_n) = \frac{\arccos(1/3)}{\pi}$ we obtain the formula of Gulibaud.

It follows from relation (3.1) that as ϵ tends to zero $\lim_{n\to\infty} N_\epsilon(M_n) = (1+o(1)) \cdot \sqrt{2\epsilon}/\pi$. This is a more precise version of Theorem 1.6. We will not reproduce here the proofs of Proposition 3.2 but rather will explain informally why, for simple majority on a large number n of players and for a small amount ϵ of noise, the function $N_\epsilon(M_n)$ behaves like $\sqrt{\epsilon}$. The median gap between the number of votes of the winning candidate and the losing one in a simple majority is close to $\sqrt{n}$. According to the central limit theorem the probability that the gap is larger than $t\sqrt{n}$ behaves like $e^{-t^2/4}$. It follows that a large gap is extremely rare while below the median gap the distribution of the gap is rather close to uniform. When we flip ϵn random votes the median gap between the number of votes for Alice that now go to Bob and votes for Bob that now go to Alice is thus close to $\sqrt{\epsilon}\sqrt{n}$, and again it is rare that this gap is much larger. In order for the noise to change the election's results we require (apart from an event with a small probability) that the original gap between the votes for Alice and Bob be around $\sqrt{\epsilon}\sqrt{n}$ and this occurs with probability of roughly $\sqrt{\epsilon}$. On the other hand, if the gap between votes for Alice and votes for Bob behaves like $\sqrt{\epsilon}\sqrt{n}$ then the probability that the noise will change the election's outcome is bounded away from zero. This argument indeed shows that $N_\epsilon(M_n)$ behaves like $\sqrt{\epsilon}$.

3.2 Multi-level games based on simple majority

The first class of examples that exhibit social chaos are based on multi-level majority where the number of levels increases. Recall the class of multi-level majority-based voting rules described inductively in the Introduction: (i) Simple majority on an odd number of at least three have one layer. (ii) Suppose that the players are divided into an odd number of three or more groups and on each group we consider a game with r layers. If G is defined by the simple majority of the outcomes then G is said to be a multi-layer majority-based simple game with $r+1$ layers.

Define $u^{(1)}(\epsilon) = u(\epsilon) = 3/2 \cdot \epsilon - 2\epsilon^2 + 5/4\epsilon^3$ and $u^{(r)}(\epsilon) = u(u^{(r-1)}(\epsilon)$, $r = 2, 3, 4, \ldots$.

Theorem 3.3. *Let G_r be an r-layer simple game based on simple majority. Then*

$$N_\epsilon(G_r) \geq u^{(r)}(\epsilon).$$

As ϵ tends to 0,

$$N_\epsilon(G_r) \geq (3/2)^r \epsilon + O(\epsilon^2).$$

Proof: The proof is by induction on r and the case $r = 1$ is given by Lemma 3.1. Consider a random vector $(x_1, x_2, \ldots, x_n) \in \Omega(n)$. The value of $v(x_1, x_2, \ldots, x_n)$ is the simple majority of the $(r-1)$ level outcomes $z_1, z_2, \ldots, z_m$. Suppose that we flip each variable x_i with probability ϵ. By the induction hypothesis each z_i is flipped with probability of at least $u^{(r-1)}(\epsilon)$ and by Lemma 3.1 the probability that v is flipped is at least $u(u^{(r-1)}(\epsilon) = u^{(r)}(\epsilon)$. When $G = T_r$ is the recursive ternary majority game with r levels we obtain that $N_\epsilon(T_r) = u^{(r)}(\epsilon)$. To prove Theorem 3.3 we need to verify that for every $\epsilon > 0$, $\lim_{r\to\infty} u^{(r)}(\epsilon) = 1/2$.

We will state separately the result for a bounded number of levels.

Theorem 3.4. *Let $\phi(\epsilon) = \phi^{(1)}(\epsilon) = \frac{\arccos(1-\epsilon)}{\pi}$ and let $\phi^{(i+1)}(\epsilon) = \phi(\phi^{(i)}(\epsilon)$, $i = 1, 2, \ldots$. Let r be a fixed positive integer. Consider a sequence (G_n) of r-level majority strong simple games where each level is based on simple majority of t_n players. (Thus, $n = t_n^r$.) Then for a fixed $\epsilon > 0$ as n tends to infinity,*

$$N_\epsilon(r) =: \lim_{n\to\infty} N_\epsilon(G_n) = \phi^{(r)}(\epsilon).$$

As ϵ tends to zero

$$N_\epsilon(r) \geq (1 + o(1))(2/\pi)\epsilon^{1/2^r}.$$

In our model, simple majority is considerably more stable in the presence of noise than a two-layer election method like the U.S. electoral system. In an electoral system with many states and many voters in each state, $N_\epsilon(G)$ is proportional, for small values of ϵ, to $\epsilon^{1/4}$. In a three-stage system of this kind $N_\epsilon(G)$ behaves like $\epsilon^{1/8}$. Is the advantage of simple majority on a two-level majority in terms of noise sensitivity relevant for evaluating the U.S. electoral method? Can we conclude that events like those in the 2000 U.S. presidential election are less likely to occur in a simple majority system? While the probabilistic models are unrealistic, it appears that the advantage of simple majority compared to a two-level electoral method is quite general.

Remark: The hierarchical voting methods considered here resemble the multi-tier system of councils ("soviets" in Russian). Lenin (among others) advocated this system during the 1917 Russian Revolution. Lenin's concept of centralized democracy is based on a hierarchical method of voting and was implemented in the Soviet Union and its satellites for party institutions, national bodies, and labor unions. (For national bodies, the method was changed in 1936.) For party institutions (which were the most important) there could be as many as seven levels. Party members in a local organization, for example, the Department of Mathematics in Budapest, elected representatives to the Science Faculty party committee who in turn elected representatives to the University council. The next levels were the council of the 5th District of Budapest, the Budapest council, the Party Congress, the Central Committee and finally the Politburo. Friedgut (1979) is a good source on the early writings of Marx, Lenin, and others, and for an analysis of the Soviet election systems that were prevalent in the 70's.

3.3 Two-level games based on supermajority

The second class of examples of social welfare functions that exhibit social chaos are based on a two-level method where the lower level is biased. Consider a two-level proper game $G = G[a, b, t]$, where t satisfies $b/2 < t < b$, defined as follows. There are $a \cdot b$ voters divided into a communities of b voters each. Given a subset S of voters, we call a community C positive if $|S \cap C| \geq t$ and negative if $|S \cap C| \leq b - t$. The subset S is a winning coalition if there are more positive communities than negative communities. Let $G_n = G[a_n, b_n, t_n]$ be a sequence of such games. In order for G_n to have strongly diminishing bias it is necessary and sufficient that the expected number of decisive coalitions tends to infinity.

Theorem 3.5. *If $G_n = G[a_n, b_n, t_n]$ has strongly diminishing bias and*

$$\lim_{n\to\infty} b_n = \infty,$$

then the sequence G_n leads to social chaos.

The proof is given in Section 9.2 in the Appendix.

