SEMIVALENTES OF POLITICAL ECONOMIC GAMES*

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The class of continuous semivalues is completely characterized for various spaces of nonatomic games.

1. Introduction. Semivalues are defined in [4] where characterization of the semivalues is given for two basic spaces: the space of all finite games, and the space of "differentiable" nonatomic games, i.e., $pNA$. In the purely economic situation, we usually encounter games in $pNA$ (or in $pNAD$); but in many political economic situations, as in the Aumann–Kurz models of power and taxation ([1],[2]), we face games which are the products of weighted majority games by games in $pNA$. These games are members of other spaces which contain $pNA$ and which we will refer to as spaces of political economic games. In this paper we will characterize all semivalues on spaces of political economic games. §3 presents a characterization of all continuous semivalues on a typical class of political economic games, followed by a detailed proof. In §4 we introduce further results without proofs. The proofs of the results in §4 are more involved than that of §3, but actually are based on the same ideas and thus we decided to omit them from our paper.

2. Preliminaries. Most of the definitions and notations are according to [3]. Let $(I, \mathcal{E})$ be a measurable space isomorphic to $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the $\sigma$-field of Borel subsets of $[0,1]$. A set function (or game) is a function $v: \mathcal{E} \to \mathbb{R}$ with $v(\emptyset) = 0$. A set function $v$ is monotonic if for all $T$ and $S$ in $\mathcal{E}$, $T \subset S \Rightarrow v(T) < v(S)$. The space of all set functions on $(I, \mathcal{E})$ that are the difference of two monotonic set functions is denoted $BV$. The space of all bounded finitely additive set functions is denoted $FA$, and its subspace of all nonatomic measures is denoted $NA$. If $Q \subset BV$ then $Q^+$ denotes the subset of $Q$ of all monotonic set functions. A mapping $\Psi: Q \to BV$ is positive if $\Psi(Q^+) \subset BV^+$. Let $G$ denote the group of automorphisms of $(I, \mathcal{E})$. Each $\theta$ in $G$ induces a linear mapping $\theta^*$ of $BV$ onto itself that is given by $(\theta^*v)(S) = v(\theta S)$ for all $S$ in $\mathcal{E}$. A subset $Q$ of $BV$ is symmetric if for each $\theta$ in $G$, $\theta^*Q \subset Q$.

Let $Q$ be a symmetric subspace of $BV$. A semivalue on $Q$ is a positive linear mapping $\psi$ from $Q$ into $FA$ that satisfies:

$$\psi \text{ is symmetric, i.e., } \psi(\theta^*) = \theta^*\psi \quad \text{for all } \theta \text{ in } G.$$  \hspace{1cm} (2.1)

$$\text{if } v \in Q \cap FA \text{ then } \psi v = v.$$ \hspace{1cm} (2.2)

The bounded variation norm of a set function $v$ in $BV$ is defined by $\|v\| = \inf(u(I) + w(I))$ where the infimum ranges over all pairs of monotonic set functions $u, w$ with $v = u - w$. A nondecreasing sequence of sets in $\mathcal{E}$ of the form $\Omega: S_0 \subset S_1 \subset \cdots \subset S_n$ is called a chain. The variation of $v$ over a chain $\Omega$ is defined by $\|v\|_\Omega = \sum_{i=1}^{n-1} |v(S_i) - v(S_{i-1})|$. It is known [3, Proposition 4.1] that $\|v\| = \sup\|v\|_\Omega$ where the supremum is.
taken over all chains \( \Omega \). If \( Q \) is a subspace of \( BV \) and \( \psi : Q \to BV \) is linear then \( \|\psi\| \) is defined as \( \sup\{||\psi v|| : v \in Q, ||v|| = 1 \} \).

The space \( pNA \) is the closed subspace of \( BV \) that is generated by powers of nonatomic measures.

Let \( \mathcal{F} \) denote the family of all measurable functions from \( I \) to \([0,1]\) (measurable with respect to the \( \sigma \)-fields \( \mathcal{E} \) and \( \mathcal{B} \)). There is a partial order on \( \mathcal{F} \) with \( w(0) = 0 \) is called an ideal set function; it is called monotonic if \( f \geq g \) implies \( w(f) \geq w(g) \). For every ideal set function \( w \) we define by \( ||w|| \) the supremum of \( \sum |w(f_i) - w(f_{i-1})| \) taken over all increasing sequences \( 0 \leq f_0 < f_1 < \cdots < f_n \leq 1 \) measurable functions. The indicator function of a set \( S \) in \( \mathcal{F} \) is denoted \( \chi_S \) i.e., \( \chi_S(s) = 1 \) if \( s \in S \) and equals 0 if \( s \not\in S \).

Let \( f \) denote the family of all measurable functions from \( I \) to \([0,1]\) (measurable with respect to the \( \sigma \)-fields \( g \) and \( \mathcal{J} \)). There is a partial order on \( f \): \( f \geq g \) if \( f(s) \geq g(s) \) for all \( s \) in \( I \). A real valued function \( w \) on \( f \) with \( w(0) = 0 \) is called an ideal set function; it is called monotonic if \( f \geq g \) implies \( w(f) \geq w(g) \). For every ideal set function \( w \) we denote by \( ||w|| \) the supremum of \( \sum |w(f_i) - w(f_{i-1})| \) taken over all increasing sequences \( 0 \leq f_0 < f_1 < \cdots < f_n \leq 1 \) measurable functions.

It is known [3, Theorem G] that there is a unique linear mapping that associates with each set function \( v \) in \( pNA \) an ideal set function \( v^* \) such that \( (v \psi)^* = v^* \psi^* \) for all \( v, \psi \) in \( pNA \) and \( \psi^* \) is monotonic wherever \( \psi \) is in \( pNA \), \( ||\psi|| = ||\psi^*|| \), and such that \( \mu^*(f) = \int f \mu \) for all \( \mu \) in \( NA \) and all \( f \) in \( \mathcal{F} \).

Denote \( v^*(t,S) = (\frac{d}{dt})v^*(t+\tau S)_{\tau=0} \). By Theorem H of [3] we know that for each \( v \) in \( pNA \) and each \( S \) in \( \mathcal{F} \), the derivative \( \partial v^*(t,S) \) exists for almost all \( t \) in \([0,1]\) and is integrable over \([0,1]\) as a function of \( t \).

We denote by \( W \) the set of nonnegative functions \( g \) in \( L_\infty([0,1]) \) with \( \int_0^1 g(t) \) dt = 1.

The characterization of the class of semivalues on \( pNA \) is given in [4, Theorem 2].

**Theorem 2.3 ([4, Theorem 2]).** For each \( g \) in \( W \) the mapping \( \psi_g : pNA \to FA \) that is given by

\[
(\psi_g v)(S) = \int_0^1 \partial v^*(t,S) g(t) \, dt
\]

is a semivalue. Moreover, every semivalue on \( pNA \) is of this form. The map \( g \to \psi_g \) of \( W \) onto the class of semivalues on \( pNA \) is a linear isometry.

Define \( DIAG \) to be the set of all \( v \) in \( BV \) satisfying: there exist a positive integer \( k \), a \( k \)-dimensional vector \( \xi \) of probability measures in \( NA \), and a neighborhood \( U \) in \( \mathbb{R}^k \) of the diagonal \([0,\xi(I)]\) such that if \( \xi(S) \in v \) then \( v(S) = 0 \). A semivalue \( \psi \) on a symmetric subspace \( Q \) of \( BV \) is diagonal if \( \psi \) is 0 for all \( v \in Q \cap DIAG \).

**Proposition 2.4.** Continuous semivalues are diagonal.

**Proof.** The proof in [7] that continuous values are diagonal does not make use of the efficiency axiom and therefore the same proof works here.

