Chapter 56

VALUES OF GAMES WITH INFINITELY MANY PLAYERS

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Abstract

This chapter studies the theory of value of games with infinitely many players.

Games with infinitely many players are models of interactions with many players. Often most of the players are individually insignificant, and are effective in the game only via coalitions. At the same time there may exist big players who retain the power to wield single-handed influence. The interactions are modeled as cooperative games with a continuum of players. In general, the continuum consists of a non-atomic part (the "ocean"), along with (at most countably many) atoms. The continuum provides a convenient framework for mathematical analysis, and approximates the results for large finite games well. Also, it enables a unified view of games with finite, countable, or oceanic player-sets, or indeed any mixture of these.

The value is defined as a map from a space of cooperative games to payoffs that satisfies the classical value axioms: additivity (linearity), efficiency, symmetry and positivity. The chapter introduces many spaces for which there exists a unique value, as well as other spaces on which there is a value.

A game with infinitely many players can be considered as a limit of finite games with a large number of players. The chapter studies limiting values which are defined by means of the limits of the Shapley value of finite games that approximate the given game with infinitely many players.

Various formulas for the value which express the value as an average of marginal contribution are studied. These value formulas capture the idea that the value of a player is his expected marginal contribution to a perfect sample of size t of the set of all players where the size t is uniformly distributed on $[0,1]$. In the case of smooth games the value formula is a diagonal formula: an integral of marginal contributions which are expressed as partial derivatives and where the integral is over all perfect samples of the set of players. The domain of the formula is further extended by changing the order of integration and derivation and the introduction of a well-crafted infinitesimal perturbation of the perfect samples of the set of players provides a value formula that is applicable to many additional games with essential nondifferentiabilities.

Keywords

games, cooperative, coalitional form, transferable utility, value, continuum of players, non-atomic

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1. Introduction

The Shapley value is one of the basic solution concepts of cooperative game theory. It can be viewed as a sort of average or expected outcome, or as an *a priori* evaluation of the players' expected payoffs.

The value has a very wide range of applications, particularly in economics and political science (see Chapters 32, 33 and 34 in this Handbook). In many of these applications it is necessary to consider games that involve a large number of players. Often most of the players are individually insignificant, and are effective in the game only via coalitions. At the same time there may exist big players who retain the power to wield single-handed influence. A typical example is provided by voting among stockholders of a corporation, with a few major stockholders and an "ocean" of minor stockholders. In economics, one considers an oligopolistic sector of firms embedded in a large population of "perfectly competitive" consumers. In all these cases, it is fruitful to model the game as one with a continuum of players. In general, the continuum consists of a non-atomic part (the "ocean"), along with (at most countably many) atoms. The continuum provides a convenient framework for mathematical analysis, and approximates the results for large finite games well. Also, it enables a unified view of games with finite, countable, or oceanic player-sets, or indeed any mixture of these.

1.1. An outline of the chapter

In Section 2 we highlight a sample of a few asymptotic results on the Shapley value of finite games. These results motivate the definition of games with infinitely many players and the corresponding value theory; in particular, Proposition 1 introduces a formula, called the diagonal formula of the value, that expresses the value (or an approximation thereof) by means of an integral along a specific path.

The Shapley value for finite games is defined as a linear map from the linear space of games to payoffs that satisfies a list of axioms: symmetry, efficiency and the null player axiom. Section 3 introduces the definitions of the space of games with a continuum of players, and defines the symmetry, positivity and efficiency of a map from games to payoffs (represented by finitely additive games) which enables us to define the value on a space of games as a linear, symmetric, positive and efficient map.

The value does not exist on the space of all games with a continuum of players. Therefore the theory studies spaces of games on which a value does exist and moreover looks for spaces for which there is a unique value. Section 4 introduces several spaces of games with a continuum of players including the two spaces of games, pNA and $bv'NA$, which played an important role in the development of value theory for non-atomic games. Theorem 2 asserts that there exists a unique value on each of the spaces pNA and $bv'NA$ as well as on pNA_∞. In addition there exists a (unique) value of norm 1 on the space $'NA$. Section 4 also includes a detailed outline of the proofs.

A game with infinitely many players can be considered as a limit of finite games with a large number of players. Section 5 studies limiting values which are defined by means of the limits of the Shapley value of finite games that approximate the given game with infinitely many players. Section 5 also introduces the asymptotic value. The asymptotic value of a game v is defined whenever all the sequences of Shapley values of finite games that "approximate" v have the same limit. The space of all games of bounded variation for which the asymptotic value exist is denoted *ASYMP*. Theorem 4 asserts that *ASYMP* is a closed linear space of games that contains $bv'M$ and the map associating an asymptotic value with each game in *ASYMP* is a value on *ASYMP*.

Section 6 introduces the mixing value which is defined on all games of bounded variation for which an analogous formula of the random order one for finite games is well defined. The space of all games having a mixing value is denoted *MIX*. Theorem 5 asserts that *MIX* is a closed linear space which contains *pNA*, and the map that associates a mixing value with each game v in *MIX* is a value on *MIX*.

Section 7 introduces value formulas. These value formulas capture the idea that the value of a player is his expected marginal contribution to a perfect sample of size t of the set of all players where the size t is uniformly distributed on $[0, 1]$. A value formula can be expressed as an integral of directional derivative whenever the game is a smooth function of finitely many non-atomic measures. More generally, by extending the game to be defined over all ideal coalitions – measurable $[0, 1]$-valued functions on the (measurable) space of players – this value formula provides a formula for the value of any game in *pNA*. Changing the order of derivation and integration results in a formula that is applicable to a wider class of games: all those games for which the formula yields a finitely additive game. In particular, this modified formula defines a value on $bv'NA$ or even on $bv'M$. When this formula does not yield a finitely additive game, applying the directional derivative to a carefully crafted small perturbation of perfect samples of the set of players yields a value formula on a much wider space of games (Theorem 7). These include games which are nondifferentiable functions (e.g., a piecewise linear function) of finitely many non-atomic probability measures.

Section 8 describes two spaces of games that are spanned by nondifferentiable functions of finitely many mutually singular non-atomic probability measures that have a unique value (Theorem 8). One is the space spanned by all piecewise linear functions of finitely many mutually singular non-atomic probability measures, and the other is the space spanned by all concave Lipschitz functions with increasing returns to scale of finitely many mutually singular non-atomic probability measures.