Remark: It can be shown by similar proofs to those we present that social chaos occurs for more general hierarchical voting methods. This applies both in the case where in each level the method is balanced and the number of levels tends to infinity and in the case where there are as little as two levels and the method is biased in the lower level. (In the Appendix we will formulate and prove a more general theorem that gives Theorem 3.5 as a special case.) Social chaos may occur also for more complicated recursive methods. For example the different communities that we considered need not be disjoint. But the analysis in such cases can be considerably harder.

4 Bias

The framework described in this paper extends to more general probabilistic distributions of voter behavior (or individual signals). We expect that the dichotomy described here between stochastically stable and chaotic behavior can also be extended, though such extensions are conceptually and mathematically more difficult. When we consider the effect of stochastic perturbations there are also various possible models for the "noise." Does it represent mistakes in counting the votes? or, perhaps, a small fraction of voters reconsidering their positions or considering whether or not to vote?

We also expect that the stochastic stability of the majority rule extends to various other probability distributions representing the voters' behavior.

Our probabilistic model assumes three properties which we will discuss separately:

(1) Independence: Voters' probabilities (or signals) are independent.

(2) No bias: For two alternatives a and b the probability that the society prefers alternative a to alternative b is $1/2$.

(3) Identical voters: Voters' probabilities (or signals) are identical.

The assumption of identical voter preference distribution is unrealistic but I expect that the basic results of this paper can be extended to cases where the assumption of identical voter behavior is relaxed while the other two assumptions are kept. We will concentrate on the other two assumptions and discuss in this section bias and in the next section dependence between voters.

The assumption of no bias is clearly an ideal assumption. A more realistic assumption in many cases of aggregated choice is the assumption of "small bias," which is defined as follows:

Definition 4.1. Consider a sequence (G_k, ν_k), where G_k is a proper simple game with n_k players and ν_k is a probability distribution on coalitions of G_k. (Alternatively, ν_k can be regarded as a probability distribution on 0-1 vectors of length n_k.) The sequence (G_k, ν_k) has *small bias* if for some constant $t > 0$, the probability $p_{\nu_k}(G_k)$ that a random coalition according to ν_k is a winning coalition satisfies

$$t \leq p_{\nu_k}(G_k) \leq 1 - t, \quad \text{for} \quad \text{every} \quad \text{k} \geq 1.$$

While it is hard to justify that a priori two candidates in an election have the same probability of winning it is often quite realistic that the a priori probability of winning is substantial for both. We will return to this issue towards the end of this section.

We will describe in this section the extension of our results to cases with bias. Bias occurs when the simple game is strong and thus neutral between the candidates but the probability distribution of signals favors one of the candidates, and also when the probability distribution is uniform but the voting method itself is described by a simple game giving advantage to one of the candidates. The results in these two types of extensions (and their combination) are similar. The dichotomy between noise-stable and noise-sensitive monotone simple games and many of the results presented in this paper can be extended to the case of independent voter behavior with a small bias.

Recall that Ω_n denote the set of $0-1$ vectors $(x_1, x_2, \dots, x_n)$ of length n. Given a real number $p > 0$, $0 < p < 1$, $\mathbf{P}_p$ will denote that product distribution on Ω_n defined by

$$\mathbf{P}_p(x_1, x_2, \dots, x_n) = p^k(1-p)^{n-k},$$

where $k = x_1 + x_2 + \cdots + x_n$. For a monotone simple game G with payoff function $v = v(x_1, x_2, \dots, x_n)$, define

$$\mathbf{P}_p(G) = \sum\{\mathbf{P}_p(x)v(x) : x \in \Omega(n)\}.$$

Let $t > 0$ and consider the following scenario: first choose the voters' signals $x_1, x_2, \dots, x_n$ at random such that $x_i = 1$ with probability p independently for $i = 1, 2, \dots, n$. Consider $S = v(x_1, x_2, \dots, x_n)$. Next let $y_i = x_i$ with probability $1-t$ and $y_i = 1 - x_i$ with probability t, independently for $i = 1, 2, \dots, n$. Let $T = v(y_1, y_2, \dots, y_n)$. Let $C_t^p(G)$ be the correlation between S and R.

Definition 4.2. Consider a sequence $(G_k, p_k)_{k=1,2,\dots}$ where G_k is a monotone simple game and $0 < p_k < 1$. The sequence (G_k, p_k) is *asymptotically noise-sensitive*, if for every $t > 0$,

$$\lim_{k\to\infty} C_t^{p_k}(G_k) = 0.$$

In the case of small bias, namely, when $\mathbf{P}_{p_k}(G_k)$ is bounded away from 0 and 1, Theorem 1.7, which asserts that simple majority is noise-stable, extends and so do Theorems 2.9 and 2.12.

If $\lim_{k\to\infty} \mathbf{P}_{p_k}(G_k) = 1$ (or 0) the sequence is *always* noise-sensitive.

Theorem 4.3. *Every sequence* (G_k, p_k) *of monotone simple games where* $\lim_{n\to\infty} \mathbf{P}_{p_k}(G_n) = 1$ *(or 0) is noise-sensitive.*

Theorem 4.3 follows from a Fourier-theoretic interpretation of noise sensitivity, which extends the formula for the uniform distribution and a certain inequality of Bonamie and Beckner.

Perhaps some explanation of the meaning of noise sensitivity in the presence of bias is in order. When there are two alternatives noise sensitivity can be described in two ways:

- (i) The result following a small perturbation is asymptotically uncorrelated with the original result.
- (ii) The probability that a small perturbation will change the collective choice tends to 1/2.

The reason that when there is no bias (i) implies (ii) is that the model assigns equal probabilities to the two outcomes. Property (i) may extend to rather general probability distributions but the dramatic conclusion in (ii) requires that there be no bias.

Assuming small bias, condition (i) implies the following:

- (iii) With a probability uniformly bounded away from zero, a small amount of noise will change the outcome of the election.

Note that for a sequence of games (G_k) with small bias in order to conclude property (iii) it is enough to assume a weaker property than noise sensitivity. Property (iii) follows if the sequence (G_k), in addition to having small bias, is *uniformly chaotic*, which is defined as follows:

Definition 4.4. Consider a sequence $(G_k, p_k)_{k=1,2,\dots}$ where G_k is a monotone proper simple game and $0 < p_k < 1$. The sequence (G_k, p_k) is *uniformly chaotic*, if there is $\alpha > 0$ so that for every $t > 0$,

$$\lim_{k\to\infty} C_t^{p_k}(G_k) < 1 - \alpha.$$

It is interesting to compare what happens to conditions (i) and (iii) when the bias get larger (maintaining the assumption of independent and identical voter preferences). In an election between Alice and Bob, when the probability of Alice winning tends to zero then so does the probability that a small random perturbation in voter behavior will change the outcome of the election. On the other hand, Theorem 4.3 asserts that in this case it is *always* true, even for the majority rule, that the correlation between outcomes before and after the noise also tends to zero. Roughly speaking, Theorem 4.3 asserts that when your probability of winning the election is small and you have the option of forcing a recount of a random fixed fraction of votes, this option almost doubles your chances of winning (for the majority rule and for any other voting rule).