Another result which will be used in the proofs of the present paper is:

**Proposition 2.5.** Let \( Q \) be a symmetric subspace of \( BV \), and let \( \psi \) be a semivalue on \( Q \). If \( \mu \in NA^+ \) and \( f \) is defined on the range of \( \mu \) with \( f \circ \mu \in Q \), then \( \psi(f \circ \mu) = a \mu \) for some constant \( a \) in \( \mathbb{R} \).

**Proof.** Follows from the proof of Proposition 6.1 in [3].

**3. Characterization of the semivalues on a class of political economic games.** In the purely economic situation, we usually encounter games in \( pNA \) (or in \( pNAD \)—the closed linear space generated by \( pNA \) and \( DIAG \)) but in many political economic situations we face games of the form \( v = uq \) where \( q \) is in \( pNA \) and \( u \) is a jump function with respect to a given \( NA \) probability measure \( \mu \), i.e.,

\[
u(S) = \begin{cases} 1 & \text{if } \mu(S) > \alpha, \\ 0 & \text{if } \mu(S) \leq \alpha. \end{cases}
\]

Such games arose for instance in models for taxation (see Aumann–Kurz [1] and [2]).
We denote by \( u \ast pNA \) the minimal linear symmetric space containing \( pNA \) and all games of the form \( uq \) where \( q \in pNA \) and \( \alpha \) is a fixed number in \((0,1)\).

**Theorem A.** For any pair \((a, g), a \in \mathbb{R}^+, g \in W\), there is a semivalue \( \psi_{(a,g)} \) on \( u \ast pNA \) such that for any \( q \in pNA \)

\[
(\psi_{(a,g)}q)(S) = \int_0^1 g(t) \partial q^*(t, S) \, dt \quad \text{and} \quad (\psi_{(a,g)}(uq))(S) = aq^*(\alpha) \mu(S) + \int_\alpha^1 g(t) \partial q^*(t, S) \, dt.
\]

Moreover, any continuous semivalue on \( u \ast pNA \) is of that form. The mapping \((a, g) \rightarrow \psi_{(a,g)}\) is 1-1 and \( \|\psi_{(a,g)}\| = \max(a, \|q\|_\infty) \).

The proof of the theorem is accomplished in several stages. First we shall state and prove a result on the range of vector of members of \( pNA \). This is a generalization of a result of Dvoretzky, Wald and Wolfowitz [5, p. 66, Theorem 4].

**Lemma 3.3.** Let \( v \) be a finite dimensional vector of measures in \( NA \), and let \( m \) be a positive integer. Then for each \( m \)-tuple \( f_1, \ldots, f_m \) of ideal sets such that \( f_1 + \cdots + f_m = 1 = X_I \), and each \( k \)-tuple \( q_1, \ldots, q_k \) of members of \( pNA \), and each \( \epsilon > 0 \) there is a partition \((T_1, \ldots, T_m)\) of \( I \), \( T_i \) in \( \mathcal{E} \) such that for all \( A \subset \{1, \ldots, m\} \) and all \( 1 < j < k \)

\[
v\left( \bigcup_{i \in A} T_i \right) = \int \left( \sum_{i \in A} f_i \right) \, dv \quad \text{and} \quad \left| q_j \left( \bigcup_{i \in A} T_i \right) - q^*_j \left( \sum_{i \in A} f_i \right) \right| < \epsilon.
\]

**Remark.** The same result holds if \( pNA \) is replaced by \( pNA' \) (replace in the proof \( II \) by \( III' \)).

**Proof.** From the definition of \( pNA \) it follows that for each \( 1 < j < k \) there exists a polynomial \( v_j \) of \( NA \)-measures; \( v_j = P(\mu_{f_1}, \ldots, \mu_{f_j}) \) with \( \|q_j - v_j\| < \epsilon \). By Theorem 4 of [5] there is a partition \((T_1, \ldots, T_m)\) of \( I \), \( T_i \) in \( \mathcal{E} \) such that for all \( 1 < i < m \), \( 1 < j < k \), and \( 1 < i < i' < m \)

\[
v(T_i) = \int f_i \, dv \quad \text{and} \quad \mu^*_j(T_i) = \int f_i \, d\mu^*_j.
\]

From the finite additivity of members of \( NA \), we deduce that for each \( A \subset \{1, \ldots, m\}, 1 < j < k \) and \( 1 < i < n \), \( v(T(A)) = \int f(A) \, dv \) and \( \mu^*_j(T(A)) = \int f(A) \, d\mu^*_j \) where \( T(A) = \bigcup_{i \in A} T_i \) and \( f(A) = \sum_{i \in A} f_i \). From the last equalities and the properties of the mapping \( v \rightarrow v^* \), it follows that also \( v_j(T(A)) = v_j^*(f(A)) \). Thus

\[
|q_j(T(A)) - q_j^*(f(A))| < |q_j(T(A)) - v_j(T(A)) + v_j^*(f(A))| + |v_j^*(f(A)) - q_j^*(f(A))| < \|q_j - v_j\| + 0 + \|v_j^* - q_j^*\|
\]

and as \( \|v^*\| = \|v\| \) for each \( v \in pNA \), \( |q_j(T(A)) - q_j^*(f(A))| < 2\epsilon \). This completes the proof of Lemma 3.3.

**Corollary 3.4.** Let \( v \) be a finite dimensional vector of measures in \( NA \), and let \( m \) be a positive integer. Then for each \( m \)-tuple \( f_1, \ldots, f_m \) of ideal sets such that \( f_1 + \cdots + f_m = 1 = X_I \), and each \( k \)-tuple \( q_1, \ldots, q_k \) of set functions in \( pNA \), and each \( \epsilon > 0 \) there is an \( m \)-tuple \( T_1, T_2, \ldots, T_m \) of sets in \( \mathcal{E} \) such that \( T_1 \subset \cdots \subset T_m \) for all \( 1 < j < k \) and all \( 1 < i < i' < m \)

\[
v(T_i) = \int f_i \, dv \quad \text{and} \quad |q_j(T_i) - q_j^*(f_i)| < \epsilon \quad \text{and} \quad |q_j(T_i - T_i) - q_j^*(f_i - f_i)| < \epsilon.
\]
PROOF. Follows by applying Lemma 3.3 to the \( m + 1 \)-tuple \( f_1, f_2 - f_1, \ldots, f_m - f_{m-1}, 1 - f_m \).

We will proceed in order to show that (3.1) and (3.2) define a unique linear symmetric operator from \( \omega \cdot \mu \) into \( FA \). For this we shall need the following lemma.

**Lemma 3.5.** If \( w = v + \sum_{i=1}^{n} (\theta^*_i u_i q_i) \) is monotonic, where \( v \in \mu \), \( q_i \in \mu \) and \( \theta_i \in G, i = 1, \ldots, n \), then for every \( S \in \mathcal{E} \) and \( \epsilon \geq 0 \) we conclude that

\[
\begin{align*}
\int_{0}^{\alpha} g(t)\partial v^*(t, S) \, dt &> 0, \\
\int_{a}^{1} g(t)\partial v^*(t, S) \, dt + \sum_{i=1}^{n} \int_{a}^{1} g(t)\partial q_i^*(t, S) \, dt &> 0 \quad \text{and} \\
\sum_{i=1}^{n} \mu(\theta_i, S)q_i^*(\alpha) &> 0.
\end{align*}
\]

**Proof.** Assume that \( w \) is monotonic. For proving (3.6) it is enough to show that for any \( t \) with \( 0 < t < \alpha \) for which \( \partial v^*(t, S) \) is defined, \( \partial v^*(t, S) > 0 \). Let \( 0 < t < \alpha \) and let \( 0 < h \) be such that \( t + h < \alpha \). For such \( t \) and \( h \), \( t + hS < t + h \) and therefore

\[
(\theta^*_i u_i)(t + hS) < (\theta^*_i u_i)(t + h) = t + h < \alpha \quad \text{for each} \quad i = 1, \ldots, n.
\]