The value is defined as a map from a space of games to payoffs that satisfies a short list of plausible conditions: linearity, symmetry, positivity and efficiency. The Shapley value of finite games satisfies many additional desirable properties. It is continuous (of norm 1 with respect to the bounded variation norm), obeys the null player and the dummy axioms, and it is strongly positive. In addition, any value that is obtained from values of finite approximations obeys an additional property called diagonality. Section 9 introduces such additional desirable value properties as strong positivity and diagonality. Theorem 9 asserts that strong positivity can replace positivity and linearity in the characterization of the value on pNA_∞, and thus strong positivity and continuity can replace positivity and linearity in the characterization of a value on *pNA*. A striking property of the value on *pNA*, the asymptotic value, the mixing value, or any of the values obtained by the formulas described in Section 7, is the diagonal property: if two games coincide on all coalitions that are "almost" a perfect sample (within ε with respect to finitely many non-atomic probability measures) of the set of players, their values coincide. Theorem 11 asserts that any continuous value is diagonal.

Section 10 studies semivalues which are generalizations of the value that do not necessarily obey the efficiency property. Theorem 13 provides a characterization of all semivalues on *pNA*. Theorems 14 and 15 provide results on asymptotic semivalues.

Section 11 studies partially symmetric values which are generalizations of the value that do not necessarily obey the symmetry axiom. Theorems 15 and 16 characterize in particular all the partially symmetric values on *pNA* and *pM* that are symmetric with respect to those symmetries that preserve a fixed finite partition of the set of players. A corollary of this characterization on *pM* is the characterization of all values on *pM*. A partially symmetric value, called a μ-value, is symmetric with respect to all symmetries that preserve a fixed non-atomic population measure μ and (in addition to linearity, positivity and efficiency), obeys the dummy axiom. Such a partially symmetric value is called a μ-value. Theorem 18 asserts the uniqueness of a μ-value on $pNA(\mu)$. Theorem 19 asserts that an important class of nondifferentiable markets have an asymptotic μ-value.

Games that arise from the study of non-atomic markets, called market games, obey two specific conditions: concavity and homogeneity of degree 1. The core of such games is nonempty. Section 12 provides results on the relation between the value and the core of market games. Theorem 20 asserts the coincidence of the core and the value in the differentiable case. Theorem 21 relates the existence of an asymptotic value of nondifferentiable market games to the existence of a center of symmetry of its core. Theorems 22 and 23 describe the asymptotic μ-value and the value of market games with a finite-dimensional core as an average of the extreme points of the core.

2. Prelude: Values of large finite games

This section introduces several asymptotic results on the Shapley value ψ of finite games. These results can serve as an introduction to the theory of values of games with infinitely many players, by motivating the definitions of games with a continuum of players and the corresponding value theory.

We start with the introduction of a representation of games with finitely many players. Let ℓ be a fixed positive integer and $f : \mathbb{R}^\ell \to \mathbb{R}$ a function with $f(0) = 0$. Given $w_1, \ldots, w_\ell \in \mathbb{R}^\ell_+$, define the n-person game $v = [f; w_1, \ldots, w_\ell]$ by

$$v(S) = f\left(\sum_{i \in S} w_i\right).$$

A special subclass of these games is weighted majority games where $\ell = 1$, $0 < q < \sum_{i=1}^n w_i$ and $f(x) = 1$ if $x \geqslant q$ and 0 otherwise. The asymptotic results relate to sequences of finite games where the function f is fixed and the vector of weights varies. If the function f is differentiable at x, the directional derivative at x of the function f in the direction $y \in \mathbb{R}^\ell$ is $f_y(x) := \lim_{\varepsilon\to 0+}(f(x+\varepsilon y) - f(x))/\varepsilon$.

The following result follows from the proofs of results on the asymptotic value of non-atomic games. It can be derived from the proof in Kannai (1966) or from the proof in Dubey (1980).

PROPOSITION 1. *Assume that f is a continuously differentiable function on $[0,1]^\ell$. For every $\varepsilon > 0$ there is $\delta > 0$ such that if $w_1, \ldots, w_n \in \mathbb{R}^\ell_+$ with (the normalization) $\sum_{i=1}^n w_i = (1, \ldots, 1)$ and $\max_{i=1}^n \|w_i\| < \delta$, then*

$$\left|\psi v(i) - \int_0^1 f_{w_i}(t, \ldots, t)\, \mathrm{d}t\right| \leqslant \varepsilon \|w_i\|,$$

where $v = [f; w_1, \ldots, w_n]$. Thus, for every coalition S of players,

$$\left|\psi v(S) - \int_0^1 f_{w(S)}(t, \ldots, t)\, \mathrm{d}t\right| < \varepsilon,$$

where $\psi v(S) = \sum_{i\in S} \psi v(i)$ and $w(S) = \sum_{i\in S} w_i$.

The limit of a sequence of games of the form $[f; w_1^k, \ldots, w_{n_k}^k]$, where f is a fixed function and $w_1^k, \ldots, w_{n_k}^k$ is a sequence of vectors in $\mathbb{R}^\ell_+$ with $\sum_{i=1}^{n_k} w_i^k = (1, \ldots, 1)$ and $\max_{i=1}^{n_k} \|w_i\| \to_{k\to\infty} 0$ can be modeled by a 'game' of the form $f \circ (\mu_1, \ldots, \mu_\ell)$ where $\mu = (\mu_1, \ldots, \mu_\ell)$ is a vector of non-atomic probability measures defined on a measurable space $(I, \mathcal{C})$. If f is continuously differentiable, the limiting value formula thus corresponds to:

$$\varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t, \ldots, t)\, \mathrm{d}t.$$

The next results address the asymptotics of the value of weighted majority games. The n-person weighted majority game $v = [q; w_1, \ldots, w_n]$ with $w_1, \ldots, w_n \in \mathbb{R}_+$ and $0 < q < \sum_{i=1}^m w_i$, is defined by $v(S) = 1$ if $\sum_{i\in S} w_i \geqslant q$ and 0 otherwise.

PROPOSITION 2 [Neyman (1981a)]. *For every $\varepsilon > 0$ there is $K > 0$ such that if $v = [q; w_1, \ldots, w_n]$ is a weighted majority game with $w_1, \ldots, w_n \in \mathbb{R}^+$, $\sum_{i=1}^n w_i = 1$ and $K \max_{i=1}^n w_i < q < 1 - K \max_{i=1}^n w_i$,*

$$\left|\psi v(S) - \sum_{i\in S} w_i\right| < \varepsilon$$

for every coalition $S \subset \{1, \ldots, n\}$.