Our next proposition, whose proof follows directly from the definition of uniform noise stability, asserts that the property of noise stability is itself stable under small changes in the probability distribution.

Let $n(k)$ be a monotone sequence of positive numbers. Consider two sequences (ν_k) and (ν'_k) where ν_k and ν'_k are probability distributions on $\Omega_{k(n)}$. We say that ν and ν' are asymptotically non-singular if for every sequence of events $S_k \in \Omega_{k(n)}$, $\lim \nu(S_k) = 0$ if and only if $\lim \nu'(S_k) = 0$.

Proposition 4.5. *Consider a family $\mathcal{G}$ of strong simple games which is uniformly noise-stable. Suppose that p_n is a sequence of probabilities such that the distributions $\mathbf{P}_{p_n}$ on Ω_n are asymptotically non-singular to the uniform distribution. Then $\mathcal{G}$ is uniformly noise-stable w.r.t. the distributions $\mathbf{P}_{p_n}$.*

The stability of uniform noise stability extends to small perturbations of the voters' profile distribution, and the distribution describing the noise does not require the assumption of independent voters.

We will now briefly deal with three or more alternatives. Let η be a probability distribution on order relations on a set A of alternatives. Given a set of n voters we will denote by $\mathbf{P}_\eta$ the probability distribution on the voters' profile where the voters' preferences are independent and are described by η. We will assume that for every two alternatives a and b the probability that

aRb when R is chosen at random according to η is strictly between 0 and 1. We will study the distribution of social preferences when the voters' profiles are distributed according to $\mathbf{P}_\eta$. Consider the probability distribution p_η defined on asymmetric relations as follows:

$$p_\eta(R) = \prod_{aRb} \eta(a,b) / \sum_R \prod_{aRb} \eta(a,b). \tag{4.1}$$

This distribution $p_\eta(R)$ represents the following scenario. For every individual the order relations among the m alternatives are distributed according to η and are independent. The distribution on the social preference is based on the following procedure. For every two alternatives a and b choose an individual $v_{a,b}$ and let the the social preference between a and b agree with the preference of this individual. Assume that all these individuals are different. The resulting distribution on the social preferences is p_η.

We will now extend the definition of social chaos to strong monotone simple games and arbitrary distributions η on the individual order relations.

Definition 4.6. A sequence $(G_k)_{k=1,2,\ldots}$ of strong simple games leads to *social chaos* w.r.t. the probability distribution η on voters' order preferences if for every three alternatives a, b, and c the probability that the society prefers alternative a to alternative c given that it prefers a to b and b to c is asymptotically the same as the a priori probability that the society prefers a to c.

Given these definitions, Theorems 1.3 and 1.8, part $(A)_k$ (and their proofs) extend. In Theorems 1.7 and Theorem 1.8, part $(B)_k$, one has to replace the uniform distribution on all asymmetric relations by the distribution p_{η_k}.

We conclude this section by a brief discussion of the assumption of small bias. The assumption of small bias is realistic and this by itself can be regarded as a surprising phenomenon. Why is it the case that so often shortly before an election we can give substantial probability for each one of two candidates to be elected? How come the probabilities that we can assign to the choices of each voter do not "sum up" to a decisive collective outcome? This seems especially surprising in view of the property of aggregation of information. It is worth noting that there are several factors that can push the situation of collective choice towards small bias:

(i) Abstention and other strategic considerations of voters: If a voter's incentive to vote is based (as is often assumed) on his chances of being pivotal, then for voters acting strategically, abstention or applying mixed strategies and not acting just based on the signals may drive the situation towards criticality. See, e.g., Feddersen and Pesendorfer (1996, 1998) where supermajority is studied, and Samet (2004) for general simple games.[6]

If there are two main candidates and a few additional candidates who are each ideologically close to one of these two, a voter may choose to vote for a minor candidate if he knows that his vote is not needed to decide between the main candidates. Again, this can push the situation between the two main candidates towards criticality.

(ii) Strategic considerations of candidates: In elections, such as those in the U.S., the candidates concentrate their efforts on swing states. This may lead to critical behavior in some of these states.

(iii) Choice out of equilibrium: Consider the difference between the following two scenarios: a) choosing between two types of products, b) choosing between two types of products when the difference in quality is reflected in their prices. In the second scenario it is more likely that we will observe critical behavior and even more so when the decision-maker's information about the difference in quality is based primarily on the difference in price.

(iv) Dependence: A major reason why the assumption of small bias is realistic is the lack of independence between voter preferences. Even a short time before an election there is a nonnegligible probability that an event will

[6] In these cases strategic behavior leads to asymptotic complete aggregation of information in the sense that if there are independent individual signals weakly biased toward the superior alternative then with probability tending to one the superior alternative will be chosen.

influence a large number of voters in the direction of one candidate.[7]

5 Dependence

Following is an example that demonstrates some of the issues that arise when the voting rule is simple majority and voter behavior is not independent. Suppose that the society is divided into a communities of b voters each. The number of voters is thus $n = ab$, which we assume is an odd number. Each voter i receives an independent signal s_i, where $s_i = 1$ with probability 1/2 and $s_i = 0$ with probability 1/2. The voters are aware of the signals of the other voters in their community and are influenced by them. Let $q > 0$ be a small real number. A voter changes his mind if he observes a decisive advantage for the other candidate in his community, i.e., if he observes an advantage where the probability of observing such an advantage or a larger one, when voter behavior is independent and uniform, is at most q.

The election's outcome as a function of the original signals $s_1, s_2, \ldots, s_n$ can be described by a strong simple game which we denote by $G[a; b; q]$. (The example will also "work" if a fraction $(1-\alpha)$ of the voters vote according to their signals and only some small fraction $\alpha > 0$ of voters change their minds as before.[8]

Let us examine the situation for a sequence (G_n) of such examples where the parameters a and b both tend to infinity, $n = ab$, q tends to zero, and $\sqrt{1/q} = o(\sqrt{m})$. (For example, take $a = b = \sqrt{n}$, and $q = n^{-1/4}$.) In this case, G_n exhibit noise sensitivity for (independent) small amounts of noise in the original *signals*. The outcomes of elections as a function of the individual signals is thus completely chaotic. Indeed, the outcomes of

[7] This type of dependence can imply failure of aggregation of information in the following sense: it is possible that every voter's signal will favor a certain candidate with probability (say) 0.6 and yet there is a substantial probability that the other candidate will be elected regardless of the number of voters.

[8] Another variant that will have similar properties is the case in which there are election polls in each community that influence some small fraction of voters.

the elections are determined (with very high probability) by the number of communities with decisive signals in favor of one of the candidates. The expected number of decisive communities is qa. The difference between the number of decisive communities favoring the two candidates behaves like $\sqrt{qa}$. Therefore, these communities contribute $b\sqrt{qa}$ votes to one of the candidates. Since the original gap in the number of favorable signals between the two candidates behaves like $\sqrt{ab}$ and by our assumption $\sqrt{am} = o(m\sqrt{qa})$ we conclude that the decisive communities determine the outcome of the elections with very high probability. From this point on, the analysis is similar to the proof of Theorem 3.5. [9]

On the other hand, this same sequence is extremely noise-stable for independent noise with respect to counting the votes! The gap between votes cast for the two candidates behaves like $b\sqrt{qa}$ so that even if a random subset of 40% of the votes are miscounted the probability that the election's outcome will be reversed is extremely small.