For any \( \epsilon > 0 \) we could apply Corollary 3.4 to the vector \( v = (\theta^*_1, \ldots, \theta^*_n, \mu) \) of nonatomic measures, the \( 2 \)-tuple \( t, t + hS \) and the set function \( v \) in \( \mu \) to show the existence of two sets \( T_1, T_2 \) in \( \mathcal{E} \) with \( T_1 \subset T_2 \) and such that \( (\theta^*_i u_i)(T_1) = (\theta^*_i u_i)(T_2) = \alpha \) for all \( i = 1, \ldots, n \) and such that \( |v^*(t) - v(T_1)| < \epsilon \) and \( |v^*(t + hS) - v(T_2)| < \epsilon \). Therefore, on the one hand, \( (\theta^*_i u_i)(T_2) = (\theta^*_i u_i)(T_1) = \alpha \) which implies that \( w(T_2) - w(T_1) = v(T_2) - v(T_1) \), and on the other hand, \( v(T_2) - v(T_1) < v^*(t + hS) - v^*(t) + 2\epsilon \). Altogether \( v^*(t + hS) - v^*(t) > w(T_2) - w(T_1) - 2\epsilon \). As \( w \) is monotonic we deduce that \( v^*(t + hS) - v^*(t) > -2\epsilon \), and as this holds for any \( \epsilon > 0 \) we conclude that \( v^*(t + hS) - v^*(t) > 0 \) and therefore \( \partial v^*(t, S) > 0 \) for any \( 0 < t < \alpha \) for which \( \partial v^*(t, S) \) is defined. This completes the proof of (3.6).

For proving (3.7) it is enough to prove that for any \( t \) with \( \alpha < t < 1 \) for which all the derivatives \( \partial v^*(t, S) \) and \( \partial q_i^*(t, S) \) exist,

\[
\partial v^*(t, S) + \sum_{i=1}^{n} \partial q_i^*(t, S) > 0.
\]

Applying Corollary 3.4 to the vector \( (\theta^*_1, \ldots, \theta^*_n, \mu) \) of nonatomic measures, the \( 2 \)-tuple \( t, t + hS \) (where \( 0 < h < 1 \)) and the members \( v, q_1, \ldots, q_n \) in \( pNA \) we have for every \( \epsilon > 0 \) two sets \( T_1, T_2 \) in \( \mathcal{E} \) such that for every \( 1 \leq i < n \), \( \alpha < t = (\theta^*_i u_i)(t) = (\theta^*_i u_i)(T_1) < (\theta^*_i u_i)(T_2) \) and \( |q_i(T_1) - q_i(T_2)| < \epsilon \), \( |q_i(T_2) - q_i(t + hS)| < \epsilon \), and \( |v^*(t) - v(T_1)| < \epsilon \), \( |v^*(t + hS) - v^*(t)| < \epsilon \). Therefore,

\[
w(T_2) - w(T_1) = v(T_2) - v(T_1) + \sum_{i=1}^{n} q_i(T_2) - q_i(T_1)
\]

\[
< v^*(t + hS) - v^*(t) + \sum_{i=1}^{n} q_i^*(t + hS) - q_i^*(t) + 2(n + 1)\epsilon.
\]

Again as this holds for all \( \epsilon > 0 \) and as \( w \) is monotonic, it follows that \( \partial v^*(t, S) + \sum_{i=1}^{n} \partial q_i^*(t, S) > 0 \), which completes the proof of (3.7).

The proof of (3.8) will make use of

**Lemma 3.9.** Let \( \mu_1, \ldots, \mu_n \) be nonatomic probability measures and \( q_1, \ldots, q_n \) set functions in \( pNA \). Then for every \( 0 < \alpha < 1 \) and every \( \epsilon > 0 \) and every \( 1 < k < n \) there

1 Along the proof \( \alpha \) and \( \mu \) stand for the fixed scalar and the probability measure, respectively, that are used in the definition of the set function \( u \).
are two sets $T_1, T_2$ in $\mathcal{E}$, $T_1 \subset T_2$ such that for all $1 < i < n$ and for all $1 < j < m$

$$|q_j(T_1) - q_j^*(\alpha)| < \epsilon, \quad |q_j(T_2) - q_j^*(\alpha)| < \epsilon$$

$$|q_j(T_2 \setminus T_1)| < \epsilon,$$

$$\mu_i(T_2 \setminus T_1) < \epsilon,$$

$$\mu_i(T_2) > \alpha > \mu_i(T_1) \quad \text{iff} \quad \mu_i = \mu_k.$$  

**Proof.** Let $K = \{1 < i < n : \mu_i = \mu_k\}$. By Lyapunov's theorem there is $T$ in $\mathcal{E}$ such that $\mu_k(T) = \mu_k(T)$ iff $\mu_i = \mu_k$. Let $b = \mu_k(T)$. Observe that for sufficiently small $\gamma > 0$, $\alpha + \gamma(T - b)$ is an ideal set and that $\mu_i^*(\alpha + \gamma(T - b)) = \alpha$ iff $i \in K$ (i.e., iff $\mu_i = \mu_k$). If $(f_i)_{i=1}^\infty$ is a sequence in $\mathcal{F}$ that converges uniformly to $f$ in $\mathcal{F}$ then for every $q$ in $pNA$, $q^*(f_i)$ converges to $q^*(f)$. (All that is needed for that conclusion is that $\mu_i^*(f_i) \rightarrow \mu^*(f)$ converges to $\mu^*(f)$ for every nonatomic measure $\mu$.) Therefore there is $\gamma > 0$ sufficiently small so that $\alpha + \gamma(T - b)$ is an ideal set and that for all $1 < j < m$, $|q_j^*(\alpha + \gamma(T - b)) - q_j^*(\alpha)| < \epsilon/3$. Fix such a $\gamma > 0$, and observe that $\beta(\alpha + \gamma(T - b)) \rightarrow \alpha + \gamma(T - b)$ as $\beta < 1$ converges to 1. Therefore for sufficiently large $\beta < 1$, for all $1 < j < m$

$$|q_j^*((1 - \beta)(\alpha + \gamma(T - b)))| < \epsilon/3,$$

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j^*(\alpha + \gamma(T - b))| < \epsilon/3,$$

and thus

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j^*(\alpha)| < 2\epsilon/3.$$  

As $\mu_i^*(\alpha + \gamma(T - b)) = \alpha$ iff $i \in K$, it follows that for sufficiently large $\beta < 1$ we also have

$$\mu_i^*(\alpha + \gamma(T - b)) > \alpha > \mu_i^*(\beta(\alpha + \gamma(T - b))) \quad \text{iff} \quad i \in K.$$  

Fix such a $\beta < 1$ and apply Corollary 3.4 to the 2-tuple $\beta(\alpha + \gamma(T - b)) < (\alpha + \gamma(T - b))$ with $\epsilon/3$, the vector $(\mu_1, \ldots, \mu_n)$ of nonatomic measures and the members $q_1, \ldots, q_m$ of $pNA$ to show the existence of $T_1, T_2 \in \mathcal{E}$ with $T_1 \subset T_2$,

$$\mu_i(T_1) = \mu_i^*(\beta(\alpha + \gamma(T - b))), \quad 1 < i < n,$$

$$\mu_i(T_2) = \mu_i^*(\alpha + \gamma(T - b)), \quad 1 < i < n,$$

$$|q_j^*(\alpha + \gamma(T - b)) - q_j(T_2)| < \epsilon/3, \quad 1 < j < m,$$

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j(T_1)| < \epsilon/3, \quad 1 < j < m,$$

$$|q_j^*((1 - \beta)(\alpha + \gamma(T - b))) - q_j(T_2 \setminus T_1)| < \epsilon/3.$$  

Altogether, we conclude that for all $1 < i < n$, $1 < j < m$

$$|q_j(T_2) - q_j^*(\alpha)| < \epsilon/3 + \epsilon/3 < \epsilon,$$

$$|q_j(T_1) - q_j^*(\alpha)| < \epsilon/3 + \epsilon/3 + \epsilon/3 < \epsilon,$$

$$|q_j(T_2 \setminus T_1)| < \epsilon/3 + \epsilon/3 < \epsilon, \quad \text{and}$$

$$\mu_i(T_2) > \alpha > \mu_i(T_1) \quad \text{iff} \quad \mu_i = \mu_k,$$

which completes the proof of Lemma 3.9.