In particular, if $v_k = [q_k; w_1^k, \dots, w_{n_k}^k]$ is a sequence of weighted majority games with $w_i^k \geqslant 0$, $\sum_{i=1}^{n_k} w_i^k \to_{k\to\infty} 1$, and $q_k \to_{k\to\infty} q$ with $0 < q < 1$, then for any sequence of coalitions $S_k \subset \{1, \dots, n_k\}$, $\psi v_k(S_k) - \sum_{i\in S_k} w_i^k \to_{k\to\infty} 0$. The limit of such a sequence of weighted majority games is modeled as a game of the form $v = f_q \circ \mu$ where μ is a non-atomic probability measure and $f_q(x) = 1$ if $x \geqslant q$ and 0 otherwise. The non-atomic probability measure μ describes the value of this limiting game:

$$\varphi v(S) = \mu(S).$$

The next result follows easily from the result of Berbee (1981).

PROPOSITION 3. *Let $w_1, \dots, w_n, \dots$ be a sequence of positive numbers with*

$$\sum_{i=1}^{\infty} w_i = 1$$

and $0 < q < 1$. There exists a sequence of nonnegative numbers ψ_i, $i \geqslant 1$, with $\sum_{i=1}^{\infty} \psi_i = 1$, such that for every $\varepsilon > 0$ there is a positive constant $\delta > 0$ and a positive integer n_0 such that if $v = [q'; u_1, \dots, u_n]$ is a weighted majority game with $n \geqslant n_0$, $|q' - q| < \delta$ and $\sum_{i=1}^{n} |w_i - u_i| < \delta$, then

$$\left|\psi v(i) - \psi_i\right| < \varepsilon.$$

Equivalently, let $v_k = [q_k; w_1^k, \dots, w_{n_k}^k]$ be a sequence of weighted majority games with $\sum_{i=1}^{n_k} w_i^k \to_{k\to\infty} 1$ and $w_j^k \to_{k\to\infty} w_j$ and $q_k \to_{k\to\infty} q$. Then the Shapley value of player i in the game v_k converges as $k \to \infty$ to a limit ψ_i and $\sum_{i=1}^{\infty} \psi_i = 1$. The limit of such a sequence of weighted majority games can be modeled as a game of the form $v = f_q \circ \mu$ where μ is a purely atomic measure with atoms $i = 1, 2, \dots$ and $\mu(i) = w_i$. The sequence ψ_i describes the value of this limiting game:

$$\varphi v(S) = \sum_{i\in S} \psi_i.$$

3. Definitions

The space of players is represented by a measurable space $(I, \mathcal{C})$. The members of set I are called *players*, those of $\mathcal{C}$ – *coalitions*. A game in coalitional form is a real-valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. For each coalition S in $\mathcal{C}$, the number $v(S)$ is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members; it is called the *worth* of S. The set of all games forms a linear space over the field of real numbers, which is denoted G. We assume that $(I, \mathcal{C})$ is isomorphic to

$([0,1], \mathcal{B})$, where $\mathcal{B}$ stands for the Borel subsets of $[0, 1]$. A game v is *finitely additive* if $v(S \cup T) = v(S) + v(T)$ whenever S and T are two disjoint coalitions. A distribution of payoffs is represented by a finitely additive game. A value is a mapping from games to distributions of payoffs, i.e., to finitely additive games that satisfy several plausible conditions: linearity, symmetry, positivity and efficiency. There are several additional desirable conditions: e.g., continuity, strong positivity, a null player axiom, a dummy player axiom, and so on. It is remarkable, however, that a short list of plausible conditions that define the value suffices to determine the value in many cases of interest, and that in these cases the unique value satisfies many additional desirable properties. We start with the notations and definitions needed formally to define the value, and set forth four spaces of games on which there is a unique value.

A game v is *monotonic* if $v(S) \geqslant v(T)$ whenever $S \supset T$; it is of *bounded variation* if it is the difference of two monotonic games. The set of all games of bounded variation (over a fixed measurable space $(I, \mathcal{C})$) is a vector space. The *variation* of a game $v \in G$, $\|v\|$, is the supremum of the variation of v over all increasing chains $S_1 \subset S_2 \subset \cdots \subset S_n$ in $\mathcal{C}$. Equivalently, $\|v\| = \inf\{u(I) + w(I)\colon\ u, w$ are monotonic games with $v = u - w\}$ $(\inf \emptyset = \infty)$. The variation defines a topology on G; a basis for the open neighborhood of a game v in G consists of the sets $\{u \in G\colon\ \|u - v\| < \varepsilon\}$ where ε varies over all positive numbers. Addition and multiplication of two games are continuous and so is the multiplication of a game and a real number. A game v has bounded variation if $\|v\| < \infty$. The space of all games of bounded variation, BV, is a Banach algebra with respect to the variation norm $\| \ \|$, i.e., it is complete with respect to the distance defined by the norm (and thus it is a Banach Space) and it is closed under multiplication which is continuous. Let $\mathcal{G}$ denote the group of automorphisms (i.e., one-to-one measurable mappings Θ from I onto I with Θ^{-1} measurable) of the underlying space $(I, \mathcal{C})$. Each Θ in $\mathcal{G}$ induces a linear mapping Θ_* of BV (and of the space of all games) onto itself, defined by $(\Theta_* v)(S) = v(\Theta S)$. A set of games Q is called *symmetric* if $\Theta_* Q = Q$ for all Θ in $\mathcal{G}$. The space of all finitely additive and bounded games is denoted *FA*; the subspace of all measures (i.e., countably additive games) is denoted *M* and its subspace consisting of all non-atomic measures is denoted *NA*. The space of all non-atomic elements of *FA* is denoted *AN*. Obviously, NA $\subset$ M $\subset$ FA $\subset$ BV, and each of the spaces NA, AN, M, FA and BV is a symmetric space. Given a set of games Q, we denote by Q^+ all monotonic games in Q, and by Q^1 the set of all games v in Q^+ with $v(I) = 1$. A map $\varphi\colon Q \to BV$ is called *positive* if $\varphi(Q^+) \subset BV^+$; *symmetric* if for every $\Theta \in \mathcal{G}$ and v in Q, $\Theta_* v \in Q$ implies that $\varphi(\Theta_* v) = \Theta_*(\varphi v)$; and *efficient* if for every v in Q, $(\varphi v)(I) = v(I)$.