The two properties of our example: The first property of chaotic behavior for noise affecting the original signal and the second property of strong stochastic stability for noise affecting individual votes seem characteristic to situations in which voter behavior depends on independent signals in a way that creates positive correlation between voters. This topic deserves further study. Note that when we consider random independent noise in the original signals, the distribution of the resulting votes is identical to the original distribution without the noise. This is not the case for random independent noise in counting the votes.

We now move from examples to more general models. A general model for voter behavior will be of a pair $< G, \nu >$ where G is a strong simple game representing a voting rule and ν is a probability distribution of voters'

[9] If $q = t/\sqrt{a}$, where t is small but bounded away from zero, then for every amount ϵ of noise in the signals, the correlation between the outcomes before and after the noise will tend to a small constant; in this case the situation depends on the original signals in a very, but not completely, chaotic manner.

preferences or signals. In this case we can consider random independent noise that can represent mistakes in counting the votes. For such a form of noise the distribution of the noisy signals will be different than the original distribution. Another form of noise can be described as follows. Given a vector $(x_1, \cdots, x_n)$ of voters' signals and another vector $y = (y_1, \dots, y_n)$ define

$$q(y) = (1-t)^{n-d(x,y)} t^{d(x,y)} \cdot \nu(y),$$

and normalize $q(y)$ to a probability distribution by setting

$$p(y) = q(y) / \sum \{q(z) : z \in \{0,1\}^n\}.$$

If the noise changes x to y with probability $p(y)$ the distribution of the noisy voters' signals is the same as the original distribution. This type of noise can be regarded as applying to the (unknown) mechanism leading to the voters' preferences distribution ν.

A somewhat more restrictive but still quite general probabilistic setting for aggregation is the following: there are some independent signals $s_1, s_2, \dots, s_t$ that may depend on the state of the world. At the stage where individuals make up their mind they are exposed to some of these signals directly and also to the emerging preferences of other individuals. The individual's final preferences are then aggregated according to some voting rule. When we study the sensitivity to noise it can make a big difference if we consider noise affecting the original signals or noise affecting the individual choices (mistakes in counting the votes).

In such a general framework there are cases, such as the example presented at the beginning of this section, when rather general simple games describe the situation even for a standard voting rule. For simplicity we can restrict ourselves to the case of simple majority rule. Suppose that the choice of the i-th individual $v_i = v_i(s_1, \dots, s_t)$ is a function of the signals $s_1, \dots, s_t$. If the v_i's are monotone and satisfy $v_i(1-s_1, 1-s_2, \dots, 1-s_t) = 1 - v_i(1-s_1, 1-s_2, \dots, 1-s_t)$ then the outcome of the elections in terms of $s_1, s_2, \dots, s_t$ is expressed by a proper simple game G (see also Kalai (2004), Section 3). If the signals are random and uniformly distributed on Ω_t then the situation can be considered within our frameworks. As we have shown, it is possible to provide natural examples in which the elections outcome as a function of the original signals will exhibit chaotic behavior. In addition to examples that are hierarchical, like those we have considered, one can also consider cases in which voters repeatedly update their positions based on the positions of friends and neighbors.

6 Abstention

In real elections it is common for a large proportion of voters not to vote. Abstention is thus the most common form of noise when it comes to real-life elections and perhaps also other forms of aggregation. It is possible to extend our model and consider monotone and neutral social welfare functions that allow individual indifference between alternatives. When there are two alternatives a and b, a social welfare function can be described by a function $v(x_1, x_2, \ldots, x_n)$ where each $x_i \in \{-1, 0, 1\}$ and also $v(x_1, x_2, \ldots, x_n) \in \{-1, 0, 1\}$. Here $x_i = 1, -1, 0$ according to whether the i-th individual prefers a or b or is indifferent between the two, or prefers b to a, respectively. For two alternatives neutrality is equivalent to $v(-x_1, -x_2, \ldots, -x_n) = -v(x_1, x_2, \ldots, x_n)$. When we consider only ± 1 variables, the function v describes a proper simple game G.[10] We assume that v is monotone, namely, if $x_i \geq y_i, i = 1, 2, \ldots, n$ then $v(x_1, x_2, \ldots, x_n) \geq v(y_1, y_2, \ldots, y_n)$. The function v determines a neutral social welfare function on every set A of m alternatives.

We can study the stochastic behavior of social preferences as we did before. Let (v_n) be a sequence of functions of the types we considered above and let (G_k) be the proper simple game associated to (v_k). We need the assumption that for every $\alpha > 0$ if a random fraction of α among voters are indifferent between the two alternatives (in other words, if they abstain)

[10] Note that the payoff function v uses the values $-1, +1$ instead of $0, 1$ to describe this game.

and the voters that do not abstain vote randomly and uniformly, then the probability of social indifference tends to zero as n tends to infinity. This implies, in particular, that the sequence (G_k) has strongly diminishing bias.

Suppose that all voters vote; then we are back in the case we studied in this paper (with the assumption of strongly diminishing bias) and the notion of noise sensitivity and social chaos extend unchanged.

Our first theorem asserts that if (G_k) is a noise-sensitive sequence of social welfare functions then the effect of abstention of even a small fraction of voters is dramatic. (Note that the game G_k does not determine the function v_k but the result applies simultaneously for all neutral extensions of the games G_k to cases of individual indifference.)

Consider the following scenario. Let R be the outcome of an election between two candidates and random voter profile $(x_1, x_2, \ldots, x_n)$ without abstention. Now consider another random voter profile $(y_1, y_2, \ldots, y_n)$ where $y_i = x_i$ with probability $1 - s$ and $y_i = 0$ with probability s. Let R' be the outcome of the election for the voter profile $(y_1, y_2, \ldots, y_n)$.

Theorem 6.1. *Consider a sequence (v_k) of functions describing a voting rule with individual indifference. Suppose that the social indifference is strongly diminishing and that the sequence (G_k) of associated proper games is noise-sensitive. Then the correlation between R and R' tends to zero as k tends to infinity.*

Our next result asserts that the property of noise sensitivity is preserved with probability 1 under abstention.

Theorem 6.2. *Consider a sequence (v_k) of functions describing a voting rule with individual indifference. Suppose that the social indifference is strongly diminishing and that the sequence (G_k) of associated proper games is noise-sensitive. Let δ be a fixed real number, $0 < \delta < 1$. Suppose that (G'_k) is a strong simple game obtained from G_k when every voter abstains with probability δ. Then with probability 1 the sequence G'_k is noise-sensitive.*

We remark that a similar result applies when "noise-sensitive" is replaced by "uniformly noise-stable."

The next theorem, which is a simple corollary of the two theorems above, asserts that if (G_k) is a noise-sensitive sequence of social welfare functions then the effect of abstention of even a small additional fraction of voters is dramatic. (Again, the result applies simultaneously for all neutral extensions of the games G_k to cases of individual indifference.)