We return now to proof of (3.8) of Lemma 3.5. Observe that it is sufficient to prove that for every $1 < k < n$ if $K(k)$ denotes the set of all $1 < i < n$ with $\theta_i^*\mu = \theta_k^*\mu$ then $\sum_{i \in K(k)} q_i^*(\alpha) > 0$. Apply Lemma 3.9 to the nonatomic probability measures $\theta_1^*\mu, \ldots, \theta_n^*\mu$ and the set functions $\nu, q_1, \ldots, q_n$ in $pNA$ to show the existence of
T₁, T₂ in $\mathcal{E}$ with $T₁ ⊂ T₂$ and such that for all $1 ≤ i ≤ n$,

\[
|v(T₁) - v^*(\alpha)| < \epsilon, \quad |v(T₂) - v^*(\alpha)| < \epsilon,
\]

\[
|q_i(T₁) - q_i^*(\alpha)| < \epsilon, \quad |q_i(T₂) - q_i^*(\alpha)| < \epsilon,
\]

\[
\theta_i^* \mu(T₂) > \alpha > \theta_i^* \mu(T₁) \quad \text{iff} \quad i ∈ K(k).
\]

Therefore

\[
w(T₂) - w(T₁) ≤ v(T₂) - v(T₁) + \sum_{i ∈ K(k)} q_i(T₂) + \sum_{i ∈ K(k)} |q_i(T₂) - q_i(T₁)|
\]

\[
< 2\epsilon + \sum_{i ∈ K(k)} q_i^*(\alpha) + 2\epsilon\eta
\]

\[
< \sum_{i ∈ K(k)} q_i^*(\alpha) + 2(n + 1)\epsilon.
\]

As this holds for every $\epsilon > 0$ the assumption that $w$ is monotonic implies that $\sum_{i ∈ K(k)} q_i^*(\alpha) > 0$ which completes the proof of Lemma 3.5.

**Lemma 3.10.** Let $g$ be in $W$ and $a$ in $R^+$. Then (3.1) and (3.2) defines (uniquely) a semivalue $\psi_{(a,g)}$ on $u * pNA$.

**Proof.** Any element $w$ in $u * pNA$ is of the form $w = v + \sum_{j=1}^n (\theta_j^* u)q_j$, $\theta_j ∈ G$, $v, q_j ∈ pNA$. By linearity and symmetry, it follows from (3.1) and (3.2) that

\[
\psi_{(a,g)} w(S) = \int_0^1 g(t) \partial v^*(t, S) dt + \sum_{i=1}^n \int_0^1 g(t) \partial q_i^*(t, S) dt
\]

\[+
\sum_{i=1}^n aq_i^*(\alpha)(\theta_i^* \mu)(S).\]  

(3.9)

We have to show that $\psi_{(a,g)}$ is well defined, i.e., that it is independent of the representation of $w$. Because of the linearity it is enough to show that if $w = 0$ then $\psi_{(a,g)} w = 0$. If $w = 0$ then by Lemma 3.5 we conclude that $\psi_{(a,g)} w(S) > 0$, and that $\psi_{(a,g)} (-w)(S) = -\psi_{(a,g)} w(S) > 0$ which means that $\psi_{(a,g)} w = 0$. Linearity and symmetry of $\psi_{(a,g)}$ follow from the definition. The finite additivity of $\partial q_i^*(t, S)$ ($q ∈ pNA$) as well as that of $\theta_i^* \mu$ implies that $\psi_{(a,g)} w$ is finitely additive. Positivity of $\psi_{(a,g)}$ follows now from Lemma 3.5 and the finite additivity of $\psi_{(a,g)} w$. Obviously $u * pNA$ is reproducing; hence that positivity of $\psi_{(a,g)}$ and the finite additivity of $\psi_{(a,g)} w$ implies that $\psi_{(a,g)} w$ is in $FA$ whenever $w$ is in $\mu * pNA$. Now let $w ∈ (u * pNA) ∩ FA$. We have to show that $\psi_{(a,g)} w = w$. Without loss of generality we may assume that $w = v + \sum_{j=1}^n (\theta_j^* \mu) q_j$ where $v ∈ pNA$ and $q_j ∈ pNA$ and $\theta_j^* \mu = \theta_j^* \mu$ iff $i = j$. First we shall show that $q_k^*(\alpha) = 0$ for each $k$, $1 ≤ k ≤ n$. Let $1 ≤ k ≤ n$ be given. Applying Lemma 3.9 to the nonatomic probability measures $\theta_1^* \mu, \ldots, \theta_n^* \mu$, the set functions $v, q_1, \ldots, q_n$ in $pNA$ we have for every $0 < \epsilon$ two sets $T₁, T₂ ∈ \mathcal{E}$, $T₁ ⊂ T₂$ and such that for all $1 ≤ i ≤ n$ and

\[
\theta_i^* \mu(T₂) > \alpha > \theta_i^* \mu(T₁) \quad \text{iff} \quad i = k,
\]

\[
|v(T₂) - v(T₁)| < \epsilon, \quad |v(T₂ \setminus T₁)| < \epsilon,
\]

\[
|q_k(T₂) - q_k(T₁)| < \epsilon \quad \text{and} \quad \theta_i^* \mu(T₂ \setminus T₁) < \epsilon.
\]

Assuming $\epsilon < \alpha$ we find that $|w(T₂ \setminus T₁)| = |v(T₂ \setminus T₁)| < \epsilon$. On the other hand,

\[
|w(T₂) - w(T₁)| ≥ q_k(T₂) - |v(T₂) - v(T₁)| - \sum_{i=1}^n |q_i(T₂) - q_i(T₁)|
\]

\[>
|q_k^*(\alpha)| - \epsilon - \epsilon - n\epsilon = |q_k^*(\alpha)| - (n + 2)\epsilon.
\]
The assumption that \( w \) is finitely additive will imply that \( \epsilon > |w(T_2 \setminus T_1)| = |w(T_2) - w(T_1)| > |q_1^*(\alpha) - (n + 2)\epsilon| \), i.e., that \( |q_1^*(\alpha)| < (n + 3)\epsilon \). As this is true for every \( 0 < \epsilon < \alpha \) we conclude that \( q_1^*(\alpha) = 0 \). Let \( S \) be in \( \mathcal{S} \), with \( \mu(\theta, S) < \alpha \). In that case \( w(S) = v(S) \), and by using the finite additivity of \( w \) and Lemma 3.3, we see that \( \psi^*(hS) = h(\psi(S)) \), and then by continuity of \( \psi^* \) we deduce that \( \psi^*(hS) = h\psi(S) \) for any real \( h < 1 \). Therefore \( \partial \psi^*(0, S) = \psi(S) \). Now, let \( 0 < t < \alpha \), and let \( S \in \mathcal{S} \) with \( \mu(\theta, S) < \alpha \). Again using Lemma 3.3 to the vector measure \( \theta_i^* \mu, 1 < i < n \), and the games \( v \in pNA \) and the 3-tuple \( hS, t, 1 - t - hS \), \( h < \alpha - t \) we have for any \( \epsilon > 0 \) a partition \( (T_1, T_2, T_3) \) of \( I \) with \( |v(T_1) - v^*(hS)| < \epsilon \), \( |v(T_2) - v^*(t)| < \epsilon \) and \( |v(T_3) - v^*(t) + hS)| < \epsilon \) and \( \partial \mu(\theta, T_1, T_2) < \alpha \). Hence \( w(T_1 \cup T_2) = v(T_1 \cup T_2), w(T_1) = v(T_1) \) and \( w(T_2) = v(T_2) \). Therefore, using the finite additivity of \( w \) we have \( \psi^*(hS) = h\psi(S) \) and therefore \( \partial \psi^*(t, S) \) exists and equals \( \psi(S) \). In a similar way, by using Lemma 3.3 to the vector measure \( \theta_i^* \mu, 1 < i < n \), and the games \( v \in pNA \) and the 3-tuple \( hS, t, 1 - t - hS, h \), \( h < 1 - t \) we can prove that for \( \alpha < t < 1 \), \( \psi^*(\theta^* \mu) = \psi(S) \) whenever \( S \) is in \( \mathcal{S} \) with \( \mu(\theta, S) < \alpha \). For \( \alpha < t < 1 \), \( \psi^*(\theta^* \mu) = \psi(S) \) whenever \( S \) is in \( \mathcal{S} \) with \( \mu(\theta, S) < \alpha \). Therefore by the finite additivity of \( w \) as well as that of \( \psi^* \) we have \( \psi^*(S) = \psi(S) \) which completes the proof of Lemma 3.10.