DEFINITION 1 [Aumann and Shapley (1974)]. Let Q be a symmetric linear subspace of BV. A *value* on Q is a linear map $\varphi\colon Q \to FA$ that is symmetric, positive and efficient.

The definition of a value is naturally extended also to include spaces of games that are not necessarily included in BV. Thus let Q be a linear and symmetric space of games.

DEFINITION 2. A *value* on Q is a linear map from Q into the space of finitely additive games that is symmetric, positive (i.e., $\varphi(Q^+) \subset BV^+$), efficient, and $\varphi(Q \cap BV) \subset FA$.

There are several spaces of games that have a unique value. One of them is the space of all games with a finite support, where a subset T of I is called a *support* of a game v if $v(S) = v(S \cap T)$ for every S in $\mathcal{C}$. The space of all games with finite support is denoted FG.

THEOREM 1 [Shapley (1953a)]. *There is a unique value on FG.*

The unique value ψ on FG is given by the following formula. Let T be a finite support of the game v (in FG). Then, $\psi v(S) = 0$ for every S in $\mathcal{C}$ with $S \cap T = \emptyset$, and for $t \in T$,

$$\psi v(\{t\}) = \frac{1}{n!} \sum_{\mathcal{R}} \left[v\left(\mathcal{P}_t^{\mathcal{R}} \cup \{t\}\right) - v\left(\mathcal{P}_t^{\mathcal{R}}\right) \right],$$

where n is the number of elements of the support T, the sum runs over all $n!$ orders of T, and $\mathcal{P}_t^{\mathcal{R}}$ is the set of all elements of T preceding t in the order $\mathcal{R}$.

Finite games can also be identified with a function v defined on a finite subfield π of $\mathcal{C}$: given a real-valued function v defined on a finite subfield π of $\mathcal{C}$ with $v(\emptyset) = 0$, the Shapley value of the game v is defined as the additive function $\psi : \pi \to \mathbb{R}$ defined by its values on the atoms of π; for every atom a of π,

$$\psi v(a) = \frac{1}{n!} \sum \mathcal{R} \left[v\left(\mathcal{P}_a^{\mathcal{R}} \cup a\right) - v\left(\mathcal{P}_a^{\mathcal{R}}\right) \right],$$

where n is the number of atoms of π, the sum runs over all $n!$ orders of the atoms of π, and $\mathcal{P}_a^{\mathcal{R}}$ is the union of all atoms of π preceding a in the order $\mathcal{R}$.

4. The value on *pNA* and *bv′NA*

Let *pNA* (*pM*, *pFA*) be the closed (in the bounded variation norm) algebra generated by NA (*M*, *FA*, respectively). Equivalently, *pNA* is the closed linear subspace that is generated by powers of measure in NA^1 [Aumann and Shapley (1974), Lemma 7.2]. Similarly, *pM* (*pFA*) is the closed algebra that is generated by powers of elements in *M* (*FA*). Let $\| \ \|_\infty$ be defined on the space of all games G by

$$\|v\|_\infty = \inf\left\{\mu(I)\colon\ \mu \in NA^+,\ \mu - v \in BV^+,\ \mu + v \in BV^+\right\} + \sup\left\{|v(S)|\colon\ S \in \mathcal{C}\right\}.$$

In the definition of $\|v\|_\infty$, we use the common convention that $\inf \emptyset = \infty$. The definition of $\| \ \|_\infty$ is an analog of the C^1 norm of continuously differentiable functions; e.g., if f is continuously differentiable on $[0, 1]$ and μ is a non-atomic probability measure, $\|f \circ \mu\|_\infty = \max\{|f'(t)|: 0 \leqslant t \leqslant 1\} + \max\{|f(t)|: 0 \leqslant t \leqslant 1\}$. Note that $\|uv\|_\infty \leqslant \|u\|_\infty \|v\|_\infty$ for two games u and v. The function $v \mapsto \|v\|_\infty$ defines a topology on G; a basis for the open neighborhood of a game v in G consists of the sets $\{u \in G: \|u - v\|_\infty < \varepsilon\}$ where ε varies over all positive numbers. The space of games pNA_∞ is defined as the $\| \|_\infty$ closed linear subspace that is generated by powers of measures in NA^1. Obviously, $pNA_\infty \subset pNA$ ($\| \|_\infty \geqslant \| \|$). Both spaces contain many games of interest. If μ is a measure in NA^1 and $f:[0,1] \to \mathbb{R}$ is a function with $f(0) = 0$, then $f \circ \mu \in pNA$ if and only if f is absolutely continuous [Aumann and Shapley (1974), Theorem C] and in that case $\|f \circ \mu\| = \int_0^1 |f'(t)|\,dt$ where f' is the (Radon–Nikodym) derivative of the function f, and $f \circ \mu \in pNA_\infty$ if and only if f is continuously differentiable on $[0,1]$. Also, if $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of NA measures with range $\{\mu(S): S \in \mathcal{C}\} =: \mathcal{R}(\mu)$, and f is a C^1 function on $\mathcal{R}(\mu)$ with $f(\mu(\emptyset)) = 0$, then $f \circ \mu \in pNA_\infty$ [Aumann and Shapley (1974), Proposition 7.1].

Let $bv'NA$ ($bv'M$, $bv'FA$) be the closed linear space generated by games of the form $f \circ \mu$ where $f \in bv' =: \{f:[0,1] \to \mathbb{R} \mid f$ is of bounded variation and continuous at 0 and at 1 with $f(0) = 0\}$, and $\mu \in NA^1$ ($\mu \in M^1$, $\mu \in FA^1$). Obviously, $pNA_\infty \subset pNA \subset bv'NA \subset bv'M \subset bv'FA$.