Now, consider the following scenario. Let R be the outcome of an election between two candidates and random voter profile $(x_1, x_2, \ldots, x_n)$ conditional on the assumption that the probability of a voter abstaining is $\alpha \geq 0$. Let $(y_1, y_2, \ldots, y_n)$ be another random voter profile where $y_i = x_i$ with probability $1 - s$ and $y_i = 0$ with probability s. Let R' be the outcome of the election for the voter profile $(y_1, y_2, \ldots, y_n)$.

Theorem 6.3. *Consider a sequence (v_k) of functions describing a voting rule with individual indifference. Suppose that the social indifference is strongly diminishing and that the sequence (G_k) of associated proper games is noise-sensitive. Then the correlation between R and R' tends to zero as k tends to infinity.*

The proofs are described in the Appendix. Similar results apply to the case of more than two alternatives.

7 Discussion

1. The superiority of majority. What are the simple games most stable under noise? It was conjectured by several authors that under several conditions which exclude individual voters having a large power, majority is (asymptotically) most stable to noise. (See a discussion and a surprising application to computer science in Khot, Kindler, Mossel and Ryan (2004)). This conjecture was recently proved by Mossel, O'Donnell and Oleszkiewicz (2005).

Definition 7.1. The sequence (G_k) of monotone simple games has *diminishing Banzhaf individual power* if

$$\lim_{n\to\infty} b_{\max}(G_k) = 0.$$

Theorem 7.2 (Mossel, O'Donnell and Oleszkiewicz (2005)). *for a sequence (G_n) of strong simple games with diminishing individual Banzhaf power,*

$$N_s(G_n) \geq (1 - o(1)) \cdot (\frac{\arccos(1-2s)}{\pi}).$$

In other words, when the individual Banzhaf power indices diminish simple majority is asymptotically most stable to noise.[11] A consequence of this theorem based on theorem 2.2 is

Corollary 7.3. *For every sequence (G_n) of strong simple games with diminishing individual Banzhaf power,*

$$p_{\mathbf{cyc}}(G_n) \leq 1/4 - (3/(2\cdot\pi)) \cdot \arcsin(1/3) + o(1).$$

In other words, for social welfare functions with diminishing Banzhaf value, simple majority is asymptotically least likely to yield a cyclic social preference relation on three alternatives.

2. Aggregation of information. Given a strong monotone simple game G on a set V of players we denote by $\mathbf{P}_p(G)$ the probability that a random set S is a winning coalition when for every player $v \in V$ the probability that $v \in S$ is p, independently for all players. Condorcet's Jury theorem asserts that for the sequence G_n of majority games on n players

$$\lim_{n\to\infty} \mathbf{P}_p(G_n) = 1, \quad \text{for every } p > 1/2. \tag{7.1}$$

This result, a direct consequence of the law of large numbers, is referred to as asymptotically complete aggregation of information. In Kalai (2004)

[11] It is crucial to fix s and let n tend to infinity. If s tends to zero as n tends to infinity the situation can be very different. For example, when $s = 1/n$ the majority function is most sensitive to noise.

it was proved that for a sequence of monotone simple games asymptotically complete aggregation of information is equivalent to diminishing Shapley-Shubik individual power.

The speed of aggregation of information for simple majority games is described by the central limit theorem. For a monotone strong simple game G, let $T_\epsilon(G) = [p_1, p_2]$ and $t_\epsilon(G) = p_2 - p_1$, where $\mathbf{P}_{p_1}(G) = \epsilon$ and $\mathbf{P}_{p_2}(G) = 1 - \epsilon$. When G_n is simple majority on n players the central limit theorem implies that $t_\epsilon(G_n) = \theta(\sqrt{n})$. It is known that for n-player games (n being an odd integer) and every ϵ, $t_\epsilon(G)$ is minimized by the simple majority game[12].

The next theorem asserts that uniform noise stability (in a slightly stronger sense than before) implies that up to a multiplicative constant the aggregation of information is optimal.

Theorem 7.4. *Let (G_k) be a sequence of strong simple games such that G_k has n_k players and let $\epsilon > 0$ be a fixed real number. Suppose that the class of games and distributions $(G_k, \mathbf{P}_p)$, $p \in T_\epsilon(G_k)$ is uniformly noise-stable. Then $t_\epsilon(G_n) = \theta(1/\sqrt{n_k})$.*

3. Indeterminacy. Consider a sequence (G_n) of strong simple games. Let $m \geq 3$ be a fixed integer, let A be a set of m alternatives, and denote by F_n the neutral social welfare function based on G_n when the set of alternatives is A. Let R be an arbitrary asymmetric relation on A. It was proved in Kalai (2004) that if (G_n) has diminishing individual Shapley-Shubik power then, for n sufficiently large, R is in the range of F_n (in other words, $p_R(G_n) > 0$) and this property is referred to there as *complete indeterminacy*. Note that social chaos (in view of Theorem 1.7) can be regarded as the ultimate form of social indeterminacy as it implies that all the probabilities $p_R(G_n)$ are asymptotically the same.

The main tool for showing that diminishing individual Shapley-Shubik power implies social indeterminacy is the equivalence between diminishing

[12] The fact that the derivative $d\mathbf{P}_p(G)/dp$ is maximized for simple majority can be found, e.g., in Friedgut and Kalai (1996).

individual Shapley-Shubik power and asymptotically complete aggregation of information. It also follows from the results of Kalai (2004) that diminishing individual Shapley-Shubik power implies diminishing individual Banzhaf power (but not vice versa). When we assume diminishing individual Banzhaf power we can expect the following strong form of complete indeterminacy.

Definition 7.5. A sequence (G_n) of strong simple games leads to *stochastically complete social indeterminacy* if for every asymmetric relation R on a set A of m alternatives

$$\liminf_{n\to\infty} p_R(G_n) > 0.$$

Theorem 7.6. *(1) A sequence (G_n) of strong simple games with diminishing Banzhaf individual power leads to stochastically complete social indeterminacy.*

(2) A sequence (G_n) of strong simple games that is uniformly chaotic leads to stochastically complete social indeterminacy.

The proof of Theorem 7.6 uses the argument used in proving Theorem 1.7 and for part (1) we require also a recent theorem of Mossel, O'Donnell and Oleszkiewicz (2005), referred to as "it ain't over until it's over". This theorem roughly asserts that for any random voter profile when a large random set of votes is counted there is still (almost surely as n tends to infinity) a probability bounded away from zero (however tiny) that either candidate will win. We defer further details to the Appendix.

4. Noise stability. This paper is mainly devoted to the chaotic behavior of noise-sensitive classes of simple games. There are interesting issues concerning the behavior of social preferences for social welfare functions based on uniformly noise-stable classes of simple games, and random uniform voter profiles. I will mention one topic worth further study: suppose that we are interested in the social preference relation between two alternatives A and B but we can cannot compare these two alternatives directly. Rather, we can compare both A and B with other alternatives $C_1, C_2, \ldots C_r$. Suppose that for all those alternatives we discover that the society prefers A to C_i and C_i to B. It seems true that as r grows we can conclude that the society prefers A to B with higher and higher probability that tends to 1 as r tends to infinity. This is unknown even for simple majority. (In contrast, it follows from Theorem 1.7 that for every fixed r and a noise-sensitive sequence (G_n) of strong simple games, the a posteriori probability that the society prefers A to B tends to 1/2.)