**Lemma 3.11.** Let \( g \) be in \( W \) and \( a \) in \( R^* \). Then the semivalue \( \psi_{(a, g)} \) on \( u^* \cdot pNA \) defined by (3.1) and (3.2) is continuous and \( \|\psi_{(a, g)}\| = \max \{a, \|g\|_{L_\infty}\} \).

**Proof.** Let \( w \) be in \( u^* \cdot pNA \). Without loss of generality we may assume that \( w = v + \sum_{i=1}^{n}(\theta_i^* w)q_i \) where \( v \) is in \( pNA \), \( q_i \in pNA \) for \( 1 < i < n \) and \( \theta_i^* \mu = \theta_j^* \mu \) if \( i = j \) (1 < i, j < n).

\[
\|\psi_{(a, g)}w\| = \sup_{S \in \mathcal{S}} |(\psi_{(a, g)}w)(S)| + |(\psi_{(a, g)}w)(1 - S)|.
\]

Therefore we have to prove that the right-hand side is at most \( \max \{a, \|g\|_{L_\infty}\} \|w\| \). As

\[
|(\psi_{(a, g)}w)(S)| = \left| a \sum_{i=1}^{n} q_i^*(\alpha)(\theta_i^* \mu)(S) + \int_{a}^{\alpha} \partial v^*(t, S) g(t) dt \right|
\]

\[
\leq a \sum_{i=1}^{n} |q_i^*(\alpha)|((\theta_i^* \mu)(S))
\]

\[
\leq \max \{a, \|g\|_{L_\infty}\} \left[ \int_{0}^{\alpha} \partial v^*(t, S) dt + \int_{a}^{1} \partial v^*(t, S) dt \right]
\]

\[
\leq \max \{a, \|g\|_{L_\infty}\} \left[ \int_{0}^{1} \partial v^*(t, S) dt + \sum_{i=1}^{n} |q_i^*(\alpha)|((\theta_i^* \mu)(S)) \right]
\]

\[
\leq \max \{a, \|g\|_{L_\infty}\} \left[ \int_{0}^{1} \partial v^*(t, S) dt + \sum_{i=1}^{n} |q_i^*(\alpha)|((\theta_i^* \mu)(S)) \right]
\]

\[
\leq \max \{a, \|g\|_{L_\infty}\} \left[ \int_{0}^{1} \partial v^*(t, S) dt + \sum_{i=1}^{n} |q_i^*(\alpha)|((\theta_i^* \mu)(S)) \right]
\]
it is sufficient to prove that
\[
\|w\| > \sum_{i=1}^{n} |q_i^*(\alpha)| + \int_{0}^{\alpha} \left( |\partial v^*(t, S)| + |\partial v^*(t, 1 - S)| \right) dt + \int_{\alpha}^{1} \left( |\partial v + \sum_{i=1}^{n} g_i|*(t, S)| + |\partial v + \sum_{i=1}^{n} g_i|*(t, 1 - S)| \right) dt.
\] (3.12)

First assume that \( v \) and \( q_i, 1 < i < n \) are polynomials in nonatomic probability measures. For every integer \( k > 2 \) we will construct a chain \( \Omega_k \) so that \( \|w\|_{\Omega_k} \) will converge as \( k \to \infty \) to the right-hand side of (3.12).

Observe that there is \( f \in \mathcal{F} \) with \((\theta_i^*\mu)^*(f) = (\theta_j^*\mu)^*(f) \) iff \( i = j \). We may assume that \( 1 < i, j < n \) are polynomials in nonatomic measures. For every integer \( k > 2 \) we will construct a chain \( \Omega_k \) so that \( \|w\|_{\Omega_k} \) will converge as \( k \to \infty \) to the right-hand side of (3.12).

Let \( v \) be the largest integer with \( l < ak \). Without loss of generality we may assume that for \( l < i < n \) there is a (unique) \( \beta_i = \beta_i(k) \) with \( 0 < \beta_i < 2/k \) and \((\theta_i^*\mu)^*(l/k + \beta_i f) = \alpha \). Obviously all the \( \beta_i \)'s are different and \( 0 < \beta_i < \beta_j < 2/k \) whenever \( 1 < i < j < n \). Define \( (g_j)_{j=1}^{2k+n-3} \) by

\[
\tilde{f}_i = \begin{cases} 
\frac{i}{2k} & \text{if } i < 2l \text{ is an even integer}, \\
\frac{i - 1 + \frac{S}{2}}{2k} & \text{if } i < 2l \text{ is an odd integer}, \\
\frac{l}{k} + \frac{g_i - 2l}{2k} & \text{if } 2l < i < 2l + n, \\
\frac{i + 3 - n}{2k} & \text{if } 2l + n < i \text{ and } i - n \text{ is an odd integer}, \\
\frac{i + 2 - n}{2k} + \frac{S}{k} & \text{if } 2l + n < i \text{ and } i - n \text{ is an even integer}.
\end{cases}
\]

Apply Corollary 3.4 to the vector \((\theta_i^*\mu, \ldots, \theta_j^*\mu)\) of nonatomic measures, the members \( v, q_1, \ldots, q_n \) of \( pNA \) and \( \epsilon = \frac{1}{k(2k + n)} \) to construct a chain \( \Omega_k : (T_j)_{j=0}^{2k+n-3} \) \((T_0 \subseteq T \subseteq \cdots \subseteq T_{2k+n-3})\) such that for all \( 1 < j < n \) and for all \( 0 < i < 2k + n - 3 \),

\[
(\theta_i^*\mu)(T_j) = (\theta_j^*\mu)(T_i), \quad |q_j(T_i) - q_i^*(\tilde{f}_i)| < \frac{1}{k(2k + n)} ,
\]

\[
|v(T_i) - v^*(\tilde{f}_i)| < \frac{1}{k(2k + n)} .
\]

Denote by \( \Omega_k^1 \) the subchain \( (T_j)_{j=0}^{2l} \), \( \Omega_k^2 \) the subchain \( (T_j)_{j=l+2}^{2l+n} \) and \( \Omega_k^3 \) the subchain \( (T_j)_{j=2l+n+1}^{2k+n-3} \). Then

\[
\|w\|_{\Omega_k} > \|w\|_{\Omega_l^1} + \|w\|_{\Omega_l^2} + \|w\|_{\Omega_l^3} .
\]

As \((\theta_j^*\mu)(T_{2l}) = (\theta_j^*\mu)(\tilde{f}_{2l}) = 1/k < \alpha \) for all \( 1 < j < n \),

\[
\|w\|_{\Omega_l^1} = \sum_{i=1}^{2l} |v(T_i) - v(T_{i-1})| < \sum_{i=1}^{2l} |v^*(\tilde{f}_i) - v^*(\tilde{f}_{i-1})| - \frac{2l}{k(2k + n)} .
\]

As \( v \) is a polynomial in nonatomic measures, \( \sum_{i=1}^{2l} |v^*(\tilde{f}_i) - v^*(\tilde{f}_{i-1})| \) converges as \( k \to \infty \) to \( \int_{S}^{S} (|\partial v^*(t, S)| + |\partial v^*(t, 1 - S)|) dt \) (see for instance pp. 45, 46 of [3] or observe that for \( 1 < i < 2l \) \( |v^*(\tilde{f}_i) - v^*(\tilde{f}_{i-1})| = |\partial v^*(i/2k, S)|/k + o(1/k) \) where \( S = S \text{ if } i \text{ is odd and } S = 1 - S \text{ if } i \text{ is even} \).