THEOREM 2 [Aumann and Shapley (1974), Theorems A, B; Monderer (1990), Theorem 2.1]. *Each of the spaces pNA, bv'NA, pNA_∞ has a unique value.*

The proof of the uniqueness of the value on these spaces relies on two basic arguments. Let $\mu \in NA^1$ and let $\mathcal{G}(\mu)$ be the subgroup of $\mathcal{G}$ of all automorphisms Θ that are μ-measure-preserving, i.e., $\mu(\Theta S) = \mu(S)$ for every coalition S in $\mathcal{C}$. Then the set of all finitely additive games u that are fixed points of the action of $\mathcal{G}(\mu)$ (i.e., $\Theta_* u = u$ for every Θ in $\mathcal{G}(\mu)$) are games of the form $\alpha\mu$, $\alpha \in \mathbb{R}$ (here we use the standardness assumption that $(I, \mathcal{C})$ is isomorphic to $([0,1], \mathcal{B})$). Therefore, if Q is a symmetric space of games and $\varphi: Q \to FA$ is symmetric, $\mu \in NA^1$, and $f:[0,1] \to \mathbb{R}$ is such that $f \circ \mu \in Q$, then $\Theta_*(f \circ \mu) = f \circ \mu$ for every $\Theta \in \mathcal{G}(\mu)$. It follows that the finitely additive game $\varphi(f \circ \mu)$ is a fixed point of $\mathcal{G}(\mu)$, which implies that $\varphi(f \circ \mu) = \alpha\mu$. If, moreover, φ is also efficient, then $\varphi(f \circ \mu) = f(1)\mu$. The spaces pNA and $bv'NA$ are closed linear spans of games of the form $f \circ \mu$. The space pNA_∞ is in the closed linear span of games of the form $f \circ \mu$ with $f \circ \mu$ in pNA_∞. Therefore, each of the spaces pNA_∞, pNA and $bv'NA$ has a dense subspace on which there is at most one value.

To complete the uniqueness proof, it suffices to show that any value on these spaces is indeed continuous. For that we apply the concept of internality. A linear subspace Q of BV is said to be *reproducing* if $Q = Q^+ - Q^+$. It is said to be *internal* if for all v in Q,

$$\|v\| = \inf\{u(I) + w(I): u, w \in Q^+ \text{ and } v = u - w\}.$$

Clearly, every internal space is reproducing. Also any linear positive mapping φ from a closed reproducing space Q into BV is continuous, and any linear positive and efficient mapping φ from an internal space Q into BV has norm 1 [Aumann and Shapley (1974), Propositions 4.15 and 4.7]. Therefore, to complete the proof of uniqueness, it is sufficient to show that each of the spaces pNA, $bv'NA$ and pNA_∞ is internal. The closure of an internal space is internal [Aumann and Shapley (1974), Proposition 4.12] and any linear space of games that is the union of internal spaces is internal. Therefore one demonstrates that each of these spaces is the union of spaces that contain a dense internal subspace. There are several dense internal subspaces of pNA [Aumann and Shapley (1974), Lemma 7.18; Reichert and Tauman (1985); Monderer and Neyman (1988), Section 5]. The proof of the internality of $bv'NA$ is more involved [Aumann and Shapley (1974), Section 8].

Note that if Q and Q' are two linear symmetric spaces of games and $Q \supset Q'$, the existence of a value on Q implies the existence of a value on Q'. However, the uniqueness of a value on one of them does not imply uniqueness on the other. There are, for instance, subspaces of pNA which have more than one value [Hart and Neyman (1988), Example 4]. There are however results that provide sufficient conditions for subspaces of pNA to have a unique value. For a coalition S we define the *restriction* of v to S, v_S, by $v_S(T) = v(S \cap T)$ for every $T \in \mathcal{C}$. A set of games M is *restrictable* if $v_S \in M$ for every $v \in M$ and for every $S \in \mathcal{C}$. Hart and Monderer [(1997), Theorem 3.6] prove the uniqueness of the value on restrictable subspaces of pNA_∞ that contain NA. This is the non-atomic version of the analogous result for finite games proved in Hart and Mas-Colell (1989) and Neyman (1989).

Existence of a value on pNA and $bv'NA$ is proved in Aumann and Shapley, Chapter I. We outline below a proof that is based on the proof of Mertens and Neyman (2001). Assume that $v = \sum_{i=1}^n f_i \circ \mu_i$ with $f_i \in bv'$ and $\mu_i \in NA^1$. Define $\varphi v(S) = \sum_{i=1}^n f_i(1)\mu_i(S)$. It follows from the continuity at 0 and 1 of each of the functions f_i that

$$\sum_{i=1}^n f_i(1)\mu_i(S) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_0^{1-\varepsilon}\left[\sum_{i=1}^n f_i\bigl(t+\varepsilon\mu_i(S)\bigr) - \sum_{i=1}^n f_i(t)\right] dt.$$

By Lyapunov's Theorem, applied to the vector measure $(\mu_1, \dots, \mu_n)$, for every $0 \leqslant t \leqslant 1-\varepsilon$ there are coalitions $T \subset R$ with $\mu_i(T) = t$ and $\mu_i(R) = t + \varepsilon\mu_i(S)$. Therefore, if v is monotonic,

$$v(R) - v(T) = \sum_{i=1}^n f_i\bigl(t+\varepsilon\mu_i(S)\bigr) - \sum_{i=1}^n f_i(t) \geqslant 0,$$

and therefore $\varphi v(S) \geqslant 0$. In particular, if $0 = \sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g_j \circ \nu_j$ where $f_i, g_j \in bv'$ and $\mu_i, \nu_j \in NA^1$, $\sum_{i=1}^n f_i(1)\mu_i = \sum_{j=1}^k g_j(1)\nu_j$, and thus also φv is well defined, i.e., it is independent of the presentation. Obviously, $\varphi v \in NA \subset FA$ and $\varphi v(I) = v(I)$ and φ is symmetric and linear. Therefore φ defines a value on the linear and symmetric subspace of $bv'NA$ of all games of the form $\sum_{i=1}^n f_i \circ \mu_i$ with n

a positive integer, $f_i \in bv'$ and $\mu_i \in NA^1$. We next show that φ is of norm 1 and thus has a (unique) extension to a continuous value (of norm 1) on $bv'NA$. For each $u \in FA$, $\|u\| = \sup_{S\in\mathcal{C}} |u(S)| + |u(S^c)|$. Therefore, it is sufficient to prove that for every coalition $S \in \mathcal{C}$, $|\varphi v(S)| + |\varphi v(S^c)| \leqslant \|v\|$. For each positive integer m let $S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_m$ and $S_0^c \subset S_1^c \subset \cdots \subset S_m^c$ be measurable subsets of S and $S^c = I \setminus S$ respectively with $\mu_i(S_j) = \frac{j}{m+1}\mu_i(S)$ and $\mu_i(S_j^c) = \frac{j}{m+1}\mu_i(S^c)$. For every $0 \leqslant t \leqslant \frac{1}{m+1}$ let I_t be a measurable subset of $I \setminus (S_m \cup S_m^c)$ with $\mu_i(I_t) = t$. Define the increasing sequence of coalitions $T_0 \subset T_1 \subset \cdots \subset T_{2m}$ by $T_0 = I_t$, $T_{2j-1} = I_t \cup S_j \cup S_{j-1}^c$ and $T_{2j} = I_t \cup S_j \cup S_j^c$, $j = 1, \ldots, m$. Obviously,