5. Other economic models. We study noise sensitivity and its chaotic consequences for social welfare functions. It would be interesting to study similar questions for other economic models and, in particular, for exchange economies and more general forms of economies. Related notions of indeterminacy, aggregation of information, pivotal agents, and power were considered for various models of economies and some analogies with results in social choice theory can be drawn. The well-known Sonnenschein-Debreu-Mantel theorem concerning demand functions for exchange economies is an indeterminacy result that implies that various dynamics for reaching equilibrium points can be chaotic. Yet, it appears that exchange economies behave in a similar way to weighted majority functions and are thus quite stable to noise in terms of initial endowments and individual demand functions. More complicated models of economies (or simple models under a complicated probabilistic environment) appear to exhibit a more chaotic behavior in the sense of this paper.

6. The asymptotic nature of our results. We would like to stress that our approach and results are asymptotic: we consider the situation when the number of individuals tends to infinity. The number of alternatives m and the noise ϵ are supposed to be fixed as n tends to infinity, so one has to be careful in drawing conclusions for numerical values of n, ϵ, and m. Moreover, the dichotomy between stochastic stability and complete chaos is based on first fixing ϵ and letting n tend to infinity and then letting ϵ tend to zero. Studying the dependence on ϵ even for multi-level majority-based rules gives a more involved picture that we briefly discussed in Section 3.2.

8 Conclusion

Social welfare functions form a simple and basic model in economic theory and political science. We have considered their probabilistic behavior under uniformly distributed voter profiles. In contrast to social welfare functions based on simple majority (or weighted majority) which demonstrate stable behavior to small stochastic perturbations, in this paper we studied a class of social welfare functions that demonstrate completely chaotic behavior and showed that this class can be characterized in surprisingly different ways. This class contains some rather natural examples. Chaotic behavior occurs in our model for hierarchical or recursive aggregation of preferences and in cases where the social preferences heavily depend on a small number of small communities with decisive views.

One interpretation of our results relates to the primary role of the majority rule. There have been many efforts to demonstrate the dominance of majority among voting rules. Analysis of sensitivity to noise gives a clear advantage to the majority rule. Our results on the stochastic robustness of majority and the chaotic nature of methods that are asymptotically orthogonal to majority point in this direction. So is the result by Mossel, O'Donnell and Oleszkiewicz (2005) on the asymptotic optimality of the majority rule in terms of sensitivity to noise. The majority rule is a very simple and very basic economic/political mechanism and an analysis of sensitivity to noise in a similar manner to ours may be an important aspect in evaluating other economic mechanisms.

Do we witness chaotic behavior in realistic economic situations of preference aggregation? A complete social chaos of the kind considered in this paper is an ideal rather than a realistic economic phenomenon. However, I would expect to find chaotic components in real-life examples of preference aggregation and in my opinion complete (stochastic) stability of the kind we observe for weighted majority is often unrealistic. Understanding the chaotic components (in the sense of this paper) in other stochastic economic models is worthy of further study.

9 Appendix

9.1 Proof of the implication $(NS) \Rightarrow (B_m)$

Consider a sequence (G_k) of strong simple games that is noise-sensitive. We already proved that for every asymmetric relation R on m alternatives

$$\lim_{k\to\infty} P_R(G_k) = 1/2^{\binom{m}{2}}.$$

Therefore, assuming noise sensitivity, property (A_m) implies property (B_m). We will now prove that noise sensitivity implies property (A_m).

Consider a random profile P and let $R = F(P)$. We need the following lemma:

Lemma 9.1. *Let (G_k) be a noise-sensitive sequence of strong simple games. Let x be a random vector in Ω_n and let $(V^1, V^2, \ldots, V^r)$ be random partitions of the players of G_k into r disjoint parts. For $j = 1, 2, \ldots, r$ let $y_i^j = x_i$ if $i \notin V_j$ and if $x_i \in V_j$ let $y_i^j = x_i$ with probability $1-t$ and $y_i^j = 1-x_i$ with probability t. (Independently for $i \in V_j$. Let $(z^0, z^1, \ldots, z^k) = (G_k(x), G_k(y^1), \ldots G_k(y^r))$. Then, as k tends to infinity, the distribution of $(z_0, z_1, \ldots, z^r)$ tends to the uniform distribution on Ω_{r+1}.*

Proof: Note that for $r = 1$ this is just the definition of noise sensitivity. Consider the following process:

(1) Choose a random set of players V_j such that each player belongs to V_j with probability $1/r$.

(2) Choose a random assignment $x_i : i \notin V_j$ of 0-1 values for the players not in V_j, and let $y_i = x_i$ for these players.

(3) Choose at random an assignment $x_i : i \in V_j$.

(4) Choose at random another assignment $v_i : i \in V$ such that $y_i = x_i$ if $i \notin V_j$ and for $i \in V_j$, $y_i = x_i$ with probability $1-t$ and $y_i = x_i$, otherwise (independently for different players).

Now consider the probability z over choices (1) and (2) that the four probabilities over choices (3) and (4) that $f(x) = \sigma_1$ and $f(y) = \sigma_2$, for $\sigma_i = 0, 1$,$i = 1, 2$, all belong to the interval $1/4 - \epsilon', 1/4 + \epsilon'$.

Noise sensitivity implies that for every $\epsilon' > 0$, z approaches 1 as k tends to infinity. In fact, this is a complicated form of the definition of noise sensitivity where the noisy bits are chosen in two stages: in the first stage a random subset of players is chosen and in the second stage a random subset of these players are chosen.

Let $\epsilon' = \epsilon/r$ and choose k large enough so that $z \geq 1-\epsilon/2^r$. The assertion of the lemma will now follow.

Q.E.D.

We return now to the proof of the implication (NS) $\Rightarrow$ (B_m). Let P be a random profile of voter preferences. Let R be the social preference relation for some random voter's profile P. The relation R is a random variable. Suppose now that a fraction $2t$ of voters, chosen at random, switch, with probability 1/2, the last two alternatives in their order preference relation. Let P' be the new profile. Let $V_{a,b}$ be the set of voters such that their two least-preferred alternatives are a and b. Each voter belongs to $V(a,b)$ with probability $1/\binom{m}{2}$. Therefore the sets $V(a,b)$ form a random partition and we can invoke Lemma 9.1. Lemma 9.1 asserts that the probability of every vector of length $\binom{m}{2}$ representing an asymmetric relation of social preferences among all the $\binom{m}{2}$ pairs $\{a,b\}$ tends to $1/2^{\binom{m}{2}}$, which is the desired result.

