Uniqueness of the Shapley Value

ABRAHAM NEYMAN*

Institute of Mathematics and Department of Economics, The Hebrew University, Jerusalem 91904, Israel

It is shown that the Shapley value of any given game \( v \) is characterized by applying the value axioms—efficiency, symmetry, the null player axiom, and either additivity or strong positivity—to the additive group generated by the game \( v \) itself and its subgames. © 1989 Academic Press, Inc.

It has often been remarked (e.g., Hart and Mas-Colell, 1988) that the standard axiomatizations (e.g., Dubey, 1975; Shapley, 1953; Young, 1985) of the Shapley value require the application of the value axioms to a large class of games (e.g., all games or all simple games). It is shown in this paper that the Shapley value of any given game \( v \) is characterized by applying the value axioms to the additive group generated by the game \( v \) itself and its subgames. This is important, particularly in applications where typically only one specific problem is considered.

Let \( N \) be a finite set. We refer to \( N \) as the set of players and to a subset \( S \) of \( N \) as a coalition. A (cooperative) game (with side payments) on the set of players \( N \) is a function \( v: 2^N \to \mathbb{R} \) satisfying \( v(\emptyset) = 0 \). For a given game \( v \) and a coalition \( T \), we refer to \( v(T) \) as the worth of \( T \). Given a game \( v \) on the set of players \( N \), and a coalition \( S \), we denote by \( v_S \) the real valued function \( v_S: 2^N \to \mathbb{R} \) given by \( v_S(T) = v(S \cap T) \). Thus, \( v_S \) is a game on the set of players \( N \) and is called the subgame of \( v \), obtained by restricting \( v \) to subsets of \( S \) only.

The set of all games on the set of players \( N \) is denoted \( G^N \), or \( G \), for short. Let \( v \in G \) and \( i, j \in N \). The players \( i, j \) are substitutes in \( v \) if for every coalition \( S \subseteq N \setminus \{i, j\} \), \( v(S \cup i) = v(S \cup j) \); player \( i \) is a null player in the game \( v \) if \( v(S \cup i) = v(S) \) for every coalition \( S \).

* Supported partially by the National Science Foundation Grant DMS 87 05294 at the State University of New York at Stony Brook and by CORE.
Let $Q$ be a set of games on the set of players $N$. A map $\Psi: Q \to \mathbb{R}^N$ is efficient if for every $v$ in $Q$, $\sum_{i \in N} \Psi_{iv} = v(N)$; it is additive if for every $v, w$ in $Q$ with $v + w \in Q$, $\Psi v + \Psi w = \Psi(v + w)$; it is symmetric if for every $v$ in $Q$ and every two players $i, j$ that are substitutes in $v$, $\Psi_i v = \Psi_j v$. A map $\Psi: Q \to \mathbb{R}^N$ obeys the null player axiom if for every $v$ in $Q$ and every player $i$ in $N$ that is a null player of $v$, $\Psi_i v = 0$. For a given game $v$ in $G$ we denote by $G(v)$ the additive group generated by the game $v$ and all of its subgames, i.e., $G(v) = \{w \in G \mid w = \Sigma k_i v_S, \text{ where } k_i \text{ are integers and } S_i \text{ coalitions}\}$.

**Theorem A.** Let $v \in G$. Any map $\Psi$ from $G(v)$ into $\mathbb{R}^N$ that is efficient, additive, and symmetric and obeys the null player axiom is the Shapley value.