$$\begin{aligned}\|v\| &\geqslant \sum_{j=1}^{2m}\left|v(T_j) - v(T_{j-1})\right| \\ &\geqslant \left|\sum_{j=0}^{m-1} v(T_{2j+1}) - v(T_{2j})\right| + \left|\sum_{j=1}^{m} v(T_{2j}) - v(T_{2j-1})\right|.\end{aligned}$$

Set $\varepsilon = \frac{1}{m+1}$. Note that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^{\varepsilon}\sum_{j=0}^{m-1}\left[\sum_{i=1}^{n} f_i\big(t + \varepsilon j + \varepsilon\mu_i(S)\big) - \sum_{i=1}^{n} f_i(t + j\varepsilon)\right]\mathrm{d}t \\ &\quad = \frac{1}{\varepsilon}\int_0^{1-\varepsilon}\left[\sum_{i=1}^{n} f_i\big(t + \varepsilon\mu_i(S)\big) - \sum_{i=1}^{n} f_i(t)\right]\mathrm{d}t \to_{m\to\infty} \varphi v(S),\end{aligned}$$

and similarly

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^{\varepsilon}\sum_{j=1}^{m}\left[\sum_{i=1}^{n} f_i(t + \varepsilon j) - \sum_{i=1}^{n} f_i\big(t + j\varepsilon - \varepsilon\mu_i(S^c)\big)\right]\mathrm{d}t \\ &\quad = \frac{1}{\varepsilon}\int_{\varepsilon}^{1}\left[\sum_{i=1}^{n} f_i(t) - \sum_{i=1}^{n} f_i\big(t - \varepsilon\mu_i(S^c)\big)\right]\mathrm{d}t \to_{m\to\infty} \varphi v(S^c).\end{aligned}$$

As

$$v(T_{2j+1}) - v(T_{2j}) = \sum_{i=1}^{n} f_i\big(t + \varepsilon j + \varepsilon\mu_i(S)\big) - \sum_{i=1}^{n} f_i(t + j\varepsilon),$$

and

$$v(T_{2j}) - v(T_{2j-1}) = \sum_{i=1}^{n} f_i(t + 2j\varepsilon) - \sum_{i=1}^{n} f_i\big(t + 2j\varepsilon - \varepsilon\mu_i(S^c)\big),$$

we deduce that for each fixed $0 \leqslant t \leqslant \varepsilon$,

$$\left|\sum_{j=0}^{m-1}\left[\sum_{i=1}^{n} f_i\left(t+\varepsilon j+\varepsilon\mu_i(S)\right)-\sum_{i=1}^{n} f_i(t+j\varepsilon)\right]\right| \\ +\left|\sum_{j=1}^{m}\left[\sum_{i=1}^{n} f_i(t+\varepsilon j)-\sum_{i=1}^{n} f_i\left(t+j\varepsilon-\varepsilon\mu_i\left(S^c\right)\right)\right]\right| \leqslant \|v\|$$

and therefore $|\varphi v(S)|+|\varphi v(S^c)| \leqslant \|v\|$. It follows that the map φ, which is defined on a dense subspace of $bv'NA$ and with values in NA, is linear, symmetric, efficient, and of norm 1, and therefore φ has a unique extension to a continuous value on $bv'NA$.

The bounded variation of the functions f_i is used in the above proof to show that the integrals $\int_a^b f(x)\,\mathrm{d}x$, $0 \leqslant a < b \leqslant 1$, are well defined. Therefore, the proof also demonstrates the result of Tauman (1979), i.e., that there is a unique value of norm 1 on $In'NA$: the closed linear space of games that is generated by games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in In' = \{f : [0,1] \to \mathbb{R} \mid f(0)=0,\ f$ integrable and continuous at 0 and 1$\}$. The integrability requirement can also be dropped. Mertens and Neyman (2001) prove the existence of a value (of norm 1) on the spaces $'AN \supset {}'NA$: the closure in the variation distance of the linear space spanned by all games $f \circ \mu$ where $\mu \in AN^1 \supset NA^1$ and f is a real-valued function defined on $[0,1]$ with $f(0)=0$ and continuous at 0 and 1. Moreover, the continuity of the function f at 0 and 1 can be further weakened by requiring the upper and lower averages of f on the intervals $[0,\varepsilon]$ and $[1-\varepsilon,1]$ to converge to $f(0)=0$ and $f(1)$, respectively.

The existence of a value (of norm 1) on pNA and $bv'NA$ also follows from various constructive approaches to the value, such as the limiting approach discussed immediately below and the value formulas approach (Section 7). Other constructive approaches include the mixing approach (Section 6) that provides a value on pNA. Each constructive approach describes a way of "computing" the value for a specific game.

The space of all games that are in the (feasible) domain of each approach will form a subspace and the associated map will be a value. Each of the approaches has its advantages and each sheds additional interpretive light on the value obtained. We start with limiting value approaches.

5. Limiting values

Limiting values are defined by means of the limits of the Shapley value of the finite games that approximate the given game.

Given a game v on (I, C) and a finite measurable field π, we denote by v_π the restriction of the game v to the finite field π. The real-valued function v_π (defined on π)

can be viewed as a finite game. The Shapley value for finite games ψv_π on π is given by

$$(\psi v_\pi)(a) = \frac{1}{n!} \sum_{\mathcal{R}} \left[v\left(\mathcal{P}_a^{\mathcal{R}} \cup a\right) - v\left(\mathcal{P}_a^{\mathcal{R}}\right) \right],$$

where n is the number of atoms of π, the sum runs over all orders $\mathcal{R}$ of the atoms of π, and $\mathcal{P}_a^{\mathcal{R}}$ is the union of all atoms preceding a in the order $\mathcal{R}$.