9.2 Proofs of properties of simple majority and multi-level majority

Proof of Theorem 1.6: Let f_{2k+1} be the majority function on the set $V = \{1, 2, \ldots, 2k+1\}$. In order to prove Lemma 3.1 we will study the Fourier-Walsh coefficients of f_{2k+1}. For a Boolean function f the coefficient $\widehat{f}(\{j\})$ by definition equals $\sum_{S\subset V, j\notin S} 2^{-n} f(S) - \sum_{S\subset V, j\in S} 2^{-n} f(S)$. Therefore, if f is monotone we obtain that

$$\widehat{f}(\{j\}) = -b_j(f)/2,$$

where $b_j(f)$ is the Banzhaf value of j in f.

For f_{2k+1} we obtain that $\widehat{f}(\{j\}) = \binom{2k}{k}/2^{2k+1}$. Recall that for a Boolean function f and for $k \ge 0$ we denoted

$$W_k(f) = \sum\{\widehat{f}^2(S) : S \subset N, |S| = k\}.$$

For every Boolean function that represents a strong simple game:
(1) $W_0(f) = \widehat{f}^2(\emptyset) = 1/4$,
(2)$\sum_{k=0}^{n} W_k^2(f) = \|f\|_2^2 = 1/2$, and
(3) $W_k(f) = 0$ for an even integer $k > 0$ (Proposition 2.1).

For simple majority on $2k+1$ voters we obtain that

$$W_1(f) = \sum_{j=1}^{2k+1} \widehat{f}^2(\{j\}) = (2k+1)\binom{2k}{k}/2^{2k+1}.$$

It is easy to verify that this expression is monotone decreasing with k. Therefore, if $k \ge 1$ the value of $W_1(f_{2k+1})$ is minimized for $k = 1$, namely, for simple majority on three voters. For $k = 1$ we get $W_1(f_3) = 3/16$. Of course, $\widehat{f_3}(S) = 0$ for $|S| > 3$ and therefore $W_k(f_3) = 0$ for every $k \ge 3$. It follows that for every $k \ge 1$, $W_1(f_{2k+1}) \le W_1(f_3)$ and $W_r(f_{2k+1}) \ge W_r(f_3)$.

These relations with Theorems 2.3 and t:arrow imply our result. Indeed, Theorem 2.3 asserts that the probability $N_\epsilon(f)$ that an ϵ-noise will change the outcome of an election is given by the formula:

$$N_\epsilon(f) = 1/2 - 4 \cdot \sum_{r=1}^{n} (1-\epsilon)^r W_r(f).$$

When we compare this expression for f_3 and f_{2k+1} the result follows from the fact that when we pass from f_3 to f_{2k+1} the weights W_i's (whose sum is constant) are shifted from the lower values of i to the larger ones. Formally, the proof is completed by the following standard lemma:

Lemma 9.2. *Let $a_1 > a_2 > \ldots, a_n > 0$ be a sequence of real numbers. Let $b = (b_1, b_2, \ldots, b_n)$ and $b' = (b'_1, b'_2, \ldots, b'_n)$ be sequences of positive real functions such that $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} b'_i$ and for every r, $1 \le r \le n$, $\sum_{i=1}^{r} b_i \ge \sum_{i=1}^{r} b'_i$ then $\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i b'_i$.*

The same argument or just applying Relation 2.7 implies part (i).

Proof of Theorem 3.5:

Consider the following class of two-level games $G(r, m, H)$. We have r committees of m players in each committee. In each committee the winner is determined (with a possibility of indifference) according to a proper monotone simple game H on m players. The society's decision is based on majority. Given a sequence $G_k = G(r_k, m_k, H_k)$ of such games we make the following assumption:

The probability $p_k = p(H_k)$ that a random coalition is winning in the game H_k satisfies:

(1) $\lim_{k\to\infty} p_k = 0$,

(2) $\lim_{k\to\infty} p_k \cdot r_k = \infty$.

Theorem 9.3. *The sequence (G_k) is noise-sensitive.*

Condition (2) implies that (G_k) has strongly diminishing bias, which is a necessary condition for noise sensitivity: The expected number of decisive communities for (G_k) is $p_k \cdot r_k$ and therefore with probability tending to one the election will be decided.

We will now describe the effect of noise.

(1) Noise sensitivity given by Theorem 4.3 implies the following properties: the probability that a nondecisive community before the noise will become decisive toward a specific candidate after the noise is $p_k(1 + o(1))$.

(2) Theorem 4.3 implies that the probability that a decisive community before the noise will remain decisive after the noise is at most $p_k(1+o(1))$. Finally, the probability that a decisive community before the noise will become decisive but in the opposite direction after the noise is at most p_k. (This is intuitively clear and can easily be verified from standard FKG inequalities.)

For the special case that H_k is supermajority game, (1) and (2) can easily be derived by direct calculations.

The proportion of communities that are decisive before the noise is negligible and therefore by (2) the effect of these communities is negligible after the noise. The outcome of the elections is decided with a probability tending to 1 by the communities that are undecided before the noise and, therefore, the probability of each candidate winning after the noise tends to 1/2.

9.3 Proof of Theorem 4.3

Theorem 4.3 is the only result in this paper that requires a nonelementary tool from harmonic analysis. We will give the proof for the case where $p_k = 1/2$, namely, for the uniform probability distribution. In this case the bias comes from the voting rule and not from the probability distribution. Corollary 2.4 (which is quite elementary) extends to the case of Boolean functions that may have large bias, as follows:

Theorem 9.4. *A sequence (f_n) of Boolean functions is noise-sensitive if and only if, for every $k > 0$,*

$$\lim_{n\to\infty} \sum_{i=1}^{k} W_k(f_n)/(\sum_{i\geq 1} W_k(f_n)) = 0. \tag{9.1}$$

Note that by the Parseval formula $\sum_{i\geq 1} W_k(f_n) = q_n(1 - q_n)$, where $q_n = \|f_n\|_2^2$ is simply the probability that f_n equals 1. (In the case that (f_n) represent a strong simple game or represent games with small bias the denominator on the right-hand side of relation (9.1) is bounded from zero and can be deleted.) In view of formula (9.1) relating the noise sensitivity to the Fourier coefficients, in order to demonstrate noise sensitivity we have to show that most of the Fourier coefficients of f_n (in terms of the 2-norm of f_n) are concentrated on large "frequencies."

For a real function $f : \Omega_n \to \mathbb{R}$, $f = \sum \widehat{f}(S)U_S$, define

$$T_\epsilon(f) = \sum \widehat{f}(S)\epsilon^{|S|}u_S.$$

The Bonamie-Beckner inequality (see, e.g., Benjamini, Kalai and Schramm (1999)) asserts that for every real function f on $\Omega(n)$,

$$\|T_\epsilon(f)\|_2 \leq \|f\|_{1+\epsilon^2}.$$

For a Boolean function f, note that for every t, $\|f\|_t^t = \mathbf{P}(f)$. It follows from the Bonamie-Beckner inequality (say, for $\epsilon = 1/2$) that for some constant K,

$$\sum_{0<S<\log(1/t)/4} \widehat{f}^2(S) \leq K\sqrt{(}q_n(1-q_n)).$$

This is sufficient to show that relation 9.1 holds even when k depends on n and is set to be $[\log(1/(q_n \cdot (1-q_n))/4]$.