**Proof.** Let $\phi$ be the Shapley value. We must prove that $\phi w = \Psi w$ for every game $w$ in $G(v)$.

For any two games $w, u$ and any two coalitions $T$ and $S$, $(w \pm u)_S = w_S \pm u_S$ and $(w_S)_T = w_{S \cap T}$. Thus, any subgame of a game in $G(v)$ is in $G(v)$. For any game $w$ in $G$ we denote by $I(w) = \{S \subseteq N \mid \exists T \subseteq S \text{ with } w(T) \neq 0\}$. Note that for any minimal coalition $S$ in $I(w)$, $w(S) \neq 0$. Also, $I(w \pm u) \subseteq I(w) \cup I(u)$ and $I(w_S) \subseteq I(w)$. We prove that $\Psi w = \phi w$ for $w$ in $Q(v)$ by induction on $|I(w)|$, the number of elements in $I(w)$. If $|I(w)| = 0$ then $w$ is the 0 game and thus, by the null player axiom $\Psi w = 0 = \phi w$. Assume that $\Psi w = \phi w$ whenever $|I(w)| \leq k$ and let $w \in G(v)$ with $|I(w)| = k + 1$. Let $S$ be a minimal element in $I(w)$. Then, $w_S$ is a unanimity game and thus, by symmetry, efficiency, and the null player axiom it follows that $\Psi_i w_S = 0 = \phi_i w_S$ for every $i \notin S$ and $\Psi_i w_S = w(S)/|S| = \phi_i w_S$ for every $i \in S$. Note that $S \notin I(w - w_S) \subseteq I(w) \cup I(w_S) \subseteq I(w)$ and thus,

$$|I(w - w_S)| < |I(w)| \quad \text{for every minimal coalition } S \in I(w). \quad (1)$$

Thus, by the induction hypothesis $\Psi(w - w_S) = \phi(w - w_S)$ and applying additivity, $\Psi w = \Psi(w_S) + \Psi(w - w_S) = \phi(w_S) + \phi(w - w_S) = \phi w$.

Our next result characterizes the Shapley value without the additivity axiom. The axiom that is added is strong positivity, which asserts that the value of each player is a monotonic function of his marginal contributions.

A map $\Psi: Q \to \mathbb{R}^N$ is strongly positive if for any player $i \in N$ and any two games, $w$ and $u$, in $Q$ with $w(S \cup i) - w(S) \geq u(S \cup i) - u(S)$, $\Psi_i w \geq \Psi_i u$.

**Theorem B.** Let $v \in G^N$. Any map $\Psi$ from $G(v)$ into $\mathbb{R}^N$ that is efficient, strongly positive, and symmetric is the Shapley value.

**Proof.** We prove that $\Psi w = \phi w$ for any $w$ in $G(v)$ by induction on $|I(w)|$. If $|I(w)| = 0$ then $w$ is the 0 game and thus, by the efficiency and
symmetry of $\Psi$, $\Psi_iw = \varphi_iw = 0$. Assume that $\Psi_w = \varphi_w$ whenever $|I(w)| \leq k$ and let $w \in G(v)$ with $|I(w)| = k + 1$. Let $\partial(w)$ denote the set of minimal coalitions in $I(w)$. Note that for every $S$ in $\partial(w)$ and every $i \in N \setminus S$, $w_S(T \cup i) - w_S(T) = 0$ and therefore by strong positivity of $\Psi$, $\Psi_i(w) = \Psi_i(w - w_S)$ which by (1) and the induction hypothesis equals $\varphi_i(w - w_S) = \varphi_i(w)$. Thus, $\Psi_iw = \varphi_iw$ for every $i \in N \setminus S$ where $S \subseteq \partial(w)$. Letting $S_0$ denote the intersection of all coalitions $S$ in $\partial(w)$, i.e., $S_0 = \cap_{S \in \partial(w)} S$, we deduce that

$$\Psi_iw = \varphi_iw \quad \text{for every } i \in N \setminus S_0. \quad (2)$$

Note that $w(T) = 0$ whenever $T \not\subseteq S_0$. Thus, every two players $i, j$ in $S_0$ are substitutes in $w$. Thus, using (1) and the efficiency of both $\Psi$ and $\varphi$ we deduce that $\Sigma_{i \in S_0} \Psi_iw = \Sigma_{i \in S_0} \varphi_iw$, and by applying the symmetry of both $\Psi$ and $\varphi$,

$$\Psi_iw = \varphi_iw \quad \text{for every } i \in S_0. \quad (3)$$

Combining (2) and (3) we conclude that $\Psi_w = \varphi_w$. 

**Remark a.** The above results show, in particular, that any additive subgroup of $G$, that is closed under the subgame operator, has a unique value. For instance, let $w$ be a given monotonic simple game. Then, $Q = \{wu | u \in G\}$ is an example of such an additive subgroup. Such spaces of games arise in models in which political approval that is described by the simple game $w$ is needed for various economic activities.

**Remark b.** It follows from the proof of Theorem B that strong positivity could be replaced by the axiom that $\Psi_iu$ depends only on the marginal contributions $u(S \cup i) - u(S)$, $S \subseteq N$, of player $i$ in $u$.

**References**