Thus

\[
\liminf_{k \to \infty} \|w\|_{\Omega_l^1} > \int_{0}^{\alpha} (|\partial v^*(t, S)| + |\partial v^*(t, 1 - S)|) dt .
\]
Similarly
\[ \liminf_{k \to \infty} \|w\|_{\Omega^2} \geq \int_\alpha^1 \left( |\partial \left( v + \sum_{j=1}^n q_j \right)^*(t, S) | + |\partial \left( v + \sum_{j=1}^n q_j \right)^*(t, 1 - S) | \right) dt. \]

We turn now to the estimation of \( \|w\|_{\Omega^2} \). For each fixed \( 1 < j < n \), \((\theta^*_\mu)_{T_{j+2l}} > \alpha > (\theta^*\mu)_{T_{j+2l-1}}\) iff \( i = j \). Thus
\[ |w(T_{j+2l}) - w(T_{j+2l-1})| > |q_j(T_{j+2l}) - v(T_{j+2l}) - v(T_{j+2l-1})| \]
\[ - \sum_{i=1}^n |q_i(T_{j+2l}) - q_i(T_{j+2l-1})|. \]

Thus \( \|w\|_{\Omega^2} \geq \sum_{j=1}^n (\theta^*_\mu)_{T_{j+2l}} - \|v\|_{\Omega^2} - \sum_{j=1}^n |q_j|_{\Omega^2} \).

For each fixed \( 1 < j < n \), \((\theta^*_\mu)_{T_{j+2l}} \to (\theta^*_\mu)_{T_{j+2l-1}}\) as \( k \to \infty \) and \( \|q_j\|_{\Omega^2} \to 0 \) and \( k \to \infty \)

and also \( \|v\|_{\Omega^2} \to 0 \) as \( k \to \infty \) and thus \( \liminf_{k \to \infty} \|w\|_{\Omega^2} \geq \sum_{j=1}^n (\theta^*_\mu)_{T_{j+2l-1}} \).

Altogether we conclude that
\[ \|w\| > \liminf \|w\|_{\Omega^2} \geq \liminf \|w\|_{\Omega^2} + \liminf \|w\|_{\Omega^2} + \liminf \|w\|_{\Omega^2} \]
\[ > \sum_{i=1}^n |q_i^*(\alpha)| + \int_0^\alpha \left( |\partial v^*(t, S) | + |\partial v^*(t, 1 - S) | \right) dt \]
\[ + \int_\alpha^1 \left( |\partial \left( v + \sum_{i=1}^n q_i \right)^*(t, S) | + |\partial \left( v + \sum_{i=1}^n q_i \right)^*(t, 1 - S) | \right) dt \]

which proves (3.12) in the case that \( v \) and \( q_i \) are polynomials in nonatomic measures.

For the general case let \( \epsilon > 0 \) and approximate \( v \) and \( q_i \) by polynomials of nonatomic measures \( \bar{v} \) and \( \bar{q}_i \) respectively with \( \|v - \bar{v}\| < \epsilon \), \( \|q_i - \bar{q}_i\| < \epsilon \), and let \( \bar{w} = \bar{v} + \sum_{i=1}^n (\theta^*_\mu)_{T_{j+2l}} \bar{q}_i \).

As \( \|\theta^*_\mu\| = 1 \) and \( \|v_1\| \leq \|v_2\| \) for all \( v_1, v_2 \) in \( BV \), \( \|\bar{w} - w\| \leq \|v - \bar{v}\| + \sum_{i=1}^n \|((\theta^*_\mu)_{T_{j+2l}}q_i - \bar{q}_i)\| < (n + 1)\epsilon \). Using Lemma 23.1 of [3] we have for all \( S \in \mathcal{E} \),
\[ \int_0^\alpha |\partial v^*(t, S) - \partial v^*(t, S)| \leq \|v - \bar{v}\| < \epsilon \]
\[ \int_\alpha^1 |\partial \left( \bar{v} + \sum_{i=1}^n \bar{q}_i \right)^*(t, S) - \partial \left( v + \sum_{i=1}^n q_i \right)^*(t, S) | dt \leq (n + 1)\epsilon. \]

Also \( |q_i^*(\alpha) - \bar{q}_i^*(\alpha)| < \|q_i^* - \bar{q}_i^*\| < \epsilon \). Altogether,
\[ (n + 1)\epsilon + \|w\| > \|\bar{w}\| > \sum_{i=1}^n |q_i^*(\alpha)| - n\epsilon + \int_0^\alpha |\partial v^*(t, S) | dt - \epsilon \]
\[ + \int_0^\alpha |\partial v^*(t, 1 - S) | dt - \epsilon \]
\[ + \int_\alpha^1 |\partial \left( v + \sum_{i=1}^n q_i \right)^*(t, S) | dt - (n + 1)\epsilon \]
\[ + \int_\alpha^1 |\partial \left( v + \sum_{i=1}^n q_i \right)^*(t, 1 - S) | dt - (n + 1)\epsilon. \]

As this is true for all \( \epsilon > 0 \), (3.12) is proved which completes the proof of Lemma 3.11.

**Proof of Theorem A.** We have already seen that for \( a \) in \( R^+ \) and \( g \) in \( W \), (3.1) and (3.2) define (uniquely) a (continuous) semivalue \( \psi_{(a,g)} \) on \( u \cdot pNA \). Now we have to show that any continuous semivalue on \( u \cdot pNA \) is of that form. Let \( \psi \) be a continuous semivalue on \( u \cdot pNA \). In particular, \( \psi \) induces a semivalue on \( pNA \) and
therefore by Theorem 2.3 there is \( g \) in \( W \) with
\[
\psi^*(S) = \int_0^1 g(t) \delta v^*(t, S) \, dt \quad \text{for each} \quad v \text{ in } pNA.
\]
(3.13)

Let \( v \) be a probability measure in \( NA \), and \( k \) a positive integer. For any \( \delta > 0 \), \( \delta < (1/2) \min(\alpha, 1 - \alpha) \) define \( F_\delta : [0, 1] \to R^+ \) by
\[
F_\delta(x) = \begin{cases} 
0 & \text{if } |x - \alpha| \geq 2\delta, \\
1 & \text{if } |x - \alpha| \leq \delta, \\
1 - 1/\delta(|x - \alpha| - \delta) & \text{if } \delta < |x - \alpha| < 2\delta,
\end{cases}
\]
and define \( \tilde{v}_\delta \) by:
\[
\tilde{v}_\delta = (F_\delta \circ v)(F_\delta \circ \mu)(\nu^k - \mu^k).
\]
(3.14)

First we shall show that
\[
(uv_\delta)\| \leq 32k\delta.
\]
(3.15)