5.1. The weak asymptotic value

The family of measurable finite fields (that contain a given coalition T) is ordered by inclusion, and given two measurable finite fields π and π' there is a measurable finite field π'' that contains both π and π'. This enables us to define limits of functions of finite measurable fields. The weak asymptotic value of the game v is defined as $\lim_\pi \psi v_\pi$ whenever the limit of $\psi v_\pi(T)$ exists where π ranges over the family of measurable finite fields that include T. Equivalently,

DEFINITION 3. Let v be a game. A game φv is said to be the *weak asymptotic value* of v if for every $S \in \mathcal{C}$ and every $\varepsilon > 0$, there is a finite subfield π of $\mathcal{C}$ with $S \in \pi$ such that for any finite subfield π' with $\pi' \supset \pi$, $|\psi v_{\pi'}(S) - \varphi v(S)| < \varepsilon$.

REMARKS. (1) Let v be a game. The game v has a weak asymptotic value if and only if for every S in $\mathcal{C}$ and $\varepsilon > 0$ there exists a finite subfield π with $S \in \pi$ such that for any finite subfield π' with $\pi' \supset \pi$, $|\psi v_{\pi'}(S) - \psi v_\pi(S)| < \varepsilon$.

(2) A game v has at most one weak asymptotic value.

(3) The weak asymptotic value φv of a game v is a finitely additive game, and $\|\varphi v\| \leqslant \|v\|$ whenever $v \in BV$.

THEOREM 3 [Neyman (1988), p. 559]. *The set of all games having a weak asymptotic value is a linear symmetric space of games, and the operator mapping each game to its weak asymptotic value is a value on that space. If ASYMP* denotes all games with bounded variation having a weak asymptotic value, then ASYMP* is a closed subspace of BV with $bv'FA \subset$ ASYMP*.*

REMARKS. The linearity, symmetry, positivity and efficiency of ψ (the Shapley value for games with finitely many players) and of the map $v \mapsto v_\pi$ imply the linearity, symmetry, positivity and efficiency of the weak asymptotic value map and also imply the linearity and symmetry of the space of games having a weak asymptotic value. The closedness of *ASYMP** follows from the linearity of ψ and from the inequalities $\|\psi v_\pi\| \leqslant \|v_\pi\| \leqslant \|v\|$. The difficult part of the theorem is the inclusion $bv'FA \subset$ *ASYMP**, on which we will comment later. The inclusion $bv'FA \subset$ *ASYMP** implies in particular the existence of a value on $bv'FA$.

5.2. The asymptotic value

The asymptotic value of a game v is defined whenever all the sequences of the Shapley values of finite games that "approximate" v have the same limit. Formally: given T in $\mathcal{C}$, a T-admissible sequence is an increasing sequence $(\pi_1, \pi_2, \ldots)$ of finite subfields of $\mathcal{C}$ such that $T \in \pi_1$ and $\bigcup_i \pi_i$ generates $\mathcal{C}$.

DEFINITION 4. A game φv is said to be the *asymptotic value* of v, if $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists and $= \varphi v(T)$ for every $T \in \mathcal{C}$ and every T-admissible sequence $(\pi_i)_{i=1}^{\infty}$.

REMARKS. (1) A game v has an asymptotic value if and only if $\lim_{k\to\infty} \psi v_{\pi_k}(T)$ exists for every T in $\mathcal{C}$ and every T-admissible sequence $\mathcal{P} = (\pi_k)_{k=1}^{\infty}$, and the limit is independent of the choice of $\mathcal{P}$.

(2) For any given v, the asymptotic value, if it exists, is clearly unique.

(3) The asymptotic value φv of a game v is finitely additive, and $\|\varphi v\| \leqslant \|v\|$ whenever $v \in BV$.

(4) If v has an asymptotic value φv then v has a weak asymptotic value, which $= \varphi v$.

(5) The *dual* of a game v is defined as the game v^* given by $v^*(S) = v(I) - v(I\setminus S)$. Then $\psi v^*_\pi = \psi v_\pi = \psi(\frac{v+v^*}{2})_\pi$, and therefore v has a (weak) asymptotic value if and only if v^* has a (weak) asymptotic value if and only if $(\frac{v+v^*}{2})$ has a (weak) asymptotic value, and the values coincide.

The space of all games of bounded variation that have an asymptotic value is denoted *ASYMP*.

THEOREM 4 [Aumann and Shapley (1974), Theorem F; Neyman (1981a); and Neyman (1988), Theorem A]. *The set of all games having an asymptotic value is a linear symmetric space of games, and the operator mapping each game to its asymptotic value is a value on that space. ASYMP is a closed linear symmetric subspace of BV which contains $bv'M$.*

REMARKS. (1) The linearity and symmetry of ψ (the Shapley value for games with finitely many players), and of the map $v \to v_\pi$, imply the linearity and symmetry of the asymptotic value map and its domain. The efficiency and positivity of the asymptotic value follows from the efficiency and positivity of the maps $v \to v_\pi$ and $v_\pi \to \psi v_\pi$. The closedness of *ASYMP* follows from both the linearity of ψ and the inequalities $\|\psi v_\pi\| \leqslant \|v\|_\pi \leqslant \|v\|$.

(2) Several authors have contributed to proving the inclusion $bv'M \subset ASYMP$. Let FL denote the set of measures with at most finitely many atoms and M_a all purely atomic measures. Each of the spaces, $bv'NA$, $bv'FL$ and $bv'M_a$ is defined as the closed subspace of BV that is generated by games of the form $f \circ \mu$ where $f \in bv'$ and μ is a probability measure in NA^1, FL^1, or M_a^1 respectively.

Kannai (1966) and Aumann and Shapley (1974) show that $pNA \subset ASYMP$. Artstein (1971) proves essentially that $pM_a \subset ASYMP$. Fogelman and Quinzii (1980) show that $pFL \subset ASYMP$. Neyman (1981a, 1979) shows that $bv'NA \subset ASYMP$ and $bv'FL \subset ASYMP$ respectively. Berbee (1981), together with Shapley (1962) and the proof of Neyman [(1981a), Lemma 8 and Theorem A], imply that $bv'M_a \subset ASYMP$. It was further announced in Neyman (1979) that Berbee's result implies the existence of a partition value on $bv'M$. Neyman (1988) shows that $bv'M \subset ASYMP$.