9.4 The results about abstention

Proof of Theorem 6.1: Consider the following process. We choose a sequence u of random signals for all voters and let S_1 be the outcome of the election had everybody voted. Next we choose a random set T of tn voters who abstain. Let S_2 be the outcome then. Let S_3 be the outcome if the votes of these tn players are determined by coin-flips independently of their original signals. Noise sensitivity for a sequence G_k of strong simple games asserts that with a large probability (namely, for a probability tending to 1 as k tends to infinity) over the choices of u and T the probability that $S_1 = S_2$ is close to 1/2. Divide the set of 2^n signals into 2^{n-1} pairs of the form $(u_1, \ldots, u_n)$ and $(1-u_1, \ldots, 1-u_n)$. Consider now a random uniform and independent choice of

(a) the set T,

(b) a pair of signals $\{u, 1-u\}$.

Based on the choices of (a) and (b), define random variables S_1, S_2, and S_3 as follows. Choose either u or $1-u$ with probability 1/2 and define S_1, S_2 and S_3 as above. Since our game is strong, for every choice of (a) and (b) both S_1 and S_2 have the value 1 with probability 1/2 and 0 with probability 1/2. Noise sensitivity implies that with a large probability over the choices (a) and (b) the probability that $S_1 = S_3$ is close to 1/2. Therefore, with a large probability the probability that $S_2 = S_3$ is close to 1. This means that for every $t > 0$ with a large probability if we choose u uniformly at random the correlation between the outcome of the elections and the outcome when a random set of tn voters abstain tends to 0.

Proof of Theorem 6.2 (sketch): We want to show that if (G_k) is a sequence of noise-sensitive strong games with indifference, $t > 0$ a real number, and if (G'_k) is obtained from (G_k) by letting a fraction of t of the players chosen at random abstain, then with probability 1 the sequence (G'_k) is noise-sensitive. We can repeat the argument of the previous theorem based on the observation that the effect of a fraction of t players abstaining is similar to the effect of a fraction of t players reconsidering their position and voting with the same probability for each candidate independently of the original signals.

9.5 It ain't over until it's over

Sketch of the proof of Theorem 7.6: In the proof of Theorem 1.7 we considered two asymmetric relations R and R' on m alternatives that differ only in the ranking of the two last alternatives and used noise sensitivity to show that the ratio $p_R(G_n)/p_{R'}(G_n)$ tends to one as n tends to infinity. If we want to show only that the ratio $p_R(G_n)/p_{R'}(G_n)$ is bounded away from 0 we need a weaker property for the sequence (G_n), referred to as [IAOUIO], which is defined as follows:

Let G be a monotone simple strong game considered as a voting rule between Alice and Bob with n voters. Let $\epsilon > 0$ and $\delta > 0$ be small real numbers. Given a set S of voters and the way these voters vote, let Q denote the probability when the remaining voters vote uniformly at random that Alice will win the elections. Let $q(G; \epsilon, \delta)$ be the probability that a random choice of a set S of $(1 - \epsilon)n$ of the voters and a random choice of the way voters in S voted the value of Q is at least δ.

Consider a sequence (G_n) of strong simple games regarded as voting rules. The sequence (G_n) has property [IAOUIO] if

- For every $\epsilon > 0$ there is $\delta > 0$ so that

$$\lim_{n\to\infty} q(G;\epsilon,\delta) = 1.$$

It can be shown using the central limit theorem that the sequence of simple majority games on n players has the property [IAOUIO]. In this case δ can be taken to be $e^{-\epsilon^{-2}}$. As indicated above the proof of Theorem 1.7 gives

Theorem 9.5. *A sequence (G_n) of strong simple games with property [IAOUIO] leads to stochastically complete social indeterminacy.*

Showing property [IAOUIO] for a uniformly chaotic sequence of games is immediate from the definition and therefore part (2) of Theorem 7.6 follows directly from the proof of Theorem 1.7. To prove the first part we need to show that diminishing Banzhaf individual power implies the property [IAOUIO]. This is much more difficult and was recently proved by Mossel, O'Donnell and Oleszkiewicz (2005) in response to a conjecture opsed by the author and Friedgut.

Theorem 9.6 (It ain't over until it's over). [Mossel, O'Donnell and Oleszkiewicz] *A sequence of strong monotone simple games (G_n) with diminishing individual Banzhaf power has property [IAOUIO].*

References

[1] Arrow, K. (1950), A difficulty in the theory of social welfare, J. of Political Economy 58, 328–346.

[2] Bell, C. E. (1981), A random voting graph almost surely has a Hamiltonian cycle when the number of alternatives is large, Econometrica 49, 1597–1603.

[3] Benjamini, I., G. Kalai and O. Schramm (1999), Noise sensitivity of Boolean functions and applications to percolation, Publ. I.H.E.S.

[4] Feddersen, T. and W. Pesendorfer (1996), The swing voter's curse, The American Economic Review 86, 408-424.

[5] Feddersen, T. and W. Pesendorfer (1998), Convicting the innocent: the inferiority of unanimous jury verdicts under strategic voting, The American Political Science Review 92, 23-35.

[6] Fishburn, P. C., W. V. Gehrlein and E. Maskin (1979), Condorcet proportions and Kelly's conjectures, Discrete Appl. Math. 1, 229–252.

[7] Friedgut, T. E. (1979), Political Participation in the USSR, Princeton University Press, Princeton, N.J.

[8] Friedgut, E. and G. Kalai (1996), Every monotone graph property has a sharp threshold, Proc. American Mathematical Society 124, 2993–3002.

[9] Gehrlein, W. V. (1997), Condorcet's paradox and the Condorcet efficiency of voting rules, Math. Japon. 45, 173–199.

[10] Holzman, R., E. Lehrer and N. Linial (1988), Some bounds for the Banzhaf index and other semivalues, Math. Oper. Res. 13, 358–363.

[11] Kalai, G. (2002), A Fourier-theoretic perspective for the Condorcet paradox and Arrow's theorem, Adv. in Appl. Math. 29, 412–426

[12] Kalai, G. (2004), Social indeterminacy, Econometrica, 72, 1565-1581.

[13] Khot, S., G. Kindler, E. Mossel and R. O'Donnell (2004), Optimal inapproximability results for MAX-CUT and other 2-variable CSPs?, preprint.

[14] Mossel, E. and R. O'Donnell (2003), On the noise sensitivity of monotone functions, Random Structures Algorithms 23, 333–350.

[15] Mossel, E., R. O'Donnell and K. Oleszkiewicz (2005), An invariance principle for functions with low influences, preprint.

[16] Peres, Y. (2004) Noise stability of weighted majority, preprint math.PR/0412377.

[17] Saari, D. G. (2001), Chaotic elections! A mathematician looks at voting, American Mathematical Society, Providence, R I.

[18] Samet, Y. (2004), Equilibria with information aggregation in sharp threshold voting rules, M.Sc. thesis, Jerusalem, 2004.

[19] Sheppard, W. (1899), On the application of the theory of error to cases of normal distribution and normal correlation, Phil. Trans. Royal Soc. London 192, 101–168.