Define \( U = \{ S \in \mathfrak{d} : 0 < \mu(S) - \alpha < 2\delta, |\nu(S) - \mu| < 4\delta \} \). Then for \( S \) in \( U \), \( \mu(S) > \alpha \) and therefore \( u(S) = 1 \) and also for \( S \) in \( U \), \( |\nu^k(S) - \mu^k(S)| < 48k \), and \( |\tilde{v}_\delta(S)| < 48k \). For every \( S \) in \( \mathfrak{d} \setminus U \) either \( \mu(S) < \alpha \) and thus \( u(S) = 0 \) or \( |\nu(S) - \mu| > 2\delta \) and thus \( \tilde{v}_\delta(S) = 0 \), or \( \mu(S) - \alpha > 2\delta \) and thus \( \tilde{v}_\delta(S) = 0 \). In any case for \( S \not\in U \), \( (uv_\delta)(S) = 0 \). Let \( \Omega : S_0 \subset S_1 \subset \cdots \subset S_L \) be a chain. Let \( i_0 \) be the first index for which \( S_{i_0} \in U \) and let \( j_0 \) be the last index for which \( S_{j_0} \in U \). Then from the definition of \( U \) it follows that \( S_i \in U \) iff \( i_0 < i < j_0 \). Therefore, as \( (uv_\delta)(S) = 0 \) whenever \( S \not\in U \), \( \| (uv_\delta) \| < 48k \) whenever \( S \in U \) we deduce that
\[
\| uv_\delta \| \leq \sum_{i=1}^L \| (uv_\delta)(S_i) - (uv_\delta)(S_{i-1}) \|.
\]
(3.16)

\[
= \sum_{i=i_0}^{j_0+1} \| (uv_\delta)(S_i) - (uv_\delta)(S_{i-1}) \|
\]
(3.17)

\[
= |uv_\delta(S_{i_0})| + \sum_{i=i_0+1}^{j_0} |uv_\delta(S_i) - (uv_\delta)(S_{i-1})|.
\]
(3.18)

\[
\leq 8\delta k + \sum_{i=i_0+1}^{j_0} |(uv_\delta)(S_i) - (uv_\delta)(S_{i-1})|.
\]
(3.19)

\[
\sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})|.
\]
(3.20)

\[
= \sum_{i=i_0+1}^{j_0} |(F_\delta \circ v)(F_\delta \circ \mu)(\nu^k - \mu^k)(S_i) - (F_\delta \circ v)(F_\delta \circ \mu)(\nu^k - \mu^k)(S_{i-1}) + (F_\delta \circ v)(F_\delta \circ \mu)(\nu^k - \mu^k)(S_{i-1}) - (F_\delta \circ v)(F_\delta \circ \mu)(\nu^k - \mu^k)(S_{i-1})|
\]
(3.21)

\[
\leq \max_{S \in U} |(F_\delta \circ v)(F_\delta \circ \mu)(S)| \sum_{i=i_0+1}^{j_0} |(\nu^k - \mu^k)(S_i) - (\nu^k - \mu^k)(S_{i-1})| + \max_{S \in U} |(\nu^k - \mu^k)(S)| \sum_{i=i_0+1}^{j_0} |(F_\delta \circ v)(F_\delta \circ \mu)(S_i) - (F_\delta \circ v)(F_\delta \circ \mu)(S_{i-1})|.
\]
But \( \max |(F_\delta \circ \nu)(F_\delta \circ \mu)(S)| \leq 1 \) and

\[
\sum_{i=1}^{n} |(p^k - \mu^k)(S_i) - (p^k - \mu^k)(S_{i-1})| \leq (p^k + \mu^k)(S_n) - (p^k + \mu^k)(S_0) < 48k \quad \text{and}
\]

\[
\max_{S \in U} |(p^k - \mu^k)(S)| \leq 4k \delta \quad \text{and} \quad \| (F_\delta \circ \nu)(F_\delta \circ \mu) \| \leq 4.
\]

Therefore \( \sum_{i=1}^{n} |v^k - \mu^k| \leq 48k + (4k \delta)4 = 24k \delta, \) hence \( \| u \| \leq 32k \delta. \) As this holds for any chain \( \Omega, (3.15) \) is proved. Define \( G_\delta : [0, 1] \to \mathbb{R}^+ \) by

\[
G_\delta(x) = \begin{cases} 
0 & \text{if } x > \alpha + 2\delta, \\
1 & \text{if } x \leq \alpha + \delta, \\
1 - \frac{1}{2}(x - \alpha - \delta) & \text{if } \alpha + \delta < x \leq \alpha + 2\delta,
\end{cases}
\]

and define \( \tilde{v}_\delta \) by

\[
\tilde{v}_\delta = (G_\delta \circ \nu)(G_\delta \circ \mu)(p^k - \mu^k).
\]

(3.16)

First observe that \( \tilde{v}_\delta \in pNA \) (although \( G \circ \nu)(G \circ \mu) \notin pNA \). Define \( D \) to be the diagonal neighborhood defined by \( D = \{ S : |\mu(S) - \nu(S)| < \delta \}. \) Let \( S \in D \) and denote \( v = p^k - \mu^k; \) then \( u(v - \tilde{v}_\delta)(S) = (v - \tilde{v}_\delta)(S), \) because if \( \mu(S) < \alpha \) and \( S \in D \) then \( \nu(S) < \alpha + \delta \) and therefore \( \tilde{v}_\delta(S) = \nu(S) \) and of course then \( u(v - \tilde{v}_\delta)(S) = 0 = (v - \tilde{v}_\delta)(S), \) and if \( \mu(S) > \alpha \) then \( u(v - \tilde{v}_\delta)(S) = (v - \tilde{v}_\delta)(S), \) and \( \nu(S) > \alpha - \delta. \)

But for \( x > \alpha + \delta, G_\delta(x) = F_\delta(x) \) which yield that \( (v - \tilde{v}_\delta)(S) = (v - \tilde{v}_\delta)(S), \) whenever \( S \in D \) with \( \mu(S) > \alpha. \) Thus we have seen that

\[
u(v - \tilde{v}_\delta) \text{ coincides with } v - \tilde{v}_\delta \text{ on a diagonal neighborhood, }\]

\( v - \tilde{v}_\delta \in pNA \) and \( u(v - \tilde{v}_\delta) \in u*pNA. \)

(3.17)

As \( \psi \) is continuous Proposition 2.4 and (3.17) implies that

\[
\psi(u(v - \tilde{v}_\delta)) = \psi(v - \tilde{v}_\delta).
\]

(3.18)

Now we claim that

\[
\partial \tilde{v}_\delta^*(t, S) = \begin{cases} 
0 & \text{if } t > \alpha + 2\delta, \\
\nu^*(t, S) & \text{if } t < \alpha + \delta.
\end{cases}
\]

(3.19)

To prove (3.19) observe that if \( t > \alpha + 2\delta \) and \( h > 0 \) then

\[
[(G_\delta \circ \nu)(G_\delta \circ \mu)(p^k - \mu^k)]^*(t) = 0 = [(G_\delta \circ \nu)(G_\delta \circ \mu)(p^k - \mu^k)]^*(t + hS)
\]

and if \( 0 < t < \alpha + \delta \) and \( h < 0 \) with \( t + h > 0 \) then

\[
[(G_\delta \circ \nu)(G_\delta \circ \mu)(p^k - \mu^k)]^*(t + hS) = \nu^*(t + hS).
\]

As \( v - \tilde{v}_\delta \) is in \( pNA, \) (3.13) and (3.18) imply that

\[
\left| \psi(v - \tilde{v}_\delta)(S) - \int_{\alpha + 2\delta}^{t} \partial \nu^*(t, S) g(t) dt \right| \leq \left| \int_{\alpha + \delta}^{\alpha + 2\delta} g(t) \partial(v - \tilde{v}_\delta)^*(t, S) dt \right| \to 0 \quad (3.20)
\]

If we let \( \delta \to 0, (3.20), (3.18) \) and (3.15) imply that

\[
\psi(u(p^k - \mu^k))(S) = \int_{\alpha}^{t} g(t) \partial(p^k - \mu^k)^*(t, S) dt.
\]

(3.21)
Observe that \( u \in u \ast pNA \). By Proposition 2.5 \( \psi u = a \mu \), and by the positivity of \( \psi \), \( a \in R^+ \). Now let \( B \) be the subset of \( pNA \) of all games \( \nu \) for which \( \psi(\nu) = \psi_{(a,g)}(\nu) \).