(3) The space of games, $bv'M$, is generated by scalar measure games, i.e., by games that are real-valued functions of scalar measures. Therefore, the essential part of the inclusion $bv'M \subset ASYMP$ is that whenever $f \in bv'$ and μ is a probability measure, $f \circ \mu$ has an asymptotic value. The tightness of the assumptions on f and μ for $f \circ \mu$ to have an asymptotic value follows from the next three remarks.

(4) $pFA \notin ASYMP$ [Neyman (1988), p. 578]. For example, $\mu^3 \notin ASYMP$ whenever μ is a positive non-atomic finitely additive measure which is not countably additive. This illustrates the essentiality of the countable additivity of μ for $f \circ \mu$ to have an asymptotic value when f is a polynomial.

(5) For $0 < q < 1$, we denote by f_q the function given by $f_q(x) = 1$ if $x \geqslant q$ and $f_q(x) = 0$ otherwise. Let μ be a non-atomic measure with total mass 1. Then $f_q \circ \mu$ has an asymptotic value if and only if μ is positive [Neyman (1988), Theorem 5.1]. There are games of the form $f_q \circ \mu$, where μ is a purely atomic signed measure with $\mu(I) = 1$, for which $f_{\frac{1}{2}} \circ \mu$ does not have an asymptotic value [Neyman (1988), Example 5.2]. These two comments illustrate the essentiality of the positivity of the measure μ.

(6) There are games of the form $f \circ \mu$, where $\mu \in NA^1$ and f is continuous at 0 and 1 and vanishes outside a countable set of points, which do not have an asymptotic value [Neyman (1981a), p. 210]. Thus, the bounded variation of the function f is essential for $f \circ \mu$ to have an asymptotic value.

(7) The set of games $DIAG$ is defined [Aumann and Shapley (1974), p. 252] as the set of all games v in BV for which there is a positive integer k, measures $\mu_1, \ldots, \mu_k \in NA^1$, and $\varepsilon > 0$, such that for any coalition $S \in \mathcal{C}$ with $\mu_i(S) - \mu_j(S) \leqslant \varepsilon$ for all $1 \leqslant i, j \leqslant k$, $v(S) = 0$. $DIAG$ is a symmetric linear subspace of $ASYMP$ [Aumann and Shapley (1974)].

(8) If μ_1, μ_2, μ_3, are three non-atomic probabilities that are linearly independent, then the game v defined by $v(S) = \min[\mu_1(S), \mu_2(S), \mu_3(S)]$ is not in $ASYMP$ [Aumann and Shapley (1974), Example 19.2].

(9) Games of the form $f_q \circ \mu$ where $\mu \in M^1$ and $0 < q < 1$ are called *weighted majority games*; a coalition S is winning, i.e., $v(S) = 1$, if and only if its total μ-weight is at least q. The measure μ is called the *weight measure* and the number q is called the *quota*. The result $bv'M \subset ASYMP$ implies in particular that weighted majority games have an asymptotic value.

(10) A *two-house weighted majority game* v is the product of two weighted majority games, i.e., $v = (f_q \circ \mu)(f_c \circ \nu)$, where $\mu, \nu \in M^1$, $0 < c < 1$, and $0 < q < 1$. If $\mu, \nu \in$

NA^1 with $\mu \neq \nu$ and $q \neq 1/2$, v has an asymptotic value if and only if $q \neq c$. Indeed, if $q < c$, $(f_q \circ \mu)(f_c \circ \nu) - (f_c \circ \nu) \in DIAG$ and therefore

$$(f_q \circ \mu)(f_c \circ \nu) \in bv'NA + DIAG \subset ASYMP,$$

and if $q = c \neq 1/2$, $(f_q \circ \mu)(f_c \circ \nu) \notin ASYMP$ [Neyman and Tauman (1979), proof of Proposition 5.42]. If μ has infinitely many atoms and the set of atoms of ν is disjoint to the set of atoms of μ, $(f_q \circ \mu)(f_c \circ \nu)$ has an asymptotic value [Neyman and Smorodinski (1993)].

(11) Let $\mathcal{A}$ denote the closed algebra that is generated by games of the form $f \circ \mu$ where $\mu \in NA^1$ and $f \in bv'$ is continuous. It follows from the proof of Proposition 5.5 in Neyman and Tauman (1979) together with Neyman (1981a) that $\mathcal{A} \subset ASYMP$ and that for every $u \in \mathcal{A}$ and $v \in bv'NA$, $uv \in ASYMP$.

The following should shed light on the proofs regarding asymptotic and limiting values.

Let π be a finite subfield of $\mathcal{C}$ and let v be a game. We will recall formulas for the Shapley value of finite games by applying these formulas to the game v_π. This will enable us to illustrate proofs regarding the limiting properties of the Shapley value ψv_π of the finite game v_π. Let $A(\pi)$, or A for short, denote the set of atoms of π. The Shapley value of the game v_π to an atom a in A is given by

$$\psi v_\pi(a) = \frac{1}{|A|!} \sum_{\mathcal{R}} \left[v\left(P_a^{\mathcal{R}} \cup a\right) - v\left(P_a^{\mathcal{R}}\right)\right],$$

where $|A|$ stands for the number of elements of A, the summation ranges over all orders $\mathcal{R}$ of the atoms in A, and $P_a^{\mathcal{R}}$ is the union of all atoms preceding a in the order R. An alternative formula for ψv_π could be given by means of a family X_a, $a \in A(\pi)$, of i.i.d. real-valued random variables that are uniformly distributed on $(0, 1)$. The values of X_a, $a \in A(\pi)$, induce with probability 1 an order $\mathcal{R}$ on A; a precedes b if and only if $X_a < X_b$. As X_a, $a \in A$, are i.i.d., all orders are equally likely. Thus, identifying sets with their indicator functions, and letting $I(X_a \leqslant X_b)$ denote the indicator of the event $X_a \leqslant X_b$, the value formula is

$$\psi v_\pi(a) = \mathrm{E}\left[v\left(\sum_{b \in A} bI(X_b \leqslant X_a)\right) - v\left(\sum_{b \in A} bI(X_b < X_a)\right)\right].$$

Thus, by conditioning on the value of X_a, and letting $S(t)$, $0 \leqslant t \leqslant 1$, denote the union of all atoms b of π for which $X_b \leqslant t$, we find that

$$\psi v_\pi(a) = \int_0^1 \mathrm{E}\big(v(S(t) \cup a) - v(S(t) \setminus a)\big)\,\mathrm{d}t.$$