By (3.21) \( v^k - \mu^k \in B \). Observe that \( u_{\nu^k} - \alpha^k u \) is in \( pNA \) and hence \( \psi(u_{\nu^k} - \alpha^k u)(S) = \int_S g(t)\delta(\nu_{\nu^k} - \alpha^k u)(t, S) dt \) and \( \psi(\alpha^k u) = \alpha^k \mu u \). Therefore it is easily verified that \( \mu^k \in B \).

But \( B \) is obviously a linear subspace of \( pNA \) and therefore as it contains \( \mu^k \) and \( v^k - \mu^k \) it contains \( v^k \) for any probability measure in \( NA \) and hence any polynomial in \( NA^+ \) measures. As both \( \psi \) and \( \psi_{(a,g)} \) are continuous and \( \|u\nu\| \leq \|u\| \|\nu\| \) it follows that \( B \) is closed, thus \( B = pNA \). Now as both \( \psi \) and \( \psi_{(a,g)} \) are linear and continuous we deduce that they coincide on \( u \ast pNA \), which completes the proof of Theorem A.

\section{Further results and remarks}

We are able to characterize the set of all continuous semivalues on many other important spaces, like \( bv'NA \) and \( bv'NA \ast pNA \). As the proof uses similar methods to those presented in the former sections we will just give a sample of results.

\textbf{Notations.} Let \( X \) be a linear subspace (not necessarily closed) of the Banach space \( bv' \) (the space of all functions \( f: [0, 1] \to R \) with \( f(0) = 0 \) such that \( f \) is of bounded variation continuous at 0 and 1, endowed with the total variation norm). We denote by \( W(X) \) the subset of the dual \( X^* \) (of the closure \( \overline{X} \) of \( X \)) of all elements \( x^* \) satisfying:

1. For each monotonic nondecreasing \( f \in \overline{X} \), \( x^*(f) > 0 \);
2. If \( \overline{X} \) contains the function \( h \) defined by \( h(x) = x \), then \( x^*(h) = 1 \).

The subspace of all absolutely continuous elements in \( bv' \) is denoted \( ac' \). For each \( 0 < x < 1 \) define \( f_x: [0, 1] \to R \) by \( f_x(y) = 0 \) iff \( y < x \) and \( f_x(y) = 1 \) iff \( y > x \). If \( \overline{X} \) contains the function \( h \) defined by \( h(x) = x \), then \( x^*(h) = 1 \).

The subspace of \( bv' \) generated by the functions \( f_x(\tilde{f}_x) \) is denoted by \( rf' \) (\( \tilde{f}' \)), and that generated by all jump functions (i.e., by \( rf' \) and \( \tilde{f}' \)) is denoted by \( r' \). If \( X \subset bv' \) we denote by \( XNA \) the linear symmetric space generated by games of the form \( f \circ \mu \), \( f \in X \) and \( \mu \) is a probability measure in \( NA \).

\textbf{Theorem 4.1.} Let \( X \) be a subspace of \( bv' \). There is a 1-1 linear isometry from \( W(X) \) onto the continuous semivalues on \( XNA \); for each \( x^* \in W(X) \) the semivalue \( \psi_{x^*} \) on \( XNA \) is given by \( \psi_{x^*}(f \circ \mu) = x^*(f) \mu \).

\textbf{Remarks.} (a) \( W(ac') = W \) and therefore Theorem 4.1 can be considered a generalization of Theorem 2.3 (\( ac'NA \) is dense in \( pNA \)).

(b) \( W(rf') \) is identified with all bounded functions \( a: (0, 1) \to R^+ \); for \( 0 < x < 1 \) \( x^*(a)(f_x) = a(x) \) and \( \|x^*(a)\| = \sup_{0 < x < 1} a(x) \). Each of the continuous semivalues on \( rf'NA \) can be extended to a semivalue on its closure: However, there are discontinuous semivalues on \( rf'NA \); they can be obtained by omitting the boundness condition on \( a \).

(c) \( W(f') \) is identified with all pairs of bounded functions \( a, b: (0, 1) \to R^+ \) for where for \( 0 < x < 1 \) \( x^*(a, b)(f_x) = a(x) \) and \( x^*(a, b)(f_x) = b(x) \). We have \( \|x^*(a, b)\| = \sup_{0 < x < 1} \{a(x, b(x))\} \).

\textbf{Notations.} If \( Q_1 \) and \( Q_2 \) are linear symmetric subspaces of \( BV \) we denote by \( Q_1 \otimes Q_2 \) the linear symmetric space generated by games of the form \( v_1v_2 \), where \( v_i \in Q_i \), \( i = 1, 2 \), and the space \( Q_1 \ast Q_2 \) is defined as the linear symmetric space generated by \( Q_1 \otimes Q_2, Q_1 \) and \( Q_2 \).

\textbf{Theorem 4.2.} For each pair \( (a, g) \), \( a: (0, 1) \to R^+ \) and \( g \in W = W(ac') \) there is a semivalue \( \psi_{(a,g)} \) on \( rj'NA \ast pNA \) given by:

\[
\psi_{(a,g)}(v) = \psi_g v \quad \text{whenever} \quad v \in pNA, \tag{4.3}
\]

\[
\psi_{(a,g)}((f_x \circ \mu)v)(S) = a(x)v^*(x)\mu(S) + \int_x^1 g(t)\delta v^*(t, S) dt, \tag{4.4}
\]

whenever \( v \in pNA \), \( 0 < x < 1 \) and \( \mu \) is a probability measure in \( NA \). The semivalue \( \psi_{(a,g)} \)
is continuous iff $a$ is bounded. Moreover, any continuous semivalue on $r^+\cap pNA$ is of that form. $\psi(a,g)$ can be extended to a semivalue on $r^+\cap pNA$ iff $a$ is bounded and then $\|\psi(a,g)\| = \max\{\sup_{0<x<1}a(x), \|g\|_{L_\infty}\}$.

**Remark.** Similar results hold for the spaces $l^\infty\cap pNA$ and $j^\infty\cap pNA$. (In the second case the semivalues are associated with triples $(a,b,g).$

**Theorem 4.5.** For each pair $(a,g), a:(0,1)\rightarrow R^+$ and $g \in L_\infty(0,1)$ there is a semivalue $\psi(a,g)$ on $r^+\cap pNA$ given by (4.4). This semivalue is continuous if and only if $a$ is bounded. Moreover, any continuous semivalue is of that form.

**Remarks.** (a) The semivalues on $r^+\cap pNA$ differ from those on $r^+\cap j^\infty\cap pNA$ since $NA \not\subseteq r^+\cap j^\infty\cap pNA$ while $NA \subseteq r^+\cap j^\infty\cap pNA$.

(b) The proof of Theorems 4.2 and 4.5 are similar to that of Theorem A.

(c) The fact that $(a,0)$ is a semivalue on $r^+\cap j^\infty\cap pNA$ is easy to prove (see Lemma 3.5(3.8)) and actually makes use only of the property of $pNA$ of having a continuous extension to ideal sets satisfying Lemma 3.3. Thus it follows that the existence of such semivalues is valid for any space of the form $r^+\cap Q$ where $Q$ has such an extension. If $Q$ is such a space satisfying: there exist $\alpha:(0,1)\rightarrow R^+ \setminus \{0\}$ s.t. for each $v \in Q$ and $0 < x < 1$, $v^*(x) = \alpha(x)v^*(1)$ then by setting $\alpha(x) = 1/\alpha(x)$, $\psi(a,0)$ is a value on $r^+\cap Q$. However, these values are discontinuous, whenever $\alpha$ is not bounded away from 0.

(d) For every $g$ in $W$ which is continuous, there is a semivalue $\psi_g$ on DIFF (for definition see [6]) that is given by: for any $v$ in DIFF and $S \in \mathcal{E}$,

$$\psi_g v(S) = \lim_{h \rightarrow 0} \left( \int_0^1 \frac{v^*(t + hS) - v^*(t)}{h} g(t) dt \right).$$

The proof is essentially the same as in Merten's proof in [6] of the existence of a value on DIFF.

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**References**